A study on deg-centric graphs

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Abstract

The deg-centric graph of a simple, connected graph $G$, denoted by $G_d$, is a graph constructed from $G$ such that, $V(G_d) = V(G)$ and $E(G_d) = \{v_iv_j : d_G(v_i,v_j) \leq \deg_G(v_i)\}$. This paper introduces and discusses the concepts of deg-centric graphs and iterated deg-centrication of a graph.

Keywords: Distance, eccentricity, deg-centric graphs, iterated deg-centric graph, deg-centrication process.

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1. Introduction

For terminology in graph theory, we refer to [5, 6, 7]. All graphs discussed in this paper are finite, simple, connected, and undirected. Without loss of generality, the vertex set of a graph $G$ of order $n$ will be $V(G) = \{v_i : 1 \leq i \leq n\}$. The order and size of $G$ are denoted by $|G|$ and $\varepsilon(G)$, respectively. Recall that the distance between two distinct vertices $v_i$ and $v_j$ in the vertex set $V(G)$ of $G$, denoted by $d_G(v_i, v_j)$, is the length of the shortest path joining them if such a path exists. Otherwise, $d_G(v_i, v_j) = \infty$. The existence of the deg-centric graphs of star graphs $K_{1,n}$ is a case of particular interest. Recall that a star graph of order $n + 1$ denoted by, $K_{1,n}$, $n \geq 0$ has $V(K_{1,n}) = \{v_i : 0 \leq i \leq n\}$ and $E(K_{1,n}) = \{v_0v_j : 1 \leq j \leq n \text{ if } n \geq 1\}$ else, $E(K_{1,0}) = \emptyset$. Note that both $K_1$ and $K_2$ are considered star graphs.

**Observation 1.1.** A connected graph $G$ which is not a star has at least two vertices $v_i, v_j$ such that, $\text{deg}_G(v_i) \geq 2$ and $\text{deg}_G(v_j) \geq 2$.

Recall that the eccentricity of a vertex $v_i \in V(G)$, denoted by $e(v_i)$, is the furthest distance from $v_i$ to some vertex of $G$. Vertices at a distance $e(v_i)$ from $v_i$ are called the eccentric vertices of $v_i$. An eccentric graph of a graph $G$, denoted by $G_e$, is obtained from the same set of vertices as $G$ with two vertices $v_i$ and $v_j$ being adjacent in $G_e$ if and only if $v_j$ is an eccentric vertex of $v_i$ or $v_i$ is an eccentric vertex of $v_j$ [1, 2, 3, 4]. The iterated eccentric graph of $G$, denoted by $G_{ek}$, is defined as the derived graph obtained by taking the eccentric graph successively $k$-times, that is, $G_{ek} = ((G_e)_{e...}e)_{(k\text{-times})}$.

Motivated by the studies on the eccentric graph and iterated eccentric graphs, we introduce a new transformed graph called deg-centric graphs. For the sake of brevity, henceforth, all the basic graph classes that are to be transformed may be mentioned as initial graphs, and the initial graphs considered in this section are simple, finite, connected and undirected.

2. Deg-centric Graphs of Graphs

**Definition 2.1.** The degree centric graph or deg-centric graph of a graph $G$ and denoted by $G_d$, is the graph with $V(G_d) = V(G)$ and $E(G_d) = \{v_iv_j : d_G(v_i, v_j) \leq \text{deg}_G(v_i)\}$. This graph transformation is called the deg-centrication of the graph $G$. Note that this process is independent of the choice of $v_i$ or $v_j$ in the above sets.
Note that it is possible that \( \deg_G(v_i) < d_G(v_i, v_j) \); hence the edge \( v_i v_j \) does not yield whilst \( \deg_G(v_j) \geq d_G(v_j, v_i) \). Thus, the edge \( v_j v_i \) yields. In this sense, the graph \( G_d \) may be considered a \textit{pseudo directed graph}.

