Abstract

Let $G$ be a graph with $q$ edges. A graph $G$ is called even vertex odd mean graph if there exist an injective labeling $f : V(G) \rightarrow \{0, 2, 4, 6, ..., 2q\}$ with an induced edge labeling $f^* : E(G) \rightarrow \{1, 3, 5, ..., 2q-1\}$ such that for each edge $uv$, $f^*(uv) = \frac{f(u)+f(v)}{2}$ is bijection. In this paper we obtain sufficient conditions for certain uniform theta graphs to be even vertex odd mean graphs.

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1. Introduction

Let \( G = (V, E) \) be a simple, connected and undirected graph with vertex set \( V(G) \) and edge set \( E(G) \). The notions and terminology that exist in this paper can be found in Harary [6]. In [9] Rajan et al. defined a generalized theta graph \( \theta(m_1, m_2, \ldots, m_n) \) is the graph obtained by taking two different isolated vertices \( u \) and \( v \) (called the end vertices of \( \theta \)) and attaching them by \( n \) internal disjoint paths of length greater than one, where \( m_i, 1 \leq i \leq n \) denote the order of internal vertices of \( \ith \) path. The two isolated vertices \( u \) and \( v \) are called north pole \( (N) \) and south pole \( (S) \) respectively. A generalized theta graph is called uniform theta graph if all internal disjoint paths have the same order i.e. \( m_i = m \) for all \( 1 \leq i \leq n \). A uniform theta graph with \( n \) internal disjoint paths, all have the same order \( m \) is denoted by \( \theta(n; m) \). In Figure 1 we show the representation of \( \theta(3; 5) \). R. Vasuki et al. defined the even vertex odd mean labeling on graph \( G = (V, E) \) as an injective vertex labeling \( f \) from \( V(G) \) to \( \{0, 2, \ldots, 2q\} \) such that the induced edge labeling \( f^* \) is defined by \( f^*(uv) = \frac{f(u) + f(v)}{2} \) for every edge \( uv \in G \). The resulting edge labels are odd distinct integers from \( \{1, 3, 5, \ldots, 2q - 1\} \) [10]. A graph that admits an even vertex odd mean labeling is said to be an even vertex odd mean graph \([1],[2],[3],[4],[7],[8],[11]\]. Labeled graphs are used in coding theory, cryptography, mathematical modelling, x-ray, crystallography, and determining optimal circuit layouts, also graph theory is important in many areas of computer science study, including networking, database management systems, and artificial intelligence [12]. Gallian provides a current survey of numerous graph labeling challenges as well as a comprehensive bibliography [5].

![Uniform theta graph \( \theta(3; 5) \)](image-url)
2. Main results

**Theorem 1.** If \( n \) is odd and \( m \geq 2 \), then \( \theta(n;m) \) is an even vertex odd mean graph.

**Proof.** Let \( G = \theta(n;m) \) be a uniform theta graph. Let \( u,v \) be the pair of end vertices of degree \( n \) and \( u_{ij} \) \((1 \leq i \leq n, 1 \leq j \leq m)\) be the internal vertices of \( n \) copies of path \( P_{(m+2)} \). Then \( |V(G)| = mn + 2 \) and \( |E(G)| = mn + n \). Define an injective function \( f : V(G) \to \{0, 2, 4, \ldots, 2q = 2mn + 2n\} \) as follows, label the end vertices \( u \) and \( v \) by \( 2mn + 2n \) and 0 respectively.

