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Further results on edge irregularity strength of some graphs

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Abstract

The focal point of this paper is to ascertain the precise value of edge irregularity strength of various finite, simple, undirected and captivating graphs, including the splitting graph, shadow graph, jewel graph, jellyfish graph and m copies of 4-pan graph.

Keywords: *Edge irregularity strength, irregular assignment, irregularity strength, jewel graph, jellyfish graph, splitting graph, shadow graph, 4-pan graph.*

MSC (2020): *05C78.*

1. Introduction

In recent years, the interest in graph labeling has grown significantly due to its relevance in solving practical problems across different disciplines. An intriguing and well-researched topic of graph theory, graph labeling has applications in a variety of fields, including computer science, mathematics, social networks and more. Now a days, graph labeling has found extensive applications across diverse domains of life. For instance, it plays a crucial role in DNA sequence analysis by facilitating the exploration of genetic information and relationships. Additionally, it is employed in image and video processing to accomplish tasks such as object detection, tracking and scene analysis. Consequently, researchers and practitioners can acquire valuable insights into the structural and combinatorial attributes of graphs, resulting in a more profound comprehension of intricate relationships and patterns present within real-world systems. The study of graph labeling revolves around assigning the non-negative integers to the vertex set $\Lambda(\Upsilon)$, edge set $\Gamma(\Upsilon)$, or other elements of a graph Υ in such a way that certain predefined properties or constraints are satisfied. When we assign labels to vertices or edges, it falls into two categories: vertex labeling or edge labeling. If both vertices and edges are labeled simultaneously, it is referred to total labeling.

The most recent Gallian survey [15] indicates a substantial amount of work has been dedicated to the field of graph labeling. Edge labeling for a graph Υ was first developed in 1988 by Chartrand et al. [13]. We call this labeling as irregular assignments because all vertices have distinct weights. Irregularity strength, $s(\Upsilon)$, is a minimum positive integer used to form irregular labeling. Similar insights into the irregularity strength of graphs are shown in the works [2], [10], [14], [20] and [21]. Vertex irregular mapping or edge mapping $\Psi : \Gamma(\Upsilon) \longrightarrow \{1, 2, 3, ..., s\}$ refers to a mapping of edges where each vertex is assigned a unique weight. The calculation of a vertex's weight involves applying the equation $wt_\Psi(c) = \Sigma\Psi(cd)$, $\forall c, d \in \Lambda(\Upsilon)$ and $cd \in \Gamma(\Upsilon)$.

In 2007, Baca et al. [12] introduced two new labelings: edge irregular total labeling and vertex irregular total labeling, as a result of Chartrand's research. Edge irregular total labeling for a graph Υ is a mapping $\Psi : \Gamma(\Upsilon) \cup \Lambda(\Upsilon) \longrightarrow \{1, 2, 3, ..., s\}$ in such a way that the total edge weights assessed by $wt_\Psi(cd) = \Psi(c) + \Psi(d) + \Psi(cd)$, $\forall c, d \in \Lambda(\Upsilon)$,

$cd \in \Gamma(\Upsilon)$, are different for all edges. Total edge irregularity strength denoted by $tes(\Upsilon)$ is a minimum positive integer used to label edges and vertices to form edge irregular total labeling. Similarly, vertex irregular total labeling $\Psi : \Gamma(\Upsilon) \cup \Lambda(\Upsilon) \longrightarrow \{1, 2, 3, ..., s\}$ is a mapping of edges and vertices of Υ in such a way that the total vertex weights calculated by $wt_{\Psi}(c) = \Psi(c) + \Sigma\Psi(cd)$, $\forall c, d \in \Lambda(\Upsilon)$, $cd \in \Gamma(\Upsilon)$, are different for all vertices. Total vertex irregularity strength denoted by $tvs(\Upsilon)$ is a minimum positive integer used to label vertices and edges to form vertex irregular total labeling. Inspired by Baca's invention, numerous researchers have deduced results concerning the two aforementioned labelings. In 2009, Anholcer et al. [7] gave a new upper bound for the total vertex irregularity strength of graphs which improved all other known upper bounds. In 2012, Mushayt et al. [9] calculated exact value of total edge irregularity strength of hexagonal grid graphs. In 2012, Ahmad et al. determined exact value of total edge irregularity strength of the strong product of two paths P_n and P_m in their paper [6], while in 2014, they found total edge irregularity strength of product of two cycles C_n and C_m in their paper [4]. In 2014, Baca et al. [11] obtained useful results regarding total edge irregularity strength of generalized prism. Several authors have made contributions to the exploration of total edge irregularity strength and total vertex irregularity strength in their papers [7], [16], [18], [19], [22], [23], [24], [25] and [26].