An example of the deg-centric graph of a path graph on 8 vertices is given in Figure 2.1.

\[ \begin{align*}
\text{(a)} & \quad P_8 \\
\text{(b)} & \quad (P_8)_d
\end{align*} \]

Figure 2.1: A path and its deg-centric graph.

\textbf{Definition 2.2.} Let \( G \) be a graph and \( G_d \) be the deg-centric graph of \( G \). Then, the \textit{iterated deg-centric graph} of \( G \), denoted by \( G_{d^k} \), is defined as the graph obtained by applying deg-centrication successively \( k \)-times. That is, \( G_{d^k} = ((G_d)_d...)_d \) (\( k \)-times).

An illustration of iterated deg-centrication of a cycle graph on 7 vertices is given in Figure 2.2.
Figure 2.2: Example of iterated deg-centrication of $C_7$.

Note that the iterated deg-centric graph of a graph may tend to be a complete graph following a finite number of successive iterations. The following theorem characterises this property.

**Observation 2.3.** The deg-centric graph of a star graph is always isomorphic to the star graph. Similarly, the deg-centric graph of a complete graph is always isomorphic to the complete graph.

**Theorem 2.4.** The iterated deg-centric graph of a graph $G$ is complete if and only if $G$ is not a star graph.

**Proof.** It is trivially true that a complete graph of order $n \geq 3$ is inherently iterated. The deg-centric graph of a graph $G$ is complete. Assume that $G$ of order $n \geq 3$ is not complete and not a star graph. Then, by observation 1.1, there exist at least two internal vertices with the degree of at least 2. Consider one such vertex, say $v_i$, with degree $k$. Let $X(v_i) = \{v_j : d_G(v_i, v_j) \leq k, v_i \neq v_j\}$. Then clearly, after the first iteration, the closed neighbourhood of $v_i$ is given by $N_{G_d}[v_i] = \{v_i\} \cup X(v_i)$. With only $v_i$ under consideration, it follows through that after say, $\ell$ iterations the closed neighbourhood $N_{G_d}[v_i] = V(G)$. Furthermore, all vertices
now have a degree of at least 2. A similar argument is valid for all vertices. Thus, completion is reached by applying the adjacency regime of Definition 2.1 to all vertices through each iteration. Hence, the iterated deg-centric graph of a graph $G$ is complete.

Conversely, assume that the iterated deg-centric graph of a graph $G$ is complete. Furthermore, assume that $G$ is a star graph $K_{1,n}$ of order greater or equal to 3. By Observation 2.3, $G_d^k \cong K_{1,n}$ for $k \geq 1$. Thus, we have a contradiction about the claim that the iterated deg-centric graph of a graph $G$ is complete. Therefore, $G$ cannot be a star graph. Hence the result. □

Let $\varphi(G)$ denote the number of iterations required to transform a graph $G$ to completion. By convention, $\varphi(K_n) = 0$, $n \geq 1$ and $\varphi(K_{1,n}) = \infty$, $n \geq 2$.

**Theorem 2.5.** The deg-centric graph of a non-star graph $G$ with $\delta(G) \geq \text{diam}(G)$ is complete.

**Proof.** Consider any vertex $v_i$ for which $\text{deg}_G(v_i) = \delta(G)$. Let

$$X(v_i) = \{v_j : d_G(v_i, v_j) \leq \delta(G), v_i \neq v_j\}.$$ 

Clearly, if $\delta(G) \geq \text{diam}(G)$, then following deg-centrization in respect of $v_i$ the resultant closed neighbourhood is $N_{G_d}[v_i] = V(G)$. Finally, because $\text{deg}_G(v_j) \geq \text{deg}_G(v_i) = \delta(G) \geq \text{diam}(G)$ the resultant closed neighbourhood of each $v_j \in V(G) \setminus \{v_i\}$ is given by, $N_{G_d}[v_j] = V(G)$. Hence, $G_d$ is complete. □

