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The Reconstruction of a Periodic Structure from its Dynamical Behaviour

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Abstract

This work is related to the inverse problem in vibration produced in a special type of mechanical structure known as periodic structure. This problem consist in determining the stiffness and mass parameter of the structure from the natural frequencies and vibrations modes. The problem concern with the inverse eigenvalue problem for a specially structured Jacobi matrix which contains the desired parameters. Sufficient conditions to be applied to the data to obtain a real system are derived and a numerical procedure is develop. Some numerical examples are presented

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1. Introduction

Many problems in physics and engineering can be classified as direct or inverse problems. The classical direct problems in these fields, are related to the analysis and description of the behaviour of a system through its properties such as density, mass, elastics constants, conductivity, damages, strength, etc. On the other hands inverse problems are related with the estimation or determination of the properties of a system, by means of their dynamical behaviour. We are interested mainly on the particular class of inverse eigenvalue problems associated with matrices. In this context, an inverse eigenvalue problem **(IEP)** concerns the reconstruction of a matrix from prescribed spectral data. Some important references are de Boor and Golub [8], Boley and Golub[5] , and Chu and Golub[7]. This work is concerned on **(IEP)** with applications to mechanical engineering, where the work pioneered by Gantmacher and Krein[9] was the basis for many further investigations. For example [10],[19],[15],[16],[17] and [18].

The study of infinitesimal free vibrations of elastic systems is of great interest in classical vibration theory. A model which has generated much interest in the literature as a prototype for vibrating structures, is a thin rod of length L with longitudinal vibration governed by the equation

$$\frac{\partial}{\partial x}\left(EA\frac{\partial u(x,t)}{\partial x}\right) = \rho A\frac{\partial^2 u(x,t)}{\partial t^2}, \quad 0 < x < L, \quad t > 0, \tag{1.1}$$

with fixed-free end conditions $u(0) = 0 = u'(L)$. Here, $A \equiv A(x)$, $E \equiv E(x)$ and $\rho \equiv \rho(x)$ are the cross-section area, Young's modulus and mass density per unit length, respectively. It is well known that for free vibration of frequency ω, the longitudinal displacement $u(x,t)$ can be written as $u(x,t) = u(x)\sin(\omega t)$, where $u \equiv u(x)$ satisfies

$$\frac{d}{dx}\left(EA\frac{du}{dx}\right) + \lambda\rho A u = 0, \quad 0 < x < L, \quad \lambda = \omega^2. \tag{1.2}$$

For convenience, we will assume that the rod is uniform with fixed-free end conditions, i.e., attached left-hand and free right-hand end.

To discretize equation (1.2), we consider the partition $P = \{x_i < x_{i+1}, i = 0, 1, \ldots, n-1\}$ of the domain $\Omega = (0, L)$ where $h \equiv \frac{1}{n} = x_{i+1} - x_i, 0 \leq i \leq n-1$. Setting $E_i \equiv E(x_i)$, $\rho_i \equiv \rho(x_i)$, $A_i \equiv A(x_i)$, $u_i \equiv u(x_i)$ and using a finite difference scheme, equation (1.2) can be reduced to

(1.3) $-k_iu_{i-1}+(k_i+k_{i+1})u_i-k_{i+1}u_{i+1}-\lambda m_iu_i=0, \quad 1\le i\le n,$

where $k_i=\dfrac{E_iA_i}{h}$ and $m_i=\rho_iA_ih$. The boundary conditions $u_0=0=u_{n+1}-u_n$ imply that equations in (1.3) can be written as

$$(K-\lambda M)u=0, \tag{1.4}$$

where $u=(u_1,u_2,\dots,u_n)^T$, M is a diagonal matrix and K a symmetric tridiagonal matrix. In equation (1.4), λ and u are known as natural frequency and vibration mode respectively.

It is well known that the discrete model of the rod, represented by equation (1.4), is a spring-mass system consisting of masses m_i connected on-line by linear springs of stiffness k_i. We denote this system by the pair (M,K). There are various inverse eigenvalue methods to reconstruct the system (M,K) from spectral data. For example, if we are given the total mass, the spectra $(\lambda_i)_1^n$ and $(\mu_i)_1^{n-1}$ of the system (M,K) with fixed-free and fixed-fixed end conditions, respectively, satisfying the interlacing property $\lambda_i<\mu_i<\lambda_{i+1}, 1\le i\le n-1$. Then, the reconstruction can be develop in the next way: first, the equation(1.4) is reduced to the standart form

$$(J-\lambda I)v=0, \tag{1.5}$$

where $J=M^{-1/2}KM^{-1/2}, \quad v=M^{-1/2}u$ and then the matrix J must be reconstructed from the given data. The matrix J is a Jacobi matrix which we define here to be a positive definite tridiagonal matrix with negative off-diagonal terms. Algorithms for reconstruct the matrix J from $(\lambda_i)_1^n$ and $(\mu_i)_1^{n-1}$ can be found in [5], [8], [12]. These algorithms are based on methods which reduce a real symmetrical matrix to the tridiagonal form, as Householder, Givens and Lanczos.