**Case(i).** When \( m \) is odd.
Let \( P^i_m \) denote the \( i \)th internal path of \( \theta(n;m) \), hence the vertices \( P^1_m \) and \( P^2_m \) are labeled as follows:

\[
\begin{align*}
  f(u_{ij}) &= \begin{cases} 
    2n(m - j) + 4n - 2, & 1 \leq j \leq m \text{ and } j \text{ is odd} \\
    2n(m - j) - 2n + 4, & 1 \leq j \leq m \text{ and } j \text{ is even}
  \end{cases} \\
  f(u_{2j}) &= \begin{cases} 
    2n(m - j) + 4n - 6, & 1 \leq j \leq m \text{ and } j \text{ is odd} \\
    2n(m - j) - 2n + 12, & 1 \leq j \leq m \text{ and } j \text{ is even}
  \end{cases}
\end{align*}
\]

Now if the vertices of the internal path \( P^{i-2}_m \) are labeled by \( f \), hence the vertices of the internal path \( P^i_m \) are labeled as follows:

\[
\begin{align*}
  f(u_{ij}) &= \begin{cases} 
    f(u_{(i-2)j}) - 8, & 3 \leq i \leq n \text{ and } i \text{ is odd} \\
    f(u_{(i-2)j}) + 16, & 3 \leq i \leq n \text{ and } i \text{ is even}
  \end{cases}
\end{align*}
\]

Clearly, the induced edge labeling \( f^* \) is obtained as follows:
Let \( E_u \) and \( E_v \) be the set of edges joining between the end points of \( n \) disjoint paths and the two isolated vertices \( u \) and \( v \) respectively. Then we denote the set of edge labels of \( E_u \) and \( E_v \) by \( f^*(E_u), f^*(E_v) \) respectively. Thus, it is observed that:

For \( 1 \leq i \leq n \)
\[
\begin{align*}
  f^*(E_u) &= 2n(m + 1) - 2i + 1, \\
  f^*(E_v) &= 2n - 2i + 1.
\end{align*}
\]

Also let \( E(P^i_m) \) be the set of all edges of the \( i \)th internal path of \( \theta(n;m) \). We denote the set of edge labels of \( E(P^i_m) \) by \( f^*(E(P^i_m)) \). So it is easy to see that:

For \( 1 \leq j \leq m - 1 \)
\[
\begin{align*}
  f^*(E(P^1_m)) &= 2nm - 2nj + 1, \\
  f^*(E(P^2_m)) &= 2nm - 2nj + 3.
\end{align*}
\]
It follows that the induced edge labels of $E(P_{m}^i)$ are given by:

$$f^*(E(P_{m}^i)) = \begin{cases} 
  f^*(E(P_{m}^1)) + 2i - 2, & 3 \leq i \leq n \text{ and } i \text{ is odd} \\
  f^*(E(P_{m}^2)) + 2i - 4, & 3 \leq i \leq n \text{ and } i \text{ is even.}
\end{cases}$$

Hence, $f$ is an even vertex odd mean labeling of $\theta(n;m)$.

**Case (ii).** When $m$ is even. Here the vertices of 1st and $(\frac{n+3}{2})$-th internal paths label as follows:

$$f(u_{1j}) = \begin{cases} 
  2n(m - j) + 4n - 2, & 1 \leq j \leq m \text{ and } j \text{ is odd} \\
  2n(m - j) + 2n, & 1 \leq j \leq m \text{ and } j \text{ is even.}
\end{cases}$$

$$f(u_{(\frac{n+3}{2})j}) = \begin{cases} 
  2n(m - j) + 2n - 4, & 1 \leq j \leq m \text{ and } j \text{ is odd} \\
  2n(m - j) + 4n - 2, & 1 \leq j \leq m \text{ and } j \text{ is even.}
\end{cases}$$

Also the remaining vertices of other internal paths label as follows:

$$f(u_{ij}) = \begin{cases} 
  f(u_{1j}) - 4i + 4, & 2 \leq i \leq \frac{n+1}{2} \\
  f(u_{(\frac{n+3}{2})j}) - 4i + 2n + 6, & \frac{n+3}{2} \leq i \leq n.
\end{cases}$$

Therefore, we can compute the induced edge labels as follows:

$$f^*(E_u) = \{2n(m + 1) - 2i + 1, 1 \leq i \leq n\}$$

$$f^*(E_v) = \{n - 2i + 2, 1 \leq i \leq \frac{n+1}{2}\} \cup \{3n - 2i + 2, \frac{n+3}{2} \leq i \leq n\}$$