**Corollary 2.6.** The deg-centric graph $G_d$ of a non-star graph $G$ with $\text{deg}_G(v_i) \geq e(v_i)$ is complete.

**Proof.** The proof follows by similar reasoning as in Theorem 2.5. □

**Theorem 2.7.** For a non-star graph $G$ with $m = \text{diam}(G)$, $\varphi(G) \leq \varphi(P_m)$.

**Proof.** The proof follows by similar reasoning as in Theorem 2.5. □

**Lemma 2.8.** Every non-star graph $G$ contains a non star spanning tree $T$ such that $\varphi(G) \leq \varphi(T)$. 


Proof. If $G$ is a tree denoted by $T$, then $G$ is not a star graph. Hence, $\varphi(G) = \varphi(T) \Rightarrow \varphi(G) \leq \varphi(T)$. Let $G$ be a graph other than a tree. Since $G$ is connected, it is known to have a spanning tree. It is also possible to always select a spanning tree $T$, which is not a star. Assume that $\varphi(T) < \varphi(G)$. Since $\varepsilon(G) > \varepsilon(T)$, $diam(T) \geq diam(G)$, and $\sum_{v_i \in V(T)} deg_T(v_i) < \sum_{v_i \in V(G)} deg_G(v_i)$, it is not possible to attain $n(n - 1) - \varepsilon(T)$ edges through transforming $T$ to completion in fewer iterations than attaining $n(n - 1) - \varepsilon(G)$ edges through transforming $G$ to completion. Hence, $\varphi(G) \leq \varphi(T)$.

Corollary 2.9. For a non-star graph $G$, $\varphi(G + e) \leq \varphi(G)$.

Proof. The result is a direct consequence of Lemma 2.8.

Theorem 2.10. For a non-star graph $G$ with a universal vertex, $\varphi(G) \leq 2$.

Proof. A spanning star exists since $\Delta(G) = n - 1$. Consider such star, say, $k_{1,n-1}$. There must by necessity exist at least one pair of pendants of $k_{1,n-1}$ which are adjacent in $G$. Otherwise, $G$ cannot be complete, which is a contradiction. Say such pairs are vertices $v_i, v_j$. Delete the pendant edge of $v_i$ and add the edge $v_iv_j$. Label the new spanning tree $H$. Verifying that $\varphi(H) = 2$ is easy. Hence, by Lemma 2.8, it follows that $\varphi(G) \leq 2$.

Corollary 2.9 permits researchers to find preliminary upper bounds for a Hamiltonian graph $G$ of order $n$ it follows that $\varphi(G) \leq \varphi(C_n)$. If a graph $G$ of order $n$ only has a Hamilton path, then $\varphi(G) \leq \varphi(P_n)$. This observation motivates the study of certain specific graph classes.

Theorem 2.11. For the non-star graphs $G$ and $H$ of same order with $\varepsilon(G) < \varepsilon(H)$, $\varphi(G) \geq \varphi(H)$.

Proof. Let graphs $G$ and $H$ be each of order $n$ and let the number of edges in graph $G$ be less than the number of edges in graph $H$, which means the total degree of graph $G$ is less than the graph $H$. Since

$$\sum_{v_i \in V(G)} deg_G(v_i) < \sum_{u_j \in V(H)} deg_H(u_j),$$

the number of new edges yielded through deg-centrication denoted by $\gamma(G_d)$ and $\gamma(H_d)$ have the relation $\gamma(G_d) < \gamma(H_d)$. Also,

$$\sum_{v_i \in V(G_d)} deg_{G_d}(v_i) < \sum_{u_j \in V(H_d)} deg_{H_d}(u_j).$$
By iterative argument as above, it implies that a greater or equal number of iterations are required for $G$ to complete compared to the number of iterations required for $H$ to be complete. Hence, $\varphi(G) \geq \varphi(H)$. 

3. Deg-centrication of Certain Graph Classes

This section will address cycle graphs (or cycles), path graphs (or paths) and certain other graph classes.

