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Examples of Morse decompositions for semigroups actions

Carlos J. Braga Barros
Universidade Estadual de Maringá, Brasil
Hélio V. M. Tozatti *
Universidade Estadual de Maringá, Brasil
and
Josiney A. Souza †
Universidade Estadual de Maringá, Brasil

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Abstract

The concepts of Morse decompositions and dynamic Morse decompositions are equivalent for flows. In this paper we show that these concepts are not equivalent for Morse decompositions of semigroup of homeomorphisms on topological spaces. We give an example of a dynamic Morse decomposition which is not a Morse decomposition on compactifications of topological spaces. Other examples of Morse decompositions are also provided.

Keywords : *Morse decomposition, dynamic Morse decomposition, one-point compactification.*

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1. Introduction

The main subject of this paper is Morse decomposition for semigroup actions on topological spaces.

Morse decompositions were introduced by Conley in [5] and [6] to study the asymptotic behavior of flows on metric spaces. Each component of a Morse decomposition is called a Morse set. A Morse set for flows have both attractive and repulsive properties and it was defined by Conley as intersections of attractors with their complementary repellers. The concept of Morse decomposition was extended for semiflows on topological spaces (see e. g. Hirch [8], and Patrão-San Martin [9],[10]). Recently, many questions on flows or semiflows were solved throughout the theory of semigroup actions (as references sources we mention [9],[10],[1],[2],[4], [7]and [11]). A generalization of Morse decomposition for semigroup actions on topological spaces is presented in [1]. In [3] it is introduced the concept of dynamic Morse decomposition for semigroups of homeomorphisms (see Definition 2 bellow). Intuitively speaking, a dynamic Morse decomposition is the residence of limit sets and cycles are not allowed. Conley in [5] had shown that the concept of Morse decomposition for flows is equivalent to the concept of dynamic Morse decomposition. In [3] it is shown that a Morse decomposition for a semigroup of homeomorphisms of a topological space is a dynamic Morse decomposition. The converse of this result was an open question. In the present article we give an example (see Example 3.3 bellow) of a dynamic Morse decomposition which is not a Morse decomposition. The environment for this example is the one-point compactification of a given topological space. In the literature, there are few examples of Morse decompositions and they are usually presented for flows in the real line. In this paper we also give examples of Morse decompositions for semigroup of homeomorphisms on topological spaces which are not flows or semiflows. The article is organized as follows. In the first section we give the definitions and results on limit sets and Morse decompositions for semigroup actions that we use forward. The second and last section is a section of examples.

2. Morse decompositions

In this section we give the definitions and the main results on Morse decompositions for semigroup actions on topological spaces which are treated in this paper. We refer to [1] and [2] for the theory of Morse decompositions for semigroup actions on topological spaces.

We start assuming that X is a topological space and S is a semigroup. An action of S on X is a mapping $\mu : S \times X \to X$ such that $\mu(s, \mu(t, x)) = \mu(st, x)$ for all $s, t \in S$ and $x \in X$. For each $s \in S$, we define the map $\mu_s : X \to X$ by $\mu_s(x) = \mu(s, x)$ for all $x \in X$. Throughout the paper, we assume that S acts on X as a semigroup of homeomorphisms, i.e., each μ_s is a homeomorphism of X.

For subsets $Y \subset X$ and $A \subset S$ we define the sets AY=$\bigcup_{s\in A} \mu_s(Y)andA^*Y = \bigcup_{s\in A} \mu_s^{-1}(Y)$.

Let Y be a subset of X. It is usual to say that

1. Y is **forward invariant** if $SY \subset Y$;
2. Y is **backward invariant** if $S^*Y \subset Y$;
3. Y is**invariant** if it is forward and backward invariant;
4. Y is **isolated invariant** if it is invariant and there is a neighborhood V of Y such that, for $x \in V$, $Sx \subset V$ and $S^*x \subset V$ implies $x \in Y$.

We fix a family $\mathcal{F}$ of subsets of S which is a filter basis on the subsets of S (i.e., $\notin \mathcal{F}$ and given $A, B \in \mathcal{F}$ there is $C \in \mathcal{F}$ with $C \subset A \cap B$). We also assume that $\mathcal{F}$ satisfies the following **translation hypothesis**: For all s$\in S$ and $A \in \mathcal{F}$ there is $B \in \mathcal{F}$ with $B \subset sA \cap As$.