Even for a simple elastic system, we see that inverse problems can have practical and valuable applications in mechanics, structural projects, and updating problems. Information on problems of higher dimensional can be found in [2],[3], [13] and [14].

In Section 2, equations governing the movement for periodic structure are presented. They are based on the Lagrangeś generalized equations and his matrix form is derived. In Section 3 we derive a numerical procedure to the reconstuction of a prototype for periodic structure. It's based on the Lanczos algorithm for the reconstruction of a tridiagonal matrix from spectral data. The algorithm requires the natural frequencies associated to periodic structures with different end conditions and his respectively total masses. Numerical examples are presented. Section 4 will be concerned with the generalization of the boundary conditions given in Section 3. From a practical point of view, the measurement of the natural frequencies may be very expensive. For this reason, in Section 5 we present a procedure for the reconstruction of a periodic structure which requires two eigenpars. Some necessary conditions are stablish and numerical examples are presented.

2. Movement Equations for Periodic Structure

The structures which we will study in this work are called periodic structures, which are defined by a series of modules which have nominally equal characteristics of stiffness, mass and load capacity and are all interconnected. Periodic structures are found in different branches of engineering. In mechanical engineering the blades of a turbine make up a structure, with ciclyc periodicity, lightly interconnected by the rotor which they are embedded. In aerospace engineering we have space platforms and radio antennas. Also, we found applications in marine, electrical and civil engineering.

Various studies have been made on this type of structure. These studies are essentialy directed to an analysis of the direct problem, that is the study of the dynamics of a periodic structure whose physical parameters are known.

Bansal[1], studied the propagation of waves in some periodic structures, and also in some irregular structures. Bendiksen[4], studied the phenomenon of the localization of the different types of vibration in structures spaced along their length. Brasil and Mazzilli [6] studied the localization of the different modes of vibration in periodic latticed girders in bridges.

On reviewing literature dealing with the inverse and direct problems, it is inmediately apparent that in comparison to the abundant literature on the direct problem, very little work has been done on the inverse problem for this type of structure.

The following is our theoretical framework for the dynamics of a periodic structure. The configuration of the system may be defined completely by a minimal number n, degree of freedom, of variable depending on time $q^i(t), (i = 1, 2, \dots, n)$ called generalized equations. Without loss of generality we can assume that kinetic energy of the system may be written as $T = T_2 + T_1 + T_0$, where

(2.1) $$T_2 = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} m^{ij}\cdot \dot{q}_i \cdot \dot{q}_j$$

is a quadratic function of the generalized velocity $\dot{q}_i\ (t)$

(2.2) $$T_1 = \sum_{i=1}^{n} G^i\cdot \dot{q}_i$$

is linear in the generalized velocity and T_0 is indepent of them. In general, the coefficients m^{ij} and G^i and the T_0 function depend no linearly on generalized coordinates and time. T_0 has a behaviour like a potential energy, to give rise to the centrifuga forces and T_1 produces Coriolis forces, whose terms are called *giroscopics.* Both forces are related to body rotatory movement. We define the Rayleigh function

(2.3) $$\mathcal{F} = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} d^{ij}\ \dot{q}_i\dot{q}_j + \sum_{i=1}^{n}\sum_{j=1}^{n} h^{ij}\ \dot{q}_i\ q_j$$

In this function, the first group terms are the dumping viscous forces where d^{ij} are the dumping coefficients, which are constant and symmetric and the second group are the circulatory forces with skew-symmetric coefficients $h^{ij} = -h^{ji}$. The remaining forces applied to the system are denoted by Q^i.To obtain the movement equation, we use the Lagrange's generalized equations. For a system with n degree of freedom $q_1, q_2, \ldots, q_n$, the equations are

(2.4) $$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial q_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} + \frac{\partial \mathcal{F}}{\partial q_i} = Q^i,\ i = 1, ..., n$$

where the Lagrangian *calL* is given by the difference between the potential and kinetic energy, $\mathcal{L} = T - V$.

Without loss of generality we assume small displacement around equilibrium point. A simple coordinate transformation may transfer the origin of the state space to agree with the equilibrium point. Then we consider movement around the equilibrium point given by $q_i(t) = \dot{q}_i\ (t) = 0$.