For ease of reference, the cycle $C_n$, $n \geq 4$ will be presented as a “circle” with the vertices $v_i$, $1 \leq i \leq n$ evenly spread and labelled clockwise. Regarding cycles, it is known by convention that $C_3$ is complete on the zeroth iteration. It is easy to verify that $\varphi(C_4) = \varphi(C_5) = 1$. Hence, order 3 served as a click order, meaning “clicking” over to $n = 4, 5$ requires an increase of 1 in the required number of iterations for $C_4, C_5$ to reach completion. Denote $n_0 = 3$. Order $n = 5$ is the next click order i.e. $n_1 = 5$, because for $n = 6, 7, 8, \ldots, 17$ the corresponding cycles have $\varphi(C_n) = 2$. It can be verified that order $n_2 = 17$ is the second click order. The aforesaid means that all cycles $C_n$, $18 \leq n \leq 257$, $n_3 = 257$ will complete on the third iteration. The path $P_2$ will be called a flat cycle, $C_2^{\text{flat}}$ for the next result.

**Theorem 3.1.** The successive click orders of a cycle $C_n$, $n \geq 3$ are given by $n_{i+1}^c = (n_i^c)^2 - 2n_i^c + 2$, where $n_0^c = 3$.

**Proof.** By symmetry considerations and without loss of generality, the reasoning of proof concerning only vertex $v_1$ as the reference will suffice. Figure 3.1a depicts $C_3$, which is the deg-centrication complete cycle on the zeroth iteration, whereas $C_4$ is not. It signals that order 3 is a click order. Dash the base edge $v_2v_3$ (see Figure 3.1b for illustration). Note that vertex $v_1$ has two flat cycles $C_2^{\text{flat}} = v_1v_2$ (to the right) and $C_2^{\text{flat}} = v_1v_3$ (to the left) “hanging” from $v_1$. Since $\text{deg}(v_1) = 2$ string a flat cycle to both $v_2$ and $v_3$ so that $\text{deg}(v_1) = 2$ flat cycles hang from $v_1$ to both the right and left. See Figure 3.1c. Clearly, $C_5$ completes in the first iteration, whereas $C_6$ cannot complete in the first iteration. It signals that order 3 is a click order (see Figure 3.1d). Also, the maximum reach $r$ from $v_1$ after the first iteration is distance 2 (therefore $r = 2$). Hence, two cycles $C_3 = v_1v_2v_3$ and $C_3 = v_1v_4v_5$ hang from $v_1$. For illustration, see Figure 3.1e. Since $\text{deg}(v_1) = 4$ string three additional cycles $C_3$ to both the left and right of $v_1$ to attain the maximum reach of $v_1$ on the first
iteration. Clearly, $r = 8$ (see Figure 3.1f). This implies that cycle $C_n$, $n = 6, 7, 8, \ldots, 16, 17$ completes the second iteration. The order $n = 17$ signals a click order because $C_{18}$ requires three iterations to be complete. The statement mentioned above is valid because vertex $v_{18}$ is beyond the maximum reach $r = 8$ from $v_1$. On completion (see Figure 3.1g), it is known that $\text{deg}(v_1) = 16$. Furthermore, the maximum reach has increased to $r = 8$. Hence, two $C_9$’s hang from $v_1$. String 7 additional $C_9$’s to both sides of $v_1$ to obtain the next click order. By induction, this procedure can be performed infinitely many times. This procedure generates a sequence of click orders, namely 3, 5, 17, 257, 65537, \ldots. The sequence is recognised as the Fermat numbers; that is, $n^c_a = 2^{(2^a)} + 1, a = 0, 1, 2, \ldots$. As a recurrence relation, the Fermat numbers are: $n^c_a = (n^c_{a-1} - 1)^2 + 1, a = 1, 2, \ldots$. For cycles in particular the recursive formula to derive the sequence of click orders is given by $n^c_{i+1} = (n^c_i)^2 - 2n^c_i + 2$, where $n^c_0 = 3$, which settles the result. \qed
Proposition 3.2. For $n \geq 4$, $\varphi(C_n) = i + 1$, $n_i^c + 1 \leq n \leq n_{i+1}^c$.

Proof. The result is a direct consequence of Theorem 3.1.