Edge irregularity and vertex irregularity were both new labels developed by Marzuki based on the previously improved motivation in [12], which were categorized as total labels with complete irregularity. Total irregularity strength for a graph Υ is denoted as $ts(\Upsilon)$. Papers [12] and [21] played a significant role in the development of outcomes pertaining to irregular total labeling.

Due to the challenges of previous findings, Ahmed et al. introduced a new concept of edge irregularity strength, denoted as $es(\Upsilon)$ in [3], which was a minimum positive integer s used to label vertices to form edge irregular labeling. In the light of this inspiration, numerous researchers discovered the edge irregularity strength of diverse graphs. In 2016, Ahmad et al. [5] investigated exact value of edge irregularity strength of different families of toeplitz graph. One year later, Mushayt et al. [8] took product of certain families of graphs with P_2 and determined their exact value of edge irregularity strength. In the same year, Imran et al. [17] gave results on edge irregularity strength of friendship graphs, cycle chains, caterpillars, star graphs and kite graphs. Ahmad et al. [1] computed edge irregularity strength of some chain graphs and the join of two graphs, and introduced a conjecture and open problems for researchers to research further. Tarawneh et al. [29] estimated edge irregularity strength of corona graphs of path P_m with P_2, P_m with K_1 and S_m with P_m, as well as the edge irregularity strength of the corona product of a cycle with isolated vertices in their paper [27]. Additionally, they explored edge irregularity strength of various graphs in papers [28], [30] and [31]. By the motivation of previous results, Zhang et al. [32] introduced some new families of comb graph, such as comb graph Ca_n, Cd_n, Ce_n, Cf_n and Cg_n, and found their exact value of edge irregularity strength in 2020.

Theorem 1.1

The following Theorem provides us a lower bound that plays a vital role in our findings:
[3] Let Υ be a simple graph with maximum degree $\triangle = \triangle(\Upsilon)$, then
$es(\Upsilon) \geq max\{\lceil \frac{|\Gamma(\Upsilon)|+1}{2} \rceil, \triangle(\Upsilon)\}$.

2. Preliminaries

In this section, we discuss the essential definitions employed within this paper.

Definition 2.1

Edge irregular mapping or vertex mapping $\Psi : \Lambda(\Upsilon) \longrightarrow \{1, 2, 3, ..., s\}$ is a mapping of vertices in such a way that all edges have distinct weights. Edges weights can be assessed by using the relation $wt_\Psi(fm) = \Psi(f) + \Psi(m)$, $\forall f, m \in \Lambda(\Upsilon)$ and $fm \in \Gamma(\Upsilon)$. Edge irregularity strength denoted by $es(\Upsilon)$ is a minimum positive integer used to label vertices to form edge irregular labeling.

Definition 2.2

The shadow graph of a star graph represented by $D_2(K_{1,\eta})$,$\eta > 1$, can be constructed using the vertex set $\Lambda(D_2(K_{1,\eta})) = \{v, w, u_\xi, x_\xi; 1 \leq \xi \leq \eta\}$ and the edge set $\Gamma(D_2(K_{1,\eta})) = \{u_\xi v, u_\xi w, v x_\xi, w x_\xi; 1 \leq \xi \leq \eta\}$. It comprises a total of $2\eta + 2$ vertices and 4η edges.

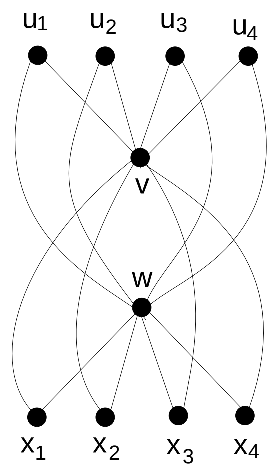

Figure 2.1: Shadow graph $D_2(K_{1,4})$.