In any way, the equilibrium implies $Q^i = 0$, that is, null generalized applied forces so that,

(2.5) $$\frac{\partial V}{\partial q_i} = 0, \quad i = 1, \ldots, n.$$

The hypothesis of small movements leads to the linearization of the movement equation. Then, the mass or inercia coefficients m^{ij} de T_2, being symmetric, are given by

$$m^{ij} = \left[\frac{\partial^2 T_2}{\partial \dot{q}_i \, \partial \dot{q}_j}\right]_{q=\dot{q}=0} \qquad i,j = 1,2,\ldots,n \tag{2.6}$$

By the other hand, the coefficients of T_1 turns

$$G^i = \sum_{i=1}^{n} \bar{g}^{ij} \, q_i \qquad j = 1,2,\ldots,n \tag{2.7}$$

where

$$\bar{g}^{ij} = \left[\frac{\partial G^i}{\partial q_i}\right]_{q=0} \qquad i,j = 1,2,\ldots,n \tag{2.8}$$

are constant coefficients and can be placed in the skew-symmetric giroscopics coefficients form

$$g^{ij} = \bar{g}^{ij} - \bar{g}^{ji} = \bar{g}^{ij} \qquad i,j = 1,2,\ldots,n. \tag{2.9}$$

Lastly, we give a develop in Taylor Series for the deformation energy U, around the equilibrium point:

$$U(q_1,q_2,\ldots,q_n) = U(q_{01},q_{02},\ldots,q_{0n}) + \sum_{i=1}^{n} \frac{\partial U}{\partial q_i} q_i + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 U}{\partial q_i \partial q_j} q_i q_j + \ldots \tag{2.10}$$

Since $U(q_1,q_2,\ldots,q_n)$ is constant and the second term is null, deleting higher terms we can approximate

$$U(q_{01},q_{02},\ldots,q_{0n}) \approx \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} k_T^{ij} q_i q_j, \tag{2.11}$$

where

$$k_T^{ij} = \frac{\partial^2 U}{\partial q_i \partial q_j} \tag{2.12}$$

are the symmetric stiffness tangent coefficients depending on the generalized coordinates. In matrix form, the system of equations for linearizated movement around a equilibrium configuration is

$$M\, \dot{u}\, (t) + (G + D)\, \dot{u}\, (t) + (K + H)\, \dot{u}\, (t) = Q(t) \tag{2.13}$$

where M is the mass matrix, G is the Cariolis or gyroscopic matrix (Skew-symmetric), D is the dumping matrix (symmetric), K is the stiffness matrix (symmetric) and H is the circulatory matrix (Skew symmetric).

3. Reconstruction of a Prototype of Periodic Structure from Spectral Data

A prototype for periodic structure, much used by investigators such as Brasil and Mazzilli [6], consists of a row of gateways, each one formed by two columns of stiffness k_i , embedded in their bases. At the top of each is a rod , with mass m_i, supported by the gateway. This type of structure is known in mechanical engineering as a *shear building*. The rod represents a pavement, with its mass weight concentrated at the ends of each column. The interconnection of the structure is represented by a series of springs of stiffness k . A prototype of this structure is showed in Figure 1

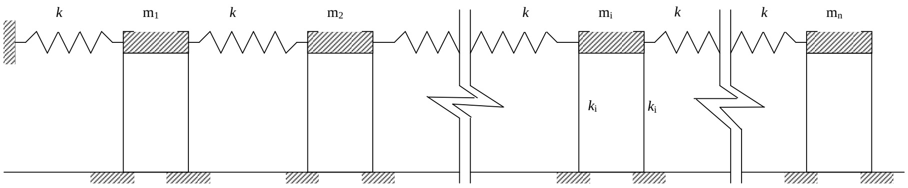

FIGURA 1

We will restrict our attention to conservatives no-giroscopic system . Moreover, we consider free vibrations undumped which permit some simplifications. In the absence of applied forces, giroscopics, viscouses dumped and circulatories, that is , Q, G, D and H nulls, the governing equation is reduced to

$$M\, \ddot{q} + Kq = 0. \tag{3.1}$$

The inverse problem for this case consist on determining the stiffness and mass parameters from the dynamical behaviour. To establish a procedure for the determination of the stiffness and mass parameters we consider (3.1) for the ith-portic:

$$m_i \ddot{q_i} = k_i q_{i-1} - (2k + k_i) q_i + k q_{i+1}, \qquad 1 \le i \le n - 1 \tag{3.2}$$

where $q_0 = q_{n+1} = 0$.