For ease of reference, the path $P_n$, $n \geq 4$ and even, then let the central vertex set of $G = P_n$ be $X = \{v_t, v_{t+1} : t = \frac{n}{2}\}$. If $n \geq 5$ and odd, then let the central vertex set of $G = P_n$ be $Y = \{v_t : t = \frac{n+1}{2}\}$. A path $P_n$, $n \geq 4$ with the vertices $v_i$, $1 \leq i \leq n$ will be depicted as shown in Figure 3.2.
Observation 3.3. For a path \( G = P_4 \), it is true that on the zeroth iteration (initial path), \( \deg_G(v_1) = 1 \) and on the first iteration, \( \deg_{G,d}(v_1) = 2 \) and on the second iteration, \( \deg_{G,d}(v_1) = 3 \). Furthermore, for a path \( G = P_n, n \geq 5 \) it is true that on the \( O^{th} \) iteration (initial path), \( \deg_G(v_1) = 1 \) then on the first iteration, \( \deg_{G,d}(v_1) = 2 \) and on the second iteration, \( \deg_{G,d}(v_1) = 3 \).

Observation 3.4. Beginning with the path \( G = P_n, n \geq 4 \) (zeroth iteration) through to any finite number of iterations, say \( q \), before completion, it follows that \( \deg_{G,d}(v_1) = 1 \) at the first iteration, \( \deg_{G,d}(v_1) = 2 \) and on the second iteration, \( \deg_{G,d}(v_1) = 4 \).

Lemma 3.5. Let \( k \geq 2 \) be the smallest number such that the edge \( v_tv_n \) yields on the \( k^{th} \) iteration. Then, \( \varphi(G) = k + 1 \).

Proof. Note that with regards to the "left half" of the path, the vertex \( v_t \) is either closer to or as close to \( v_n \) as any other vertex in the "left half". Hence, for the sake of reasoning, \( v_t \) is pivotal. If the edge \( v_tv_n \) yields on the \( k^{th} \) iteration, then the edge \( v_1v_t \) either has yielded prior or yields on the \( k^{th} \) iteration. Note that the induced subgraph \( \{v_1, v_2, v_3, \ldots, v_t\} \) is complete. From Observation 3.3 it is clear that \( \deg_{G,d}(v_1) \) suffices to yield the edge \( v_1v_n \) in the \( (k + 1)^{th} \) iteration. Therefore \( \varphi(G) = k + 1 \). □

Question: Is it not possible for some paths (dependent on \( n \)) for which, at the smallest \( k^{th} \) iteration, both edges \( v_tv_n \) and \( v_1v_n \) yield? If so, the result of Lemma 3.5 must change to \( \varphi(G) \leq k + 1 \). However, the proof of Theorem 3.1 essentially considers paths where a "base" edge of the cycle remains dashed and thus, is not required in the proof arguments. Those mentioned above, together with Observation 3.4, suffices to exclude the case. It leads to the following theorem, which requires no extensive proof.

Theorem 3.6. For path \( P_n, n \geq 3 \) the successive click orders are given by:

\[
n_{i+1}^c = (n_i^c)^2 - 2n_i^c + 2, \quad \text{where} \quad n_0^c = 3.
\]

Proof. Since \( P_3 \) is a star, \( \varphi(P_3) = \infty \). However, \( \varphi(P_4) = 2 \). So order 3 signal a click order. Put \( n_0^c = 3 \). Hereafter, the argument of proof is similar to that in the proof of Theorem 3.1. □

Note that as a direct consequence of Theorem 3.1, \( \varphi(P_n) = i + 1 \), \( n_i^c + 1 \leq n \leq n_{i+1}^c \), for \( n \geq 4 \). Similarly, by Theorem 2.5, we have \( \varphi(W_n) = \)
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1, for \( n \geq 4 \). Also, given Corollary 2.9, a graph \( G \) of order \( n + 1 \) with a spanning subgraph, which is a wheel \( W_n \), has \( \varphi(G) = 1 \). Furthermore, it is immediate from Theorem 2.5 that, for \( a, b \geq 2 \), \( \varphi(K_{a,b}) = 1 \).

