Proyecciones Journal of Mathematics
Vol. 29, N^o 1, pp. 9-15, May 2010.
Universidad Católica del Norte
Antofagasta - Chile
DOI: 10.4067/S0716-09172010000100002

SOLVABILITY OF COMMUTATIVE RIGHT-NILALGEBRAS SATISFYING $(b(aa))a = b((aa)a)^*$

IVÁN CORREA
IRVIN ROY HENTZEL
and
ALICIA LABRA
UNIVERSIDAD DE CHILE, CHILE
Received : July 2009. Accepted : December 2010

Abstract

We study commutative right-nilalgebras of right-nilindex four satisfying the identity $(b(aa))a = b((aa)a)$. Our main result is that these algebras are solvable and not necessarily nilpotent. Our results require characteristic $\neq 2, 3, 5$.

2000 Mathematics Subject Classification : *17 A30.*

Key Words : *Polynomial identity: Nilpotency; Solvability; Right-nilalgebra.*

*Supported by FONDECYT 1060229, FIBAS 20/09 and FONDECYT 1070243

1. Introduction

Let A be a (possibly infinite dimensional) commutative right-nilalgebra of right-nilindex four over a field of characteristic $\neq 2,3,5$ satisfying the identity

$$(b(aa))a = b((aa)a) \tag{1}$$

Right-nilindex four means that A satisfies the identity

$$((aa)a)a = 0. \tag{2}$$

Identity (1) appeared as an omitted special case in Hentzel and Labra[1]. In that paper they studied commutative algebras satisfying the polynomial identity

$$(b(aa))a + tb((aa)a) = 0$$

for $t \neq 1, -1$. It is immediate that every commutative algebra that satisfies this identity satisfies $(t+1)((aa)a)a = 0$. When $t \neq -1$ the algebra automatically satisfies (2). When $t = -1$ we get identity (1). We impose identity (2) because it is not a consequence of (1).

2. Preliminary Results

Linearizing completely (1) we get:

$$\begin{aligned} 0 = \quad g(a,b,c,d) = \quad & +(a(bc))d - a((bc)d) \\ & +(a(cd))b - a((cd)b) \\ & +(a(db))c - a((db)c). \end{aligned}$$

Linearizing completely (2) we get:

$$\begin{aligned} 0 = \quad f(a,b,c,d) = \quad & +((ab)c)d + ((ab)d)c \\ & +((ac)b)d + ((ac)d)b \\ & +((ad)b)c + ((ad)c)b \\ & +((bc)a)d + ((bc)d)a \\ & +((bd)a)c + ((bd)c)a \\ & +((cd)a)b + ((cd)b)a. \end{aligned}$$

Proposition 1. *Let A be a commutative algebra (possibly infinite dimensional) over a field of characteristic $\neq 2,3$. Then A satisfies identities (1) and (2) if and only if A satisfies the identity*

$$((ba)a)a + b((aa)a) = 0. \tag{3}$$

Proof. We first assume that A satisfies the identities (1) and (2). From $0 = f(b,a,a,a) = 6((ba)a)a + 3(b(aa))a + 3b((aa)a)$ we get:

$$2((ba)a)a + (b(aa))a + b((aa)a) = 0. \qquad (4)$$

Using (1) and characteristic $\neq 2$, we get (3).

Now assume that A satisfies identity (3). We replace b by a in (3) and use characteristic $\neq 2$ to obtain identity (2). Linearizing (2) we get (4). Subtracting two times identity (3) from identity (4) gives identity (1). This proves the Proposition.

3. Main Section

Lemma 2. *Let A be a commutative algebra (possibly infinite dimensional) over a field of characteristic $\neq 2,3$ satisfying identities (1) and (2). Then every subalgebra generated by a single element a of A is nilpotent of index at most 5. The only monomials that are not necessarily zero are: $a, aa, (aa)a$ and $(aa)(aa)$.*

Proof. Let a be an element of A. We will prove that all the (non associative) monomials that have degree ≥ 5 are zero. The proof proceeds by induction using degree 5 as the start.

The degree 5 monomials are: $m_1 = (((aa)a)a)a$, $m_2 = ((aa)(aa))a$, and $m_3 = ((aa)a)(aa)$.

Monomial m_1 is zero from (2). Using (1), (3) and (2): $m_2 = ((aa)(aa))a = (aa)((aa)a) = -(((aa)a)a)a = 0$. Using commutativity, (3), and (2): $m_3 = ((aa)a)(aa) = (aa)((aa)a) = -(((aa)a)a)a = 0$.

The degree 6 monomials which are not immediately zero by (2) and induction are: $m_4 = ((aa)(aa))(aa)$ and $m_5 = ((aa)a)((aa)a)$.

Using (3) and (2): $m_5 = ((aa)a)((aa)a) = -((((aa)a)a)a)a = 0$. The identity $f(a,a,aa,aa)$ establishes a dependence relation between $m_4 = ((aa)(aa))(aa)$ and other monomials of degree 6. Since these other degree 6 monomials are zero, m_4 is zero as well.