If we assume harmonic solutions of frequency w , $q = zsen(wt)$, we have $\ddot{q_i} = -\lambda z_i sen(wt), \lambda = -w^2$. The number λ is knowed by *natural frequency* and q is called *vibration mode*. Substituting in the last equations set, we have

$$k z_{i-1} - (2k + k_i) z_i + k z_{i+1} = \lambda m_i z_i, \quad 1 \le i \le n \tag{3.3}$$

with $z_0 = z_{n+1} = 0$. In the matrix form,

$$Kz = \lambda M z \tag{3.4}$$

where

$$\mathrm{K}=\begin{pmatrix} -(2k+k_1) & k & & & \\ k & -(2k+k_2) & k & & \\ & k & -(2k+k_3) & \ddots & \\ & & \ddots & \ddots & k \\ & & & k & -(2k+k_n) \end{pmatrix},$$

$M = diag\left\{m_1, m_2, \ldots, m_n\right\}$, and $z = (z_1, z_2, \ldots, z_n)^T$. The generalized eigenvalue equation (3.4) may be reduced to

$$(J - \lambda I)x = 0 \tag{3.5}$$

where $x = M^{1/2}z$ and $J = M^{-1/2}KM^{-1/2}$. The J matrix is defined by

$$J = \begin{pmatrix} \frac{-(2k+k_1)}{m_1} & \frac{k}{\sqrt{m_1 m_2}} & & & \\ \frac{k}{\sqrt{m_1 m_2}} & \frac{-(2k+k_2)}{m_2} & \frac{k}{\sqrt{m_2 m_3}} & & \\ & \frac{k}{\sqrt{m_2 m_3}} & \frac{-(2k+k_3)}{m_3} & \ddots & \\ & & \ddots & \ddots & \frac{k}{\sqrt{m_{n-1} m_n}} \\ & & & \frac{k}{\sqrt{m_{n-1} m_n}} & \frac{-(2k+k_n)}{m_n} \end{pmatrix} \tag{3.6}$$

It is well known [8],[12] that a unique Jacobi matrix J can be reconstructed from his spectrum $(\lambda_i)_1^n$ and the spectrum $(\mu_i)_1^{n-1}$of its principal submatrix $\bar{J}$ if the spectra interlace strictly $\lambda_i < \mu_i < \lambda_{i+1}, 1 \leq i \leq n-1$ and the Lanczos method can be applied in a suitable way to obtain the matrices K and M.

The inverse problem for periodic structures is concerned with finding a matrix par (M, K) such that equation (3.4) has given eigenvalues $(\lambda_i)_1^n$. Let consider the auxiliar periodic system $(\bar{M}, \bar{K})$ with eigenvalues $(\mu_i)_1^{n-1}$ having the last portic fixed. The prototype of this structure is showed in Figure 2:

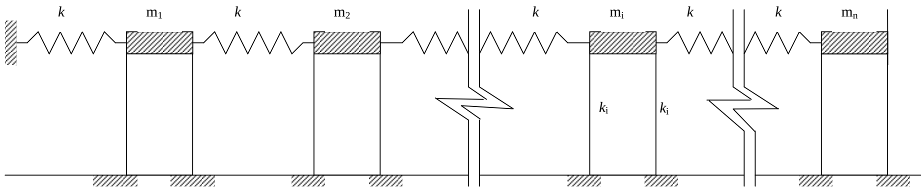

FIGURA 2

We must to reconstruct $2n+1$ parameters $(k_i, m_i)_1^n$ with the interconection stiffness k, and we have $2n-1$ data corresponding to the eigenvalues $(\lambda_i)_1^n$ and $(\mu_i)_1^{n-1}$. The unknown parameters can be determined uniquely, for example, if the total masses $m_T = \sum_{i=1}^n m_i$ and $\bar{m}_T = \sum_{i=1}^{n-1} \bar{m}_i$ are given.

If $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ and $\beta_1, \beta_2, \beta_3, \ldots, \beta_{n-1}$ are the diagonal and off-diagonal terms, we have the relations

$$\alpha_i = \frac{2k + k_i}{m_i} \qquad i = 1, \ldots, n \tag{3.7}$$

$$\beta_i = \frac{-k}{\sqrt{m_i m_{i+1}}} \qquad i = 1, \ldots, n-1. \tag{3.8}$$

Let θ_i defined by

$$\theta_i = \begin{cases} \left(\dfrac{\prod\limits_{\substack{j=i+1 \\ j\ \ odd}}^{n-1} \beta_j^2}{\prod\limits_{\substack{j=2 \\ j\ \ even}}^{n-2} \beta_j^2} \right) m_n & i \quad even \\ \left(\dfrac{\prod\limits_{\substack{j=2 \\ j\ \ even}}^{n-2} \beta_j^2}{\prod\limits_{\substack{j=i \\ j\ \ odd}}^{n-1} \beta_j^2} \right) / \ m_n & i \quad odd \end{cases} \qquad i = 1, 2, \ldots, n \tag{3.9}$$

which may be calculated setting $m_n = m_T - \overline{m}_T$.