Also the set of edge labels of $E(P_{m}^1)$ and $E(P_{m}^{(\frac{n+3}{2})})$ are given as follows:

$$f^*(E(P_{m}^1)) = 2n(m - j) + 2n - 1, 1 \leq j \leq m - 1$$

$$f^*(E(P_{m}^{(\frac{n+3}{2})})) = 2n(m - j) + 2n - 3, 1 \leq j \leq m - 1.$$}

Clearly, the edge labels of $E(P_{m}^i)$ are given by:

$$f^*(E(P_{m}^i)) = \begin{cases} 
  f^*(E(P_{m}^1)) - 4i + 4, & 2 \leq i \leq \frac{n+1}{2} \\
  f^*(E(P_{m}^{(\frac{n+3}{2})})) - 4i + 2n + 6, & \frac{n+5}{2} \leq i \leq n.
\end{cases}$$

Hence, $f$ is an even vertex odd mean labeling of $\theta(n;m)$. \hfill \Box

**Example 2.1.** The even vertex odd mean labeling $f$ of $\theta(7;5)$ and $\theta(9;6)$ are given in Figure 2 and Figure 3 respectively.
Even vertex odd mean labeling of uniform theta graphs

Figure 2.1: An even vertex odd mean labeling of $\theta(7; 5)$

Figure 2.2: An even vertex odd mean labeling of $\theta(9; 6)$
Theorem 2. If $n$ is even and $m$ is odd or $n = 4$ and $m$ is even, then
$	heta(n;m)$ is an even vertex odd mean graph.

Proof. Consider $G = \theta(n;m)$ is a uniform theta graph where $n$ is even and $m \geq 1$. Then we can define an injective function $f : V(G) \rightarrow \{0, 2, 4, ..., 2q = 2mn + 2n\}$ as follows, label the end vertices $u$ and $v$ by $2mn + 2(n - m) + 2$ and $0$ respectively.

Case(i). When $n$ is even and $m$ is odd.
In this case the vertices of the internal paths $u_{ij}(1 \leq i \leq n, 1 \leq j \leq m)$ label as the following, first we label the vertices of the paths $P_m^1, P_m^2$ by:

$$f(u_{1j}) = \begin{cases} 2mn + 2(n - m) + 2j - 2, & 1 \leq j \leq m \text{ and } j \text{ is odd} \\ 2mn + 2(n - m) + 2j + 2, & 1 \leq j \leq m \text{ and } j \text{ is even.} \end{cases}$$

$$f(u_{2j}) = \begin{cases} 2mn + 2(n - m) - 2j - 2, & 1 \leq j \leq m \text{ and } j \text{ is odd} \\ 2mn + 2(n - m) - 2j + 2, & 1 \leq j \leq m \text{ and } j \text{ is even.} \end{cases}$$

Now, for $1 \leq i \leq \frac{n}{2}$ and $\frac{n}{2} + 3 \leq i \leq n$ if the vertices of the path $P_m^{i-2}$ are labeled, then the vertices of the path $P_m^{i}$ are labeled as follows:

$$f(u_{ij}) = f(u_{(i-2)j}) - 4m - 4.$$ 

Also for $i = \frac{n}{2} + 1, \frac{n}{2} + 2$ the vertices of the internal two paths $P_m^\frac{n}{2}+1$ and $P_m^\frac{n}{2}+2$ are labeled as follows:

$$f(u_{ij}) = \begin{cases} f(u_{(i-2)j}) - 4m - 4, & 1 \leq j \leq m \text{ and } j \text{ is odd} \\ f(u_{(i-2)j}) - 4m - 8, & 1 \leq j \leq m \text{ and } j \text{ is even.} \end{cases}$$