The following definition was introduced in [1] Definitions 2.3 and 2.16 and generalizes the definitions of limit sets, attractors and repellers for flows and semiflows.

1. The ω**-limit set** of $V \subset X$ for the family $\mathcal{F}$ is defined as $\omega(V, \mathcal{F}) = \bigcap_{A\in\mathcal{F}} cls(AV)$.
2. The ω^***-limit set** of $V \subset X$ is defined as $\omega^*(V, \mathcal{F}) = \bigcap_{A\in\mathcal{F}} cls(A^*V)$.
3. An $\mathcal{F}$**-attractor** is a set $\mathcal{A} \subset X$ which admits a neighborhood V such that $\omega(V, \mathcal{F}) = \mathcal{A}$.
4. An $\mathcal{F}$**-repeller** is a set $\mathcal{R} \subset X$ which admits a neighborhood U such that $\omega^*(U, \mathcal{F}) = \mathcal{R}$.
5. The **complementary repeller** of the $\mathcal{F}$-attractor $\mathcal{A}$ is the set

   A$^* = \{x \in X : \omega(x, \mathcal{F}) \cap \mathcal{A} = \emptyset\}$.

The next definition (see [2] Definitions 5.1 and 5.2) generalizes the definitions of Morse decompositions (and finest Morse decomposition) for flows introduced by Conley in [5] and [6].

Definition 2.1. *1. Let $A_0 = \emptyset \subset A_1 \subset \cdots \subset A_n = X$ be an increasing sequence of $\mathcal{F}$-attractors. Define $\mathcal{C}_{n-i} = \mathcal{A}_{i+1} \cap \mathcal{A}_i^*$, for $i = 0, ..., n-1$. The collection $\mathcal{M} = \{\mathcal{C}_1, ..., \mathcal{C}_n\}$ is called an* $\mathcal{F}$-**Morse decomposition**.

2. *Each component $\mathcal{C}_i$ is called an* $\mathcal{F}$-**Morse set**.

3. *An $\mathcal{F}$-Morse decomposition $\mathcal{M} = \{\mathcal{C}_1, \cdots, \mathcal{C}_n\}$ is said to be* **finer** *than a $\mathcal{F}$-Morse decomposition $\mathcal{M}' = \{\mathcal{C}'_1, \cdots, \mathcal{C}'_m\}$ if for each $\mathcal{F}$-Morse set $\mathcal{C}'_j$ there exists an $\mathcal{F}$-Morse set $\mathcal{C}_i$ with $\mathcal{C}_i \subset \mathcal{C}'_j$.*

4. *An $\mathcal{F}$-Morse decomposition is called the* **finest** *$\mathcal{F}$-Morse decomposition if it is finer than all $\mathcal{F}$-Morse decompositions.*

Now we include the main properties of $\mathcal{F}$-Morse decompositions which are going to be used in this paper.

Proposition 2.1. *Assume that X is a compact topological space. An $\mathcal{F}$-Morse decomposition $\mathcal{M} = \{\mathcal{C}_1, ..., \mathcal{C}_n\}$ has the following properties:*

1. *The components $\mathcal{C}_i$ are nonempty, pairwise disjoint, isolated invariant and compact sets.*

2. *$\mathcal{C}_1$ is an $\mathcal{F}$-repeller and $\mathcal{C}_n$ is an $\mathcal{F}$-attractor.*

3. *For $x \in X$, one has $\omega^*(x, \mathcal{F}) \subset \mathcal{C}_i$ and $\omega(x, \mathcal{F}) \subset \mathcal{C}_j$ for some $i, j \in \{1, ..., n\}$. In particular, $\omega^*(x, \mathcal{F}), \omega(x, \mathcal{F}) \subset i = 1n \bigcup \mathcal{C}_i$ for all $x \in X$.*

4. *(No-cycle condition). Suppose there are $\mathcal{C}_{j_0}, \mathcal{C}_{j_1}, ..., \mathcal{C}_{j_l}$ and $x_1, ..., x_l \in X \setminus i = 1n \bigcup \mathcal{C}_i$ with $\omega^*(x_k, \mathcal{F}) \subset \mathcal{C}_{j_{k-1}}$ and $\omega(x_k, \mathcal{F}) \subset \mathcal{C}_{j_k}$, for $k = 1, ..., l$; then $\mathcal{C}_{j_0} \neq \mathcal{C}_{j_l}$.*

See [2] Proposition 5.2 and [3] Theorem 1.