From the equation $\sum\limits_{i=1}^{n} m_i = m_T$, it follows that

$$\theta_1 k^2 + \theta_2 + \theta_3 k^2 + \ldots + \theta_{n-1} k^2 + m_n = m_T \tag{3.10}$$

and the k parameter can be calculated from $\mathrm{k}^2 = \dfrac{m^T - \sum\limits_{\substack{i=2 \\ i\ even}}^{n-2} \theta_i}{\sum\limits_{\substack{i=1 \\ i\ odd}}^{n-1} \theta}$(3.10)

From the equations (3.9) and (3.10) we can obtain the relation

$$m_i = \begin{cases} \theta_i & i \quad even \\ \theta_i k^2 & i \quad odd \end{cases} \qquad i = 1, 2, \ldots, n-1 \tag{3.11}$$

Hence, a procedure for the reconstruction of the $(k_i, m_i)_1^n$ and k parameters can be summarized as follows :

Algorithm 1. *Reconstruction for periodic structure*

Input: *The spectra* $(\lambda_i)_1^n$ *and the total mass* m_T *of periodic system* (M, K) . *The spectra* $(\mu_i)_1^{n-1}$ *and the total mass* $\bar{m}_T$ *of the auxiliar system* $(\bar{M}, \bar{K})$.

Output: *The* $(k_i, m_i)_1^n$ *and* k *parameters of the periodic system* (M, K)

Compute

1. $J = M^{-1/2}KM^{-1/2}$ from $(\lambda_i)_1^n$ and $(\mu_i)_1^{n-1}$, by using Lanczos Algorithm
2. $m_n = m_T - \bar{m}_T$
3. θ_i from (3.9), for $i = 1, 2, \dots, n$
4. k from (3.10)
5. m_i from (3.11), for $i = 1, 2, \dots, n$
6. k_i from (3.7), for $i = 1, 2, \dots, n$

3.1. Example 1

Let $n = 15$ and $k = 10$ and the mass parameter are all equal to 5.

i	k_{exact}	λ_i	μ_i	k_{approx}	$\log(e_k)$	m_{approx}	$\log(e_m)$
1	0.4	0.16583	0.17679	0.39999	-11.40851	4.99999	-14.92302
2	0.5	0.39298	0.42992	0.50000	-11.89679	5.00000	-15.24923
3	0.2	0.76532	0.84865	0.19999	-10.13392	4.99999	-14.92302
4	0.5	1.26991	1.41795	0.50000	-12.29289	5.00000	-15.24923
5	0.8	1.86872	2.11357	0.79998	-10.93819	4.99999	-14.61493
6	0.1	2.56020	2.84169	0.09999	-9.22920	4.99999	-15.24923
7	0.9	3.29587	3.68465	0.89999	-11.55415	4.99999	-15.37059
8	0.8	4.08251	4.52047	0.30000	-13.04529	4.99999	-15.94238
9	0.2	4.85680	5.31421	0.20000	-10.11928	5.00000	-15.73653
10	0.8	5.62153	6.11440	0.80001	-11.18423	5.00000	-15.24923
11	0.1	6.31497	6.77213	0.10000	-10.52989	4.99999	-16.20189
12	0.3	6.92755	7.32214	0.30000	-11.49684	5.00000	-17.64735
13	0.5	7.41867	7.73908	0.50000	-11.30631	5.00000	-17.41829
14	0.9	7.78527	8.00368	0.90000	-12.77611	5.00000	-17.18463
15	0.2	8.01379	$*$	0.20000	-12.28987	$*$	$*$

where k_{exact} are the exact stiffness, k_{approx} are the approximates stiffness, $e_k = \dfrac{|k_{exact} - k_{approx}|}{|k_{exact}|}$, and $e_m = \dfrac{|m_{exact} - m_{approx}|}{|m_{exact}|}$.

From (3.9), it's clear that θ_i parameters are positives while $m_n > 0$. Moreover, from (3.11), we have $m_i > 0$, for $i = 1, 2, \dots, n-1$. For the stiffness parameters , from (3.7) we have that a necessary and sufficient conditions for $k_i > 0$ is $\alpha_i > 2k/m_i$.

4. Reconstruction from arbitrary fixed portic

In Section 3, we reconstruct the mass and stiffness parameters from the frequencies corresponding to sinusoidal forcing at an end. In [11], the reconstruction of a vibratory system from its frequency response at an interior point was presented. For this purpose, the author considers the reconstruction of a $n \times n$ Jacobi matrix partitioned as

$$\text{J}=\begin{bmatrix} B & b_m & \\ b_m & a_{m+1} & b_{m+1} \\ & b_{m+1} & C \end{bmatrix}$$

where $B \in \mathbf{R}^{mxm}, C \in \mathbf{R}^{pxp}$ and $p = n - m - 1$. The J matrix may be reconstructed from the sets $(\lambda_i)_1^n$, $(\mu_i)_1^m$ and $(\nu_i)_1^p$ corresponding to the eigenvalues of J, B, and C respectively. Within themselves these sets of eigenvalues must be distinct, being two cases:

(a) all the $(\mu_i)_1^m$ and $(\nu_i)_1^p$ are distinct; if they are arranged in ascending order and relabelled $(\tilde{\mu}_i)_1^n$, they will satisfy $\lambda_i < \tilde{\mu}_i < \lambda_{i+1}, 1 \leq i \leq n-1$.