Definition 2.3

The splitting graph represented by $S'(K_{1,\eta})$, $\eta > 1$, can be constructed using the vertex set $\Lambda(S'(K_{1,\eta})) = \{u, y, v_\xi, x_\xi; 1 \le \xi \le \eta\}$ and the edge set $\Gamma(S'(K_{1,\eta})) = \{x_\xi u, uv_\xi, v_\xi y; 1 \le \xi \le \eta\}$. It comprises a total of $2\eta + 2$ vertices and 3η edges.

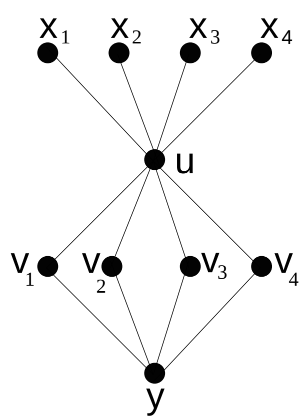

Figure 2.2: Splitting graph $S'(K_{1,4})$.

Definition 2.4

Jellyfish graph denoted by $J_{\eta,\eta}$, $\eta > 1$, can be constructed using the vertex set $\Lambda(J_{\eta,\eta}) = \{u, v, x, z, x_\xi, y_\xi; 1 \leq \xi \leq \eta\}$ and the edge set $\Gamma(J_{\eta,\eta}) = \{ux, uz, xv, xz, zv, vy_\xi, x_\xi u; 1 \leq \xi \leq \eta\}$. It comprises a total of $2\eta + 4$ vertices and $2\eta + 5$ edges.

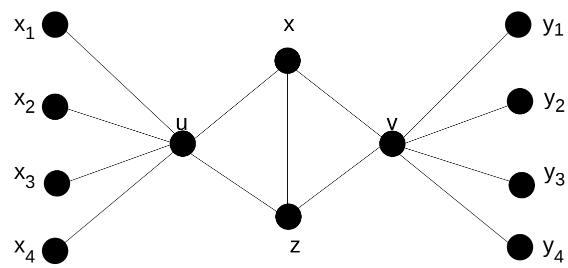

Figure 2.3: Jellyfish graph $J_{4,4}$.

Definition 2.5

The jewel graph denoted by J_η, $\eta \geq 1$, can be constructed using the vertex set $\Lambda(J_\eta) = \{x, y, z, w_\xi; 1 \leq \xi \leq \eta + 1\}$ and the edge set $\Gamma(J_\eta) = \{yz, zx, xw_\xi, yw_\xi, zw_1; 1 \leq \xi \leq \eta + 1\}$. It comprises a total of $\eta + 4$ vertices and $2\eta + 5$ edges.

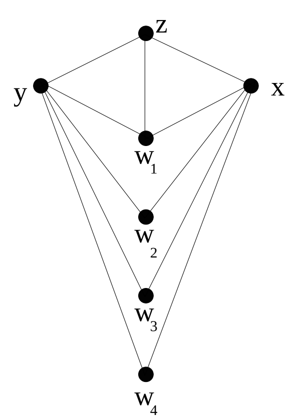

Figure 2.4: Jewel graph J_3.

Definition 2.6

4−Pan graph with m copies is obtained by joining a cycle graph C_4, to each vertex of path graph P_m,$m \geq 2$, with a bridge and is denoted by $\Upsilon = 4 - P_m$, $m \geq 2$. It is obtained by vertex set $\Lambda(4 - P_m) = \{q_m, r_m, s_m, t_m, u_m; 1 \leq m \leq \eta\}$ and the edge set $\Gamma(4-P_m) = \{q_m q_{m+1}; 1 \leq m \leq \eta - 1\} \bigcup \{q_m r_m, r_m s_m, s_m t_m, t_m u_m, u_m r_m; 1 \leq m \leq \eta\}$. It comprises a total of 5η vertices and $6\eta - 1$ edges.

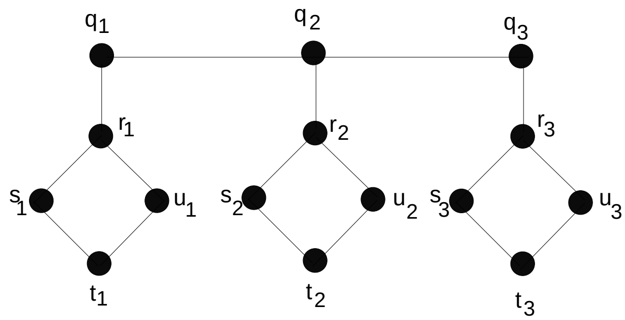

Figure 2.5: 3 copies of 4-Pan graph.