Now, we recall the definition of dynamic $\mathcal{F}$-Morse decomposition (see [3] Definition 6).

Assume that X is a compact topological space. A **dynamic $\mathcal{F}$-Morse decomposition** is a collection $\mathcal{M} = \{\mathcal{C}_1, ..., \mathcal{C}_n\}$ of nonempty, pairwise disjoint, isolated invariant and compact sets such that:

1. For all $x \in X$, one has $\omega^* (x, \mathcal{F}), \omega (x, \mathcal{F}) \subset i = 1n\bigcup \mathcal{C}_i$; and
2. (No-cycle condition) Suppose there are $\mathcal{C}_{j_0}, \mathcal{C}_{j_1}, ..., \mathcal{C}_{j_l}$ and $x_1, ..., x_l \in X \setminus i = 1n\bigcup \mathcal{C}_i$ with $\omega^* (x_k, \mathcal{F}) \subset \mathcal{C}_{j_{k-1}}$ and $\omega (x_k, \mathcal{F}) \subset \mathcal{C}_{j_k}$, for $k = 1, ..., l$; then $\mathcal{C}_{j_0} \neq \mathcal{C}_{j_l}$.

It is known from [3] Theorem 1 that an $\mathcal{F}$-Morse decomposition is a dynamic $\mathcal{F}$-Morse decomposition. On the other hand, a dynamic $\mathcal{F}$-Morse decomposition may not satisfy the properties 2 and 3 in Proposition 2.1. Later, in this paper we give an example of a dynamic $\mathcal{F}$-Morse decomposition which is not a $\mathcal{F}$-Morse decomposition.

Finally, we discuss $\mathcal{F}$-limit sets on one-point compactifications of topological spaces. Suppose that X is a locally compact, noncompact Hausdorff space, and let $Y = X \cup \{\infty\}$ be the one-point compactification of X. We recall that the basic open neighborhoods of ∞ are the sets of the form $\{\infty\} \cup (X \setminus K)$, where K is a compact set in X. It is well known that open sets in X are also open in Y. We observe that, for a subset $N \subset X$, one has $cls_Y (N) \cap X = cls_X (N)$.

Let $\mu : S \times X \to X$ be an action of the semigroup S on X. This action can be extended to Y, as follows. For each $s \in S$, we define the map $\bar{\mu}_s : Y \to Y$ as

$$\bar{\mu}_s (y) = \begin{cases} \mu_s (y), & \text{for } y \in X \\ \infty, & \text{for } y = \infty \end{cases} . \tag{2.1}$$

The map $\bar{\mu}_s$ is a homeomorphism of Y such that its restriction to X coincides with μ_s.

The $\mathcal{F}$-limit sets on the compactification Y of a subset V is denoted by $\bar{\omega} (V, \mathcal{F})$ and $\bar{\omega}^* (V, \mathcal{F})$. For a subset V in the topological space X, we have $\bar{\omega} (V, \mathcal{F}) \cap X = \bigcap_{A \in \mathcal{F}} cls_Y (AV) \cap X = \bigcap_{A \in \mathcal{F}} cls_X (AV) = \omega (V, \mathcal{F})$.

Analogously, $\bar{\omega}^* (V, \mathcal{F}) \cap X = \omega^* (V, \mathcal{F})$.

3. Examples

In this section we give some examples of $\mathcal{F}$-Morse decompositions for semigroup of homeomorphisms of topological spaces.

We start with an elementary non trivial example.

Example 3.1. *Consider the set $X = \{1, 2, 3, 4, 5\}$ with the topology*

$$\mathcal{T} = \{, \{5\}, \{1, 2\}, \{3, 4\}, \{1, 2, 5\}, \{3, 4, 5\}, \{1, 2, 3, 4\}, X\}.$$

Take the one-to-one function $\phi : X \to X$ defined by $\phi(1) = 2, \phi(2) = 1, \phi(3) = 4, \phi(4) = 3, \phi(5) = 5$.