(b) one or more pairs μ_j, ν_k are identical; now $\mu_j = \nu_k = \lambda_l$ where $l = j + k$. There can be more than one such pair.

We apply the above reconstruction to the Jacobi matrix (3.6). This means that the auxiliar periodic system correspond to the original system with the $(r+1)$-th portic fixed, which may be considered like a unconnected composed by a fixed-fixed portic system with masses $(m_i)_1^r$ and a fixed-free portic system with masses $(m_i)_{r+2}^n$. Let $(\mu_i)_1^r$ the natural frequencies of the fixed-fixed system associated to the eigenvalue problem

$$K_r z^{(r)} = \mu M_r z^{(r)} \tag{4.1}$$

or, equivalently

$$(J_r - \mu I_r)x^{(r)} = 0 \tag{4.2}$$

where $J_r = M_r^{\frac{-1}{2}} K_r M_r^{\frac{-1}{2}}$, $\quad x^{(r)} = M_r^{\frac{1}{2}} z^{(k)}$.

If $(\nu_i)_1^p$, $p = n-r+1$ are the natural frequencies of the fixed-free system, then the eigenvalue equation is

$$K_p z^{(p)} = \nu M_p z^{(p)} \tag{4.3}$$

$$(J_p - \nu I_p)x^{(p)} = 0 \tag{4.4}$$

where $J_p = M_p^{\frac{-1}{2}} K_p M_p^{\frac{-1}{2}}$, $\quad x^{(p)} = M_p^{\frac{1}{2}} z^{(p)}$. The mass and stiffness matrix are

$$K = \begin{pmatrix} K_r & \vdots & ke_r & \vdots & \bigcirc \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ ke_r^T & \vdots & -(2k + k_{r+1}) & \vdots & k\widetilde{e}_1^T \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \bigcirc & \vdots & k\widetilde{e}_1 & \vdots & K_p \end{pmatrix}$$

with $e_r = (0, 0, \ldots, 1)^T \in \Re^r$, $\quad \widetilde{e}_1 = (1, 0, \ldots, 0)^T \quad \in \Re^p$ and

$$M = \begin{pmatrix} M_r & \vdots & \overrightarrow{\bigcirc} & \vdots & \bigcirc \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \overrightarrow{\bigcirc} & \vdots & m_{r+1} & \vdots & \overrightarrow{\bigcirc} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \bigcirc & \vdots & \overrightarrow{\bigcirc} & \vdots & M_p \end{pmatrix}_{n \times n}$$

The Jacobi matrix is

$$J = \begin{pmatrix} J_r & \vdots & b_r e_r & \vdots & \bigcirc \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ b_r e_r^T & \vdots & a_{r+1} & \vdots & b_{r+1}\widetilde{e}_1^T \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \bigcirc & \vdots & b_{r+1}\widetilde{e}_1 & \vdots & J_p \end{pmatrix}$$

which may be reconstructed from the spectra $(\lambda_i)_1^n$, $(\gamma_i)_1^r$, $(\nu_i)_1^p$, $\quad p = n - r + 1$, satisfying the above mentioned conditions (a) or (b). The above reconstruction is a generalization from that in Section 3. These reconstructions may be used to solve some detection problems in simples peridiodic structures as soon as the isolation of discrete systems. It consist in isolating the natural frequencies which lies in some resonance band.

5. Reconstruction from two eigenpairs

The above reconstruction requires the measurements of $2n-1$ natural frequencies, n for the fixed-free system and $n-1$ for the fixed-fixed configuration. These, and particularly the fixed-fixed frequencies, are difficult to measure, specially if n is large. For this reason we seek some other way of reconstructing a periodic structure from limited low frequency modal data. The eigenvalue equation (3.4) can be rewritten so that the $(z_i)_1^n$ appears in the matrices and the $(k_i, m_i)_1^n$ in the vectores. Let $\mathbf{\Theta}_k = [k_1, k_{2,\ldots,}k_n]^T$, $\mathbf{\Theta}_m = [m_1, m_2, \ldots, m_n]^T$, $z = [z_1, z_{2,\ldots,}z_n]^T$, $z^* = \left[z_{1,}^* z_{2,\ldots,}^* z_n^*\right]^T$ and E be the matrix