3. Main Results

In this section, we demonstrate the outcomes of our computations.

Theorem 3.1. Let $D_2(K_{1,\eta})$ be a shadow graph, then $es(D_2(K_{1,\eta})) = 2\eta + 1$, $\eta > 1$.

Proof: Let $D_2(K_{1,\eta})$ be a shadow graph. Our task is to demonstrate that $es(D_2(K_{1,\eta})) = 2\eta + 1$. The lower bound, $es(D_2(K_{1,\eta})) \geq 2\eta + 1$, is obtained by Theorem 1.1. To establish the converse, we have to prove that $es(D_2(K_{1,\eta}))$ does not exceed $2\eta + 1$. To achieve this, define a vertex labeling $\Psi : \Lambda(D_2(K_{1,\eta})) \rightarrow \{1, 2, 3, ..., 2\eta + 1\}$ such that

$$\Psi(u_\xi) = \xi,\ 1 \leq \xi \leq \eta,$$

$$\Psi(x_\xi) = \eta + \xi,\ 1 \leq \xi \leq \eta,$$

$$\Psi(v) = 1,$$

$$\Psi(w) = 2\eta + 1, \ \eta > 1.$$

The edge weights are calculated as follows:

$$wt_\Psi(u_\xi v) = \Psi(u_\xi) + \Psi(v) = \xi + 1, \ 1 \le \xi \le \eta,$$

$$wt_\Psi(u_\xi w) = \Psi(u_\xi) + \Psi(w) = \xi + 2\eta + 1, \ 1 \le \xi \le \eta,$$

$$wt_\Psi(v x_\xi) = \Psi(v) + \Psi(x_\xi) = \xi + \eta + 1, \ 1 \le \xi \le \eta,$$

$$wt_\Psi(w x_\xi) = \Psi(w) + \Psi(x_\xi) = \xi + 3\eta + 1, \ 1 \le \xi \le \eta.$$

The aforementioned computations show that all vertex labels are at most $2\eta+1$, and all edges possess distinct weights. The labeling Ψ provides the upper bound on $es(D_2(K_{1,\eta}))$, i.e $es(D_2(K_{1,\eta})) \le 2\eta + 1$. Combining with the lower bound, we conclude that $es(D_2(K_{1,\eta})) = 2\eta+1$. This brings us to the end of the proof.

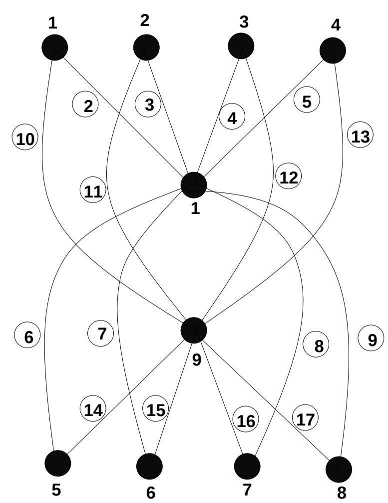

Figure 3.1: Irregular labeling on shadow graph $D_2(K_{1,4})$.

Theorem 3.2. Let $S'(K_{1,\eta})$ be a splitting graph, then $es(S'(K_{1,\eta})) = 2\eta$, $\eta > 1$.

Proof: Let $S'(K_{1,\eta})$ be a splitting graph. Our task is to demonstrate that $es(S'(K_{1,\eta})) = 2\eta$. The lower bound, $es(S'(K_{1,\eta})) \geq 2\eta$, is obtained by Theorem 1.1. To establish the converse, we have to prove that $es(S'(K_{1,\eta}))$ does not exceed 2η. To achieve this, define a vertex labeling $\Psi : \Lambda(S'(K_{1,\eta})) \rightarrow \{1, 2, 3, ..., 2\eta\}$ such that

$$\Psi(x_\xi) = \xi,\ 1 \leq \xi \leq \eta,$$

$$\Psi(v_\xi) = \eta + \xi,\ 1 \leq \xi \leq \eta,\ \eta > 1,$$

$$\Psi(u) = 1,$$

$$\Psi(y) = \eta + 1,\ \eta > 1.$$

The edge weights are calculated as follows:

$$wt_\Psi(x_\xi u) = \Psi(x_\xi) + \Psi(u) = \xi + 1,\ 1 \leq \xi \leq \eta,$$

$$wt_\Psi(uv_\xi) = \Psi(u) + \Psi(v_\xi) = \xi + \eta + 1,\ 1 \leq \xi \leq \eta,$$

$$wt_\Psi(v_\xi y) = \Psi(v_\xi) + \Psi(y) = \xi + 2\eta + 1,\ 1 \leq \xi \leq \eta.$$

The aforementioned computations show that all vertex labels are at most 2η, and all edges possess distinct weights. The labeling Ψ provides the upper bound on $es(S'(K_{1,\eta}))$, i.e $es(S'(K_{1,\eta})) \leq 2\eta$. Combining with the lower bound, we conclude that $es(S'(K_{1,\eta})) = 2\eta$. This brings us to the end of the proof. $\square$

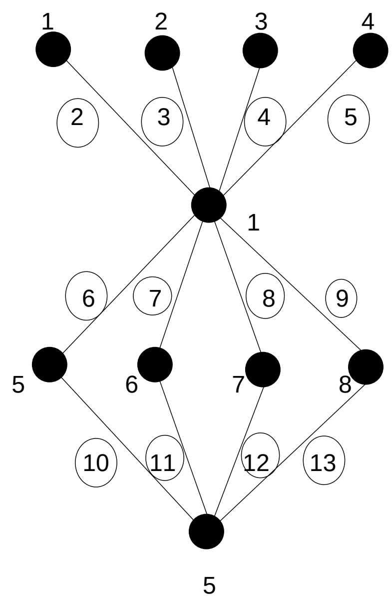

Figure 3.2: Irregular labeling on splitting graph $S'(K_{1,4})$.

Theorem 3.3. Let J_η be a jewel graph, then $es(J_\eta) = 2\eta + 3$, $\eta \geq 1$.

Proof: Let J_η be a jewel graph. Our task is to demonstrate that $es(J_\eta) = 2\eta + 3$. The lower bound, $es(J_\eta) \geq 2\eta + 3$, is obtained by Theorem 1.1. To establish the converse, we have to prove that $es(J_\eta)$ does not exceed $2\eta + 3$. To achieve this, define a vertex labeling $\Psi : \Lambda(J_\eta) \rightarrow \{1, 2, 3, ..., 2\eta + 3\}$ such that

$$\Psi(x) = 1,$$

$$\Psi(y) = 2\eta + 3, \ \eta \geq 1,$$

$$\Psi(w_\xi) = \eta + 2 - \xi, \ 1 \leq \xi \leq \eta + 1,$$

$$\Psi(z) = \eta + 2, \ \eta \geq 1.$$

The edge weights are calculated as follows:

$$wt_\Psi(xw_\xi) = \Psi(x) + \Psi(w_\xi) = \eta + 3 - \xi,\ 1 \le \xi \le \eta + 1,$$

$$wt_\Psi(yw_\xi) = \Psi(y) + \Psi(w_\xi) = 3\eta - \xi + 5,\ 1 \le \xi \le \eta + 1,$$

$$wt_\Psi(zw_1) = \Psi(z) + \Psi(w_1) = 2\eta + 3,\ \eta \ge 1,$$

$$wt_\Psi(yz) = \Psi(y) + \Psi(z) = 3\eta + 5,\ \eta \ge 1,$$

$$wt_\Psi(zx) = \Psi(z) + \Psi(x) = \eta + 3,\ \eta \ge 1.$$

The aforementioned computations show that all vertex labels are at most $2\eta+3$, and all edges possess distinct weights. The labeling Ψ provides the upper bound on $es(J_\eta)$, i.e $es(J_\eta) \le 2\eta + 3$. Combining with the lower bound, we conclude that $es(J_\eta) = 2\eta + 3$. This brings us to the end of the proof. □

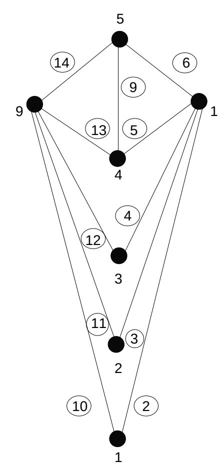

Figure 3.3: Irregular labeling on jewel graph J_3.