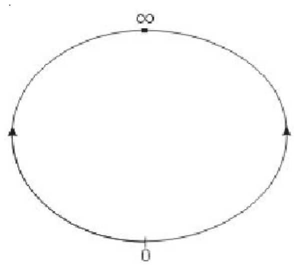
It is easy to verify that ϕ is continuous and $\phi^2 = id_X$. Consider the transformation group $S = \{id_X, \phi\}$. As usual S acts on X by $\mu(\gamma, x) = \gamma(x)$ for $\gamma \in S$. Fix the family $\mathcal{F} = \{S\}$ to compute the $\mathcal{F}$-Morse decomposition. It is immediate that the family $\mathcal{F}$ is a filter basis on the subsets of S and satisfies the translation hypothesis. We also have

$\omega(\{5\}, \mathcal{F}) = cls(S\{5\}) = \{5\}$ *and* $\omega(\{1,2,5\}, \mathcal{F}) = cls(S\{1,2,5\}) = \{1,2,5\}$.

Hence, $\{5\}$ and $\{1,2,5\}$ are $\mathcal{F}$-attractors. The $\mathcal{F}$-Morse decomposition associated to the increasing sequence $\subset \{5\} \subset \{1,2,5\} \subset X$ is the ordered collection $\mathcal{M} = \{\{5\}, \{1,2\}, \{3,4\}\}$.

Now, we give an example of a $\mathcal{F}$-Morse decomposition where the topological space X is the one-point compactification of the real line. It is interesting that in this example the action is not defined by a flow.

Example 3.2. *Let $S = R_*^+$ be the multiplicative group of positive real numbers, an take the action $\mu : S \times R \to R$ of S on R where $\mu(s,x) = sx$ is the product of s and x. Note that each map μ_s is a homeomorphism of R. We observe that zero is a fixed point and the open intervals $(-\infty, 0)$ and $(0, +\infty)$ are invariant subsets for the action of S. We will give an example of $\mathcal{F}$-Morse decomposition for the family $\mathcal{F} = \{(a, +\infty) \subset S : a > 0\}$. Let $X = R \cup \{\infty\}$ be the one-point compactification of R. The picture below illustrates the trajectories of this action.*



We observe that $\{\infty\}$ is an $\mathcal{F}$-attractor and $\{0\}$ is its complementary repeller. Thus, $\mathcal{M} = \{\{0\}, \{\infty\}\}$ is an $\mathcal{F}$-Morse decomposition associated to the increasing sequence of $\mathcal{F}$-attractors $\{\} \subset \{\infty\} \subset X$.

In the following, we give an example of a dynamic $\mathcal{F}$-Morse decomposition which is not an $\mathcal{F}$-Morse decomposition.

Example 3.3. *Let $S = R_*^+$ be the multiplicative group of positive real numbers, an take the action $\mu_1 : S \times [0, +\infty) \to [0, +\infty)$ of S on the interval $[0, +\infty)$ where $\mu_1 (s, x) = sx$ is the product of s and x. Note that each map $(\mu_1)_s$ is a homeomorphism of $[0, +\infty)$. We observe that zero is a fixed point and the open interval $(0, +\infty)$ is an invariant subset for this action. Now, we can take the restriction $\mu : S \times (0, +\infty) \to (0, +\infty)$. We fix the family $\mathcal{F} = \{(a, +\infty) \subset S : a > 0\}$. It is not difficult to show that $\mathcal{F}$ is a filter basis on the subsets of S and satisfies the translation hypothesis. Take the one-point compactifications $X_1 = (0, +\infty) \cup \{\infty_1\}$ and $X_2 = [0, +\infty) \cup \{\infty_2\}$ of $(0, +\infty)$ and $[0, +\infty)$, respectively. Take the product action $\varphi : S \times X_1 \times X_2 \to X_1 \times X_2$ of S on $X_1 \times X_2$ defined by*

$\varphi (s, x_1, x_2) = (\bar{\mu}_s (x_1), \bar{\mu}_s (x_2))$, $s \in S, (x_1, x_2) \in X_1 \times X_2$, where $\bar{\mu}_s$ is the extension as in the equation 2.1 at the end of the first section. Since the map φ_s is a homeomorphism on each coordinate it is a homeomorphism of $X_1 \times X_2$. Now, we consider the subspace X of $X_1 \times X_2$ defined by

$X{=}X_1 \times X_2 \setminus ((0, +\infty) \times \{\infty_2\})$.