According to Theorem 1. we can compute the induced edge labels as follows:

$$f^*(E_u) = \begin{cases} (m + 1)(2n - i) - m + 2, & 1 \leq i \leq n \text{ and } i \text{ is odd} \\ (m + 1)(2n - i) + 1, & 1 \leq i \leq n \text{ and } i \text{ is even.} \end{cases}$$

$$f^*(E_v) = \begin{cases} (m + 1)(n - i) + m, & 1 \leq i \leq n \text{ and } i \text{ is odd} \\ (m + 1)(n - i) + 1, & 1 \leq i \leq n \text{ and } i \text{ is even.} \end{cases}$$

$$f^*(E(P_m^1)) = 2nm + 2(n - m) + 2j + 1, \quad 1 \leq j \leq n - 1$$

$$f^*(E(P_m^2)) = 2nm + 2(n - m) - 2j - 1, \quad 1 \leq j \leq n - 1.$$ 

Suppose the edge of the path $P_m^{i-2}$ are obtained, hence for $1 \leq i \leq \frac{n}{2}$ and $\frac{n}{2} + 3 \leq i \leq n$ the edge labels of $P_m^{i}$ are labeled as follows:

$$f^*(E(P_m^i)) = f^*(E(P_m^{i-2})) - 4m - 4.$$
For \( i = \frac{n}{2} + 1, \frac{n}{2} + 2 \) the edge of the two internal paths \( E(P_{m}^{n+1}) \) and \( E(P_{m}^{n+2}) \) are labeled by:
\[
f^*(E(P_{m}^{i})) = f^*(E(P_{m}^{i-2})) - 4m - 6.
\]

Therefore, \( f \) is an even vertex odd mean labeling of \( \theta(n;m) \).

**Case(ii).** When \( n = 4 \) and \( m \) is even.

Here the vertices of the internal paths \( P_{m}^{1}, P_{m}^{2}, P_{m}^{3} \) and \( P_{m}^{4} \) are labeled as follows:
\[
\begin{align*}
f(u_{1j}) &= \begin{cases}
6m + 2j + 10, & 1 \leq j \leq m \text{ and } j \text{ is odd} \\
6m + 2j + 6, & 1 \leq j \leq m \text{ and } j \text{ is even.}
\end{cases} \\
f(u_{2j}) &= \begin{cases}
6m - 2j + 10, & 1 \leq j \leq m \text{ and } j \text{ is odd} \\
6m - 2j + 6, & 1 \leq j \leq m \text{ and } j \text{ is even.}
\end{cases} \\
f(u_{3j}) &= \begin{cases}
2m + 2j + 6, & 1 \leq j \leq m \text{ and } j \text{ is odd} \\
2m + 2j - 2, & 1 \leq j \leq m \text{ and } j \text{ is even.}
\end{cases} \\
f(u_{4j}) &= \begin{cases}
2m + 4, & j = 1 \\
2m - 2j + 2, & 2 \leq j \leq m.
\end{cases}
\]

Clearly, the edge labels of the four internal paths are given as follows:
\[
\begin{align*}
f^*(E(P_{m}^{1})) &= 6m + 2j + 9 \\
f^*(E(P_{m}^{2})) &= 6m - 2j + 7 \\
f^*(E(P_{m}^{3})) &= 6m - 2j + 3. \\
f^*(E(P_{m}^{4})) &= \begin{cases}
2m + 1, & j = 1 \\
2m - 2j + 1, & 2 \leq j \leq m.
\end{cases}
\]

Thus, \( f \) is an even vertex odd mean labeling of \( \theta(n;m) \). For other even values of \( m \) the problem becomes NP-hard problems.  

**Example 2.2.** The even vertex odd mean labeling \( f \) of \( \theta(8;7) \) and \( \theta(4;12) \) are given in Figure 4 and Figure 5 respectively.
Figure 2.3: An even vertex odd mean labeling of $\theta(8; 7)$

Figure 2.4: An even vertex odd mean labeling of $\theta(4; 12)$
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