$$\mathrm{E}=\begin{pmatrix} -2 & 1 & 0 & \ldots & 0 \\ 1 & -2 & 1 & \ldots & 0 \\ 0 & 1 & -2 & \ddots & \ddots \\ \vdots & & \ddots & \ddots & 1 \\ 0 & 0 & \ldots & 1 & -2 \end{pmatrix}.\text{Thus, equation (3.4) is rewritten}$$

$$kE\mathrm{z} - \mathbf{I}\mathrm{z}\mathbf{\Theta}_k - \lambda \mathrm{z}\ \mathbf{\Theta}_m = 0 \tag{5.1}$$

Let consider the two eigenpairs $(\lambda, z), (\lambda^*, z^*)$. Equation (5.1) yields

$$kE\mathrm{z} - \mathbf{I}\mathrm{z}\mathbf{\Theta}_k - \lambda \mathrm{z}\ \mathbf{\Theta}_m \ = kE\mathrm{z}^* - \mathbf{I}\mathrm{z}^*\mathbf{\Theta}_k - \lambda \mathrm{z}^*\ \mathbf{\Theta}_m \tag{5.2}$$

where $\mathbf{I}$ is the identity matrix. This implies for the last equation of (5.2),

$$kz_{n-1} - 2kz_n - k_n z_n - \lambda m_n z_n = kz_{n-1}^* - 2kz_n^* - k_n z_n^* - \lambda^* m_n z_n^* = 0. \tag{5.3}$$

Let $a_i = k(z_{i-1} - 2z_i)$ and $a_i^* = k(z_{i-1}^* - 2z_i^*)$, for $i = 1, 2, \ldots, n$. Then from, (5.3) we obtain the two equalities

$$a_n - k_n z_n \ = \ \lambda m_n z_n \tag{5.4}$$
$$a_n^* - k_n z_n^* \ = \ \lambda^* m_n z_n^* \tag{5.5}$$

which, from the first one, we have for the m_n

$$m_n = \frac{a_n}{\lambda z_n} - \frac{k_n}{\lambda}. \tag{5.6}$$

Substituting m_n into the second equation of (5.3), we get

$$a_n^* - k_n z_n^* = \lambda^* m_n z_n^* = \lambda^* \left(\frac{a_n}{\lambda z_n} - \frac{k_n}{\lambda}\right) z_n^* = \frac{\lambda^*}{\lambda}\left(\frac{a_n z_n^*}{z_n} - k_n z_n^*\right) \tag{5.7}$$

and then obtain the next formulae to k_n

$$k_n = \frac{\lambda^* z_n^* a_n - \lambda z_n a_n^*}{z_n(\lambda^* z_n^* - \lambda z_n^*)} \tag{5.8}$$

Then, from equation (5.3) we can write in matrix form,

$$\begin{bmatrix} z_i & \lambda z_i \\ z_i^* & \lambda^* z_i^* \end{bmatrix} \begin{bmatrix} k_i \\ m_i \end{bmatrix} = \begin{bmatrix} a_i + k z_{i+1} \\ a_i^* + k z_{i+1}^* \end{bmatrix} \qquad i = 1, 2, \ldots, n \tag{5.9}$$

We find that the solution is unique if the determinants of the left-hand side matrix of equation (5.7) are non-zero for $i = 1, \ldots, n-1$: i.e., $\lambda^* z_i^* z_i - \lambda z_i z_i^* \neq 0, \quad i = 1, \ldots, n-1$.We summarized the determination of the stiffness and mass parameters in the

Algorithm 2. *Reconstruction for periodic structure from two eigenpairs*

Input: *The eigenpairs* $(\lambda, z), (\lambda^*, z^*)$ *of periodic system* (M, K) *. The* k *parameter.*

Output: *The* $(k_i, m_i)_1^n$ *parameters of the periodic system* (M, K)

Compute

1. $a_n = k(z_{n-1} - 2z_n)$ *and* $a_n^* = k(z_{n-1}^* - 2z_n^*)$
2. k_n *from (5.8)*
3. m_n *from (5.6)*
4. $(k_i, m_i)_1^{n-1}$ *from (5.9)*

The conditions to obtain a realistic system from this procedure may be found in terms of the spectral information. The next proposition is a sufficient condition to reconstruct a positive parameter k_n. This condition is given in terms of the eigenvalues and the elements of the respective eigenvector.