The only degree 7 monomial which is not zero by induction and (2) is $m_6 = ((aa)(aa))((aa)a)$.

Using (3) and induction: $m_6 = ((aa)(aa))((aa)a) = -((((aa)(aa))a)a)a = 0$.

The only degree 8 monomial which is not zero by induction and (2) is $m_7 = ((aa)(aa))((aa)(aa))$.

The identity $f(a,a,aa,(aa)(aa))$ establishes a dependence relation between $m_7 = ((aa)(aa))((aa)(aa))$ and other monomials of degree 8. Since all the other monomials are zero, m_7 is zero as well.

Monomials of degree higher than 8 are zero by induction because one of the factors will have degree greater than 4.

Lemma 3. If A is commutative algebra that satisfies (1) and (2), then $48((aa)a)((bb)b) = 0$ for all $a, b \in A$.

Proof. Let us consider the following monomials:

$$
\begin{array}{ll}
b_1 = ((((aa)a)b)b)b & b_{14} = ((((ab)a)b)a)b \\
b_2 = (((aa)(ab))b)b & b_{15} = ((((ab)a)b)b)a \\
b_3 = (((aa)a)(bb))b & b_{16} = ((((ab)b)a)a)b \\
b_4 = ((((aa)b)a)b)b & b_{17} = (((ab)b)(aa))b \\
b_5 = (((aa)b)(ab))b & b_{18} = ((((ab)b)a)b)a \\
b_6 = ((aa)b)((ab)b) & b_{19} = ((((bb)a)a)a)b \\
b_7 = ((((aa)b)b)a)b & b_{20} = (((bb)a)(aa))b \\
b_8 = (((aa)(bb))a)b & b_{21} = ((((bb)a)a)b)a \\
b_9 = ((((aa)b)b)b)a & b_{22} = ((((bb)a)b)a)a \\
b_{10} = (((aa)(bb))b)a & b_{23} = ((((bb)b)a)a)a \\
b_{11} = (((aa)b)(bb))a & b_{24} = (((bb)b)(aa))a \\
b_{12} = ((aa)b)((bb)a) & b_{25} = ((aa)a)((bb)b) \\
b_{13} = ((((ab)a)a)b)b. &
\end{array}
$$

Straightforward calculations show that the following linear combination of identities f and g produces the result that $48((aa)a)((bb)b) = 0$:

$$
\begin{array}{lcl}
+27f((aa)a, b, b, b) & = & 162b_1 + 81b_3 + 81b_{25} \\
+30f((aa)b, a, b, b) & = & 60b_4 + 60b_5 + 60b_6 + 60b_7 + 60b_9 + 30b_{11} + 30b_{12} \\
+7f((bb)b, a, a, a) & = & 42b_{23} + 21b_{24} + 21b_{25} \\
-30f(aa, a, b, b)b & = & -60b_1 - 60b_2 - 60b_4 - 60b_7 - 30b_8 - 60b_{17} - 30b_{20} \\
-15f(aa, b, b, b)a & = & -90b_9 - 45b_{10} - 45b_{24} \\
-11f(bb, a, a, a)b & = & -33b_3 - 33b_8 - 66b_{19} \\
-16(f(a, a, a, b)b)b & = & -48b_1 - 48b_4 - 96b_{13} \\
-6(f(a, a, b, b)a)b & = & -12b_7 - 24b_{14} - 24b_{16} - 12b_{19} \\
+18(f(a, a, b, b)b)a & = & 36b_9 + 72b_{15} + 72b_{18} + 36b_{21} \\
+2g((aa)a, b, b, b) & = & 6b_3 - 6b_{25}
\end{array}
$$

$$\begin{array}{lcl}
-9g((aa)b,a,b,b) &=& -18b_5+18b_6-9b_{11}+9b_{12}\\
+16g((bb)b,a,a,a) &=& 48b_{24}-48b_{25}\\
-6g(b,(aa)b,a,b) &=& 6b_5-6b_6+6b_7-6b_9\\
-72g(b,(ab)b,a,a) &=& -72b_6+144b_{16}+72b_{17}-144b_{18}\\
-39g(b,(bb)a,a,a) &=& -39b_{12}+78b_{19}+39b_{20}-78b_{21}\\
+6g(aa,a,b,b)b &=& 12b_2+6b_8-12b_{17}-6b_{20}\\
-3g(a,aa,b,b)b &=& -6b_4+6b_7+3b_8-3b_{20}\\
-54g(a,a,a,bb)b &=& -54b_3+54b_8\\
-24g(b,aa,b,b)a &=& 24b_{10}-24b_{24}\\
-21g(b,a,a,bb)a &=& 21b_{10}-21b_{11}+42b_{21}-42b_{22}\\
-48g(b,ab,a,a)b &=& 48b_2-48b_5+96b_{13}-96b_{14}\\
+14(g(a,b,b,b)a)a &=& 42b_{22}-42b_{23}\\
+18(g(b,a,a,a)b)b &=& -54b_1+54b_4\\
-60(g(b,a,a,b)a)b &=& 120b_{14}-120b_{16}\\
+36(g(b,a,a,b)b)a &=& -72b_{15}+72b_{18}
\end{array}$$

This proves the Lemma.