Theorem 3.4. Let $J_{\eta,\eta}$ be a jellyfish graph, then $es(J_{\eta,\eta}) = \eta + 3$, $\eta > 1$.

Proof: Let $J_{\eta,\eta}$ be a jellyfish graph. Our task is to demonstrate that $es(J_{\eta,\eta}) = \eta + 3$. The lower bound, $es(J_{\eta,\eta}) \geq \eta + 3$, is obtained by Theorem 1.1. To establish the converse, we have to prove that $es(J_{\eta,\eta})$ does not exceed $\eta + 3$. To achieve this, define a vertex labeling $\Psi : \Lambda(J_{\eta,\eta}) \to \{1, 2, 3, ..., \eta + 3\}$ such that

$$\Psi(x_\xi) = \xi,\ 1 \leq \xi \leq \eta,$$

$$\Psi(y_\xi) = \xi,\ 1 \leq \xi \leq \eta - 1,$$

$$\Psi(u) = 1,$$

$$\Psi(x) = \eta + 1,\ \eta > 1,$$

$$\Psi(z) = \eta + 2,\ \eta > 1,$$

$$\Psi(v) = \eta + 3,\ \eta > 1,$$

$$\Psi(y_\eta) = \eta + 3,\ \eta > 1.$$

The edge weights are calculated as follows:

$$wt_\Psi(x_\xi u) = \Psi(x_\xi) + \Psi(u) = \xi + 1,\ 1 \leq \xi \leq \eta,$$

$$wt_\Psi(ux) = \Psi(u) + \Psi(x) = \eta + 2,$$

$$wt_\Psi(uz) = \Psi(u) + \Psi(z) = \eta + 3,$$

$$wt_\Psi(xv) = \Psi(x) + \Psi(v) = 2\eta + 4,$$

$$wt_\Psi(xz) = \Psi(x) + \Psi(z) = 2\eta + 3,$$

$$wt_\Psi(vz) = \Psi(v) + \Psi(z) = 2\eta + 5,$$

$$wt_\Psi(vy_\xi) = \Psi(v) + \Psi(y_\xi) = \xi + \eta + 3,\ 1 \leq \xi \leq \eta.$$

The aforementioned computations show that all vertex labels are at most $\eta + 3$, and all edges possess distinct weights. The labeling Ψ provides the upper bound on $es(J_{\eta,\eta})$, i.e $es(J_{\eta,\eta}) \leq \eta + 3$. Combining with the lower bound, we conclude that $es(J_{\eta,\eta}) = \eta + 3$. This brings us to the end of the proof.

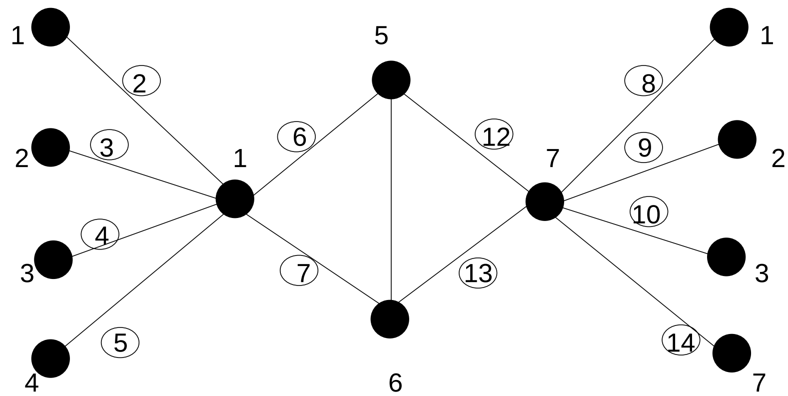

Figure 3.4: Irregular labeling on jellyfish graph $J_{4,4}$.

Theorem 3.5. Let $4-P_m$ be a 4−Pan graph with $m \geq 2$ copies, then $es(4-P_m) = 3m$.