Proposition 1. *Let* $(\lambda, z), (\lambda^*, z^*)$ *eigenpairs of the periodic system* (M, K) *with stiffness conectivity* k *. Let* $a_n = k(z_{n-1} - 2z_n)$ *and* $a_n^* = k(z_{n-1}^* - 2z_n^*)$. *If one of the following conditions is satisfied*

1. $\dfrac{\lambda^* z_n^*}{z_n} > \dfrac{a_n^*}{a_n}$ *and* $z_n > 0$ *and* $\lambda^* z_n^* > \lambda z_n^*$

2. $\dfrac{\lambda^* z_n^*}{\lambda z_n} > \dfrac{a_n^*}{a_n}$ *and* $z_n < 0$ *and* $\lambda^* z_n^* < \lambda z_n^*$

3. $\dfrac{\lambda^* z_n^*}{\lambda z_n} < \dfrac{a_n^*}{a_n}$ *and* $z_n > 0$ *and* $\lambda^* z_n^* < \lambda z_n^*$

4. $\dfrac{\lambda^* z_n^*}{\lambda z_n} < \dfrac{a_n^*}{a_n}$ *and* $z_n < 0$ *and* $\lambda^* z_n^* > \lambda z_n^*$

Then the n^{th} *stiffness* k_n *is positive.*

In order to get conditions for $m_i > 0$ and $k_i > 0$, for $i = 1, 2, \ldots, n-1$, let us $\xi_i = a_i + kz_{i+1}$, and $\xi_i^* = a_i^* + kz_{i+1}^*$ for $i = 1, 2, \ldots, n-1$. The solution of systems equations (5.9) can be wrritten $k_i = \dfrac{\delta_i}{\tau_i}$ and $m_i = \dfrac{\delta_i^*}{\tau_i}$ where $\delta_i = \begin{vmatrix} \xi_i & \lambda z_i \\ \xi_i^* & \lambda^* z_i^* \end{vmatrix}$, $\delta_i^* = \begin{vmatrix} z_i & \xi_i \\ z_i^* & \xi_i^* \end{vmatrix}$ and $\tau_i = \begin{vmatrix} z_i & \lambda z_i \\ z_i^* & \lambda^* z_i^* \end{vmatrix}$. Having introduced this preliminary notation, we are able to establish the next proposition:

Proposition 2. *Let* $(\lambda, z), (\lambda^*, z^*)$ *eigenpairs of the periodic system* (M, K) *with stiffness conectivity* k *. Then, for* m_i *and* k_i *we have*

1. $k_i > 0$ *if one of the following conditions is satisfied*

   1. $\delta_i > 0$ and $\tau_i > 0$
   2. $\delta_i < 0$ and $\tau_i < 0$

2. $m_i > 0$ *if one of the following conditions is satisfied*

   1. $\delta_i^* > 0$ and $\tau_i > 0$
   2. $\delta_i^* < 0$ and $\tau_i < 0$

6. Example 2

1. $k = 1$

i	$stiffness$	$error$	$mass$	$error$
1	1.1	−11.5891	3.00000	−10.4311
2	2.5	−10.4315	3.00000	−11.0310
3	1.8	−11.0373	3.00000	−11.2537
4	4.3	−11.2519	3.00000	−10.9385
5	3.2	−10.1335	3.00000	−10.8744
6	5.2	−11.8153	3.00000	−11.1423
7	2.4	−11.2564	3.00000	−11.1707
8	1.1	−11.8126	3.00000	−10.4107
9	2.3	−11.0372	3.00000	−11.3701
10	2.1	−10.8454	3.00000	*

2. $k = 1$

i	$stiffness$	$error$	$mass$	$error$
1	1.0000	−9.6338	1.00000	−9.8274
2	1.0000	−10.2313	1.00000	−10.1345
3	1.0000	−10.1239	1.00000	−10.2532
4	1.0000	−9.8978	1.00000	−10.4988
5	1.0000	−10.2340	1.00000	−9.13734
6	1.0000	−10.0243	1.00000	−10.11832
7	1.0000	−9.8462	1.00000	−10.14211
8	1.0000	−9.4235	1.00000	−9.83674
9	1.0000	−10.2356	1.00000	−10.1313
10	1.0000	−10.0438	1.00000	−9.92743
11	1.0000	−10.0273	1.00000	−9.82742
12	1.0000	−10.2298	1.00000	−10.2113
13	1.0000	−9.7163	1.00000	−10.2837
14	1.0000	−10.1129	1.00000	−10.0103
15	1.0000	−9.7754	1.00000	*

This paper has been concerned with the reconstruction of the mass and stiffness parameters for a structure periodic from vibration response data. A technique which recover this parameters is detalied, taking account the practical difficulties which appears in practical measurements. The technique is based is classical and robust methods used for inverse eigenvalue problem for Jacobi matrices. Additional information may be changed instead the total masses m_T and $\overline{m}_T$. This suggest us to propose a different approach, which consists in a multidimensional search strategy and apply it to detection problems and isolating of frequencies.

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