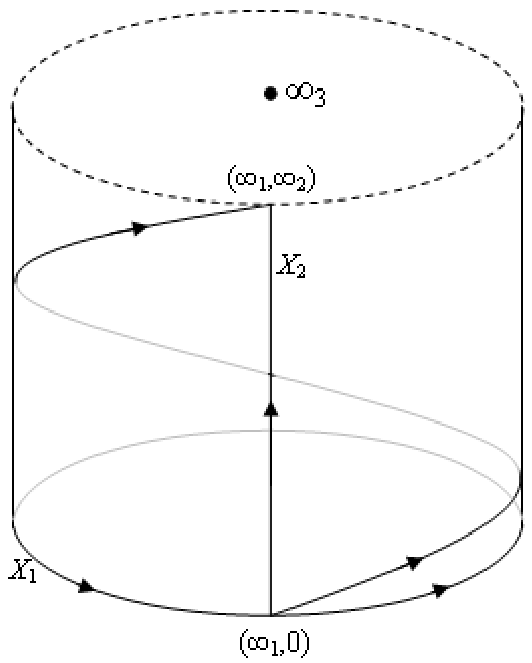
Note that X is a locally compact, noncompact Hausdorff space. Since X is an invariant subset of $X_1 \times X_2$ we have that S acts on X as a restriction of the mapping φ above. Let $Y = X \cup \{\infty\}$ be the one-point compactification of X and $\bar{\varphi} : S \times Y \to Y$ the action of S extended to Y as in the equation 2.1. Note that this action admits three fixed points: $(\infty_1, 0)$, (∞_1, ∞_2), and ∞. We denote the sets $\mathcal{C}_1 = X_1 \times \{0\}$, $\mathcal{C}_2 = \{(\infty_1, \infty_2)\}$, and $\mathcal{C}_3 = \{\infty_3\}$. We claim that $\mathcal{M} = \{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3\}$ is a dynamic $\mathcal{F}$-Morse decomposition in Y. Indeed, it is immediate that M is a collection of nonempty, pairwise disjoint, invariant and compact sets. Take the neighborhood $X_1 \times [0, 1)$ of $\mathcal{C}_1$. For $(x_1, x_2) \in X_1 \times [0, 1) \setminus \mathcal{C}_1$, we have $S (x_1, x_2) = X_1 \times [0, +\infty) X_1 \times [0, 1)$. Hence, $\mathcal{C}_1$ is an isolated invariant set. Take the neighborhood $V = ((0, 1) \cup (1, +\infty) \cup \{\infty_1\}) \times (1, +\infty) \cup \{(\infty_1, \infty_2)\}$ of $\mathcal{C}_2$. For $(x_1, x_2) \in V \backslash \mathcal{C}_2$, we have $S (x_1, x_2) = Sx_1 \times (0, +\infty) V$. Hence, $\mathcal{C}_2$ is isolated invariant. Now, take the neighborhood $Y \setminus K$ of $\mathcal{C}_3$, where K is the compact set $(X_1 \times [0, 1]) \cup \{(\infty_1, \infty_2)\}$ in X. For $(x_1, x_2) \in (Y \setminus K) \setminus \mathcal{C}_3 = X \setminus K$, we have $S (x_1, x_2) = Sx_1 \times (0, +\infty) (Y \setminus K)$. Thus, $\mathcal{C}_3$ is an isolated invariant set. It remains to show the items 1 and 2 of Definition 2. Let $x = (x_1, x_2) \in Y \setminus i = 13 \bigcup \mathcal{C}_i$. It is enough to show that $\bar{\omega} (x, \mathcal{F}) = \{(\infty_1, \infty_2), \infty_3\}$ and $\bar{\omega}^* (x, \mathcal{F}) = \{(\infty_1, 0)\}$