Theorem 4. *Let A be a commutative algebra over a field of characteristic $\neq 2,3$ that satisfies identities (1) and (2). Let W be the linear subspace of A generated by the elements of the form $J(a,b,c)$, where $J(a,b,c) = (ab)c+(bc)a+(ca)b$, with a,b,c in A. Then W is an ideal of A with $W^2 = 0$.*

Proof. From Lemma 3 and characteristic $\neq 2,3$, we get that $((aa)a)((bb)b)$ is an identity in A. Linearizing this identity we get that $J(a,b,c)\cdot J(d,e,f) = 0$ for any elements a,b,c,d,e,f in A. This proves that $W^2 = 0$.

On the other hand, we see that for every a and b in A, the product $b((aa)a)$ is in $J(A,A,A) \subset W$. This proof follows below.

$$\begin{array}{ll}
& J(b,aa,a)-J(ba,a,a)\\
= & (b(aa))a+((aa)a)b+(ab)(aa)-((ba)a)a-(aa)(ba)-(a(ba))a\\
= & (b(aa))a+b((aa)a)+(ba)(aa)-((ba)a)a-(ba)(aa)-((ba)a)a\\
= & 4b((aa)a) \text{ using (1) and (3).}
\end{array}$$

Hence, $b((aa)a)$ is in W for every a and b in A, which proves Theorem 4.

Theorem 5. *Let A be a (possibly infinite dimensional) commutative algebra over a field of characteristic $\neq 2,3,5$ satisfying identities (1) and (2). Then A is solvable.*

Proof. Using Theorem 5 we get that the quotient A/W is an algebra that satisfies the polynomial identity $J(a,b,c) = 0$. In particular, A/W

satisfies the identity $a^3 = 0$. It is known that every commutative algebra that satisfies this identity is a Jordan algebra. Hence, A/W is a Jordan nilalgebra of nilindex three. From Zelmanov and Skosyrskii[5], we get that A/W is solvable. Since $W^2 = 0$, we have that A/W and W are solvable. Therefore, from Schafer[3] (Proposition 2.2, p.18), we get that A is solvable, which proves Theorem 5. The proof of Zelmanov and Skosyrskii[5] uses characteristic $\neq 2, 3, 5$.

We remark that algebras with the hypotheses of Theorem 5 are not necessarily nilpotent. In fact, the algebra A constructed by Zhevlakov in [4](Example 1, p.82), is a commutative algebra that satisfies the polynomial identity $a^3 = 0$ and is solvable with $A^2A^2 = 0$ but is not nilpotent. It is easy to prove that A satisfies identity (1).

We wish to remark that the computer algebra system ALBERT[2] indicates that any commutative algebra A satisfying the identity $(xx)x = 0$ will also satisfy: $(((AA)(AA))((AA)(AA)))A = 0$. This indicates that $((AA)(AA))((AA)(AA))$ annihilates the whole algebra and that A is solvable of index ≤ 5.

This implies that a commutative algebra satisfying identities (1) and (2) will be be solvable of index ≤ 6. ALBERT shows that for these algebras $(((AA)(AA))((AA)(AA)))A$ need not be zero. This shows that they are not all solvable of index ≤ 4. This does not rule out the possibility that they might all be solvable of index ≤ 5. This work with ALBERT indicates that the assumption of characteristic $\neq 5$ may be unnecessary.

Acknowledgements

The authors wish to thank the referee for suggestions and comments that improved the presentation of the paper.

References

[1] I. R. Hentzel, A. Labra, Generalized Jordan algebras, Linear Alg. and its Applications 422, pp. 326-330, (2007).

[2] D. P. Jacobs, D. Lee, S. V. Muddana, A. J. Offut, K. Prabhu, T. Whiteley, *Albert's User Guide*. Department of Computers Science, Clemson University, (1993).

[3] R. D. Schafer, An introduction to Nonassociative Algebras, Academic Press, New York - San Francisco - London, (1966).

[4] K. A. Zhevlakov, A. M. Slinko, I. P. Shestakov, A. I. Shirshov, Rings that are nearly associative, Academic Press, New York - San Diego - San Francisco, (1982).

[5] E. I. Zelmanov, V. G. Skosyrskii, Special Jordan nil algebras of bounded index, Algebra and Logik 22 (6), pp. 626-635, (1983).

Iván Correa
Departamento de Matemática
Universidad Metropolitana de Ciencias de la Educación
J. P. Alessandri 774 - Santiago - Chile
e-mail: ivan.correa@umce.cl

Roy Hentzel
Department of Mathematics
Iowa State University
Ames - Iowa 50011-2064
e-mail: hentzel@iastate.edu

and

Alicia Labra
Departamento de Matemáticas
Facultad de Ciencias
Universidad de Chile, Casilla 653, Santiago - Chile
e-mail: alimat@uchile.cl