Proof: Let $4-P_m$ be a 4−Pan graph with m copies. Our task is to demonstrate that $es(4-P_m) = 3m$. The lower bound, $es(4-P_m) \geq 3m$, is obtained by Theorem 1.1. To establish the converse, we have to prove that $es(4-P_m)$ does not exceed $3m$. To achieve this, define a vertex labeling $\Psi : \Lambda(4-P_m) \to \{1, 2, 3, ..., 3m\}$ such that

$$\Psi(q_m) = \tfrac{m^3-9m^2+35m-24}{3}, 1 \leq m \leq \eta,$$

$$\Psi(r_m) = \tfrac{m^3-9m^2+35m-24}{3}, 1 \leq m \leq \eta,$$

$$\Psi(s_m) = \tfrac{m^2-m+4}{2}, 1 \leq m \leq \eta,$$

$$\Psi(t_m) = \tfrac{-m^3+9m^2-8m+18}{6}, 1 \leq m \leq \eta,$$

$$\Psi(u_m) = \tfrac{-m^3+9m^2-8m+18}{6}, 1 \leq m \leq \eta.$$

The edge weights are calculated as follows:

$$wt_\Psi(q_m q_{m+1}) = \Psi(q_m) + \Psi(q_{m+1}) = \tfrac{2m^3-15m^2+55m-21}{3},\ 1 \leq m \leq \eta - 1,$$

$$wt_\Psi(q_m r_m) = \Psi(q_m) + \Psi(r_m) = \tfrac{2}{3}(m^3 - 9m^2 + 35m - 24), 1 \leq m \leq \eta,$$

$$wt_\Psi(r_m s_m) = \Psi(r_m) + \Psi(s_m) = \tfrac{2m^3-15m^2+67m-36}{6},\ 1 \leq m \leq \eta,$$

$$wt_\Psi(s_m t_m) = \Psi(s_m) + \Psi(t_m) = \tfrac{-m^3+12m^2-11m+30}{6},\ 1 \leq m \leq \eta,$$

$$wt_\Psi(t_m u_m) = \Psi(t_m) + \Psi(u_m) = \tfrac{-m^3+9m^2-8m+18}{3},\ 1 \leq m \leq \eta,$$

$$wt_\Psi(u_m r_m) = \Psi(u_m) + \Psi(r_m) = \tfrac{m^3-9m^2+62m-30}{6},\ 1 \leq m \leq \eta.$$

The aforementioned computations show that all vertex labels are at most $3m$, and all edges possess distinct weights. The labeling Ψ provides the upper bound on $es(4-P_m)$, i.e $es(4-P_m) \leq 3m$. Combining with the lower bound, we conclude that $es(4-P_m) = 3m$. This brings us to the end of the proof. □

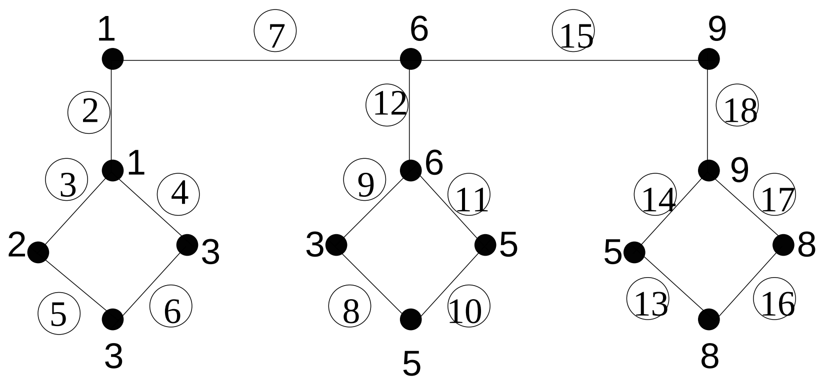

Figure 3.5: Irregular labeling on $4-P_3$.

Conflicts of Interest

The authors collectively state that there are no conflicts of interest among them.

4. Conclusion

In this paper, we determined the exact value of edge irregularity strength of shadow graph, splitting graph, jewel graph, jellyfish graph and m copies of 4-pan graph. These findings have significant implications for various applications, such as network design and optimization. Continuing to push the boundaries of graph theory research, the approach employed in this work will be utilized as a base for investigating the edge irregularity strength of other intricate graphs. Moving forward, a promising future direction will be to find the edge irregularity strength of different families of Deep Neural Network(DNN). Determining the edge irregularity strength of the graphs mentioned above has the potential to yield valuable insights in the field of graph theory.

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