$$\begin{aligned}
\bar{\omega}(x,\mathcal{F}) &= \bigcap_{a>0} cls_Y((a,+\infty)x) = \bigcap_{a>0} cls_Y((ax_1,+\infty)\times(ax_2,+\infty)) \\
&= \bigcap_{a>0} ([ax_1,+\infty)\cup\{\infty_1\})\times([ax_2,+\infty))\cup\{(\infty_1,\infty_2),\infty_3\} \\
&= \{(\infty_1,\infty_2),\infty_3\},
\end{aligned}$$

and

$$\begin{aligned}
\bar{\omega}^*(x,\mathcal{F}) &= \bigcap_{a>0} cls_Y((a,+\infty)^* x) = \bigcap_{a>0} cls_Y((0,1/a)x) \\
&= \bigcap_{a>0} cls_Y((0,x_1/a)\times(0,x_2/a)) \\
&= \bigcap_{a>0} ((0,x_1/a]\cup\{\infty_1\})\times[0,x_2/a] \\
&= \{(\infty_1,0)\}.
\end{aligned}$$

The picture bellow illustrates the trajectories, the limit sets, and the dynamic $\mathcal{F}$-Morse decomposition $\mathcal{M}$.



Now, we claim that $\mathcal{M}$ is not an $\mathcal{F}$-Morse decomposition. From Proposition 2.1, it is enough to show that $\mathcal{C}_2$ and $\mathcal{C}_3$ are neither $\mathcal{F}$-attractors nor $\mathcal{F}$-repellers. Indeed, take the open neighborhood $V = ((0,b) \cup (c,+\infty) \cup \{\infty_1\}) \times (d,+\infty) \cup \{(\infty_1,\infty_2)\}$ of $\mathcal{C}_2$, where $b < c$.

We have

$$\begin{aligned}\bar{\omega}(V,\mathcal{F}) &= \bigcap_{a>0} cls_Y((a,+\infty)V) = \bigcap_{a>0} cls_Y(X_1 \times (ad,+\infty) \cup \{(\infty_1,\infty_2)\}) \\ &= \bigcap_{a>0} X_1 \times [ad,+\infty) \cup \{(\infty_1,\infty_2),\infty_3\} \\ &= \{(\infty_1,\infty_2),\infty_3\},\end{aligned}$$

and $\bar{\omega}^*(V,\mathcal{F}) = \bigcap_{a>0} cls_Y((0,1/a)V) = cls_Y(X_1 \times (0,+\infty) \cup \{(\infty_1,\infty_2)\}) = Y$.Hence, $\mathcal{C}_2$ is neither an $\mathcal{F}$-attractor nor an $\mathcal{F}$-repeller. Take an open neighborhood $U = X_1 \times (b,+\infty) \cup \{\infty_3\}$ of $\mathcal{C}_3$.

We have

$$\begin{aligned}\bar{\omega}(U,\mathcal{F}) &= \bigcap_{a>0} cls_Y(X_1 \times (ab,+\infty) \cup \{\infty_3\}) \\ &= \bigcap_{a>0} X_1 \times [ab,+\infty) \cup \{(\infty_1,\infty_2),\infty_3\} \\ &= \{(\infty_1,\infty_2),\infty_3\},\end{aligned}$$

and $\bar{\omega}^*(U,\mathcal{F}) = \bigcap_{a>0} cls_Y(X_1 \times (0,+\infty) \cup \{\infty_3\}) = Y$.Hence, $\mathcal{C}_2$ is neither an $\mathcal{F}$-attractor nor an $\mathcal{F}$-repeller.

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Carlos J. Braga Barros
Departamento de Matemática
Universidade Estadual de Maringá
Maringá-PR Brasil 87020-900
e-mail : cjbbarros@uem.br

Hélio V. M. Tozatti
Departamento de Matemática
Universidade Estadual de Maringá
Maringá-PR Brasil 87020-900
e-mail : hvlomuert@hotmail.com

and

Josiney A. Souza
Departamento de Matemática
Universidade Estadual de Maringá
Maringá-PR Brasil 87020-900
e-mail : jasouza3@uem.br