A non-trivial double star, denoted by \( S_{a,b} \), is obtained by joining the centers of two non-trivial star graphs \( k_{1,a} \), \( a \geq 1 \) and \( k_{1,b} \), \( b \geq 1 \) with an edge.

**Proposition 3.7.** For \( a, b \geq 1 \), \( \varphi(S_{a,b}) = 2 \).

**Proof.** Assume the star \( k_{1,a} \) has \( V(k_{1,a}) = \{v_0, v_1, v_2, \ldots, v_a\} \) and the star \( k_{1,b} \) has \( V(k_{1,b}) = \{u_0, u_1, u_2, \ldots, u_b\} \). On the first iteration all the edges \( v_0u_i, 1 \leq i \leq b \) and \( u_0v_j, 1 \leq j \leq a \) establishes. Since \( \delta(G_d) = 2 \) and \( \text{diam}(G_d) = 2 \) the graph \( G_d^2 \) is complete (see Theorem 2.5).

An example of iterated deg-centrication of a double star is given in Figure 3.3c.

![Figure 3.3: Iterative deg-centric graphs of a double star \( S_{a,b} \).](image)

**Proposition 3.8.** For the Petersen graph \( G \), \( \varphi(G) = 1 \).

**Proof.** Follows directly from Theorem 2.5. \( \square \)

A helm graph, denoted by \( H_{1,n} \), \( n \geq 3 \) is a graph obtained by attaching a pendant vertex to every rim vertex of a wheel graph \( W_n \), \( n \geq 4 \).

**Proposition 3.9.** For \( n \geq 3 \), \( \varphi(H_{1,n}) = 2 \).
**Proof.** Let the vertices of the wheel be $v_0, v_1, v_2, \ldots, v_n$. Let the numerically corresponding pendant vertices be $u_1, u_2, \ldots, u_n$. Verifying that after the first deg-centrication iteration, the induced subgraph $\langle V(W_n) \rangle \cong K_{n+1}$ is easy. Furthermore, all edges $u_i v_j$, $1 \leq i \leq n$, $0 \leq j \leq n$ exist. Since $\delta(G_d) \geq 4$ and $\text{diam}(G_d) = 2$ the result follows from Theorem 2.5.

Note that for $G \cong H_{1,n}$, $n \geq 3$, the graph $G_d$ belongs to the family of split graphs. This observation leads to the following result.

**Theorem 3.10.** For any non-star graph $G \neq K_1$ with at least two pendant vertices, $\varphi(G) \geq 2$.

**Proof.** Assume $G$ has at least two pendant vertices from the vertex set $v_1, v_2, v_3, \ldots, v_k$. Let $X$ be the set of pendant vertices. At best, the induced subgraph $\langle V(G) \rangle \setminus X$ can be complete on the first iteration. However, then $G_d$ is a split graph. Therefore, $\varphi(G) \geq 2$. 

A **closed helm graph**, denoted by $CH_{1,n}$, $n \geq 3$ is a graph obtained by joining the pendant vertices of a helm graph.

**Proposition 3.11.** Let $G$ be a closed helm graph. Then,

1. $\varphi(G) = 1$, for $n = 3, 4, 5, 6, 7$.
2. $\varphi(G) = 2$, for $n \geq 8$.

**Proof.**

(i) For all $CH_{1,n}$, $n \geq 3$, $\delta(CH_{1,n}) = 3$. For $n = 3, 4, 5, 6, 7$ the diameter is bounded by $\text{diam}(CH_{1,n}) \leq 3$. Since $\delta(CH_{1,n}) \geq \text{diam}(CH_{1,n})$ the result follows from Theorem 2.5.

(ii) For $n \geq 8$ we have $\delta(CH_{1,n}) = 3$ and $\text{diam}(CH_{1,n}) = 4$. Thus, $\varphi(CH_{1,n}) > 1$. By Theorem 2.7, $\varphi(CH_{1,n}) \leq \varphi(P_4) = 2$. Therefore, the result.

A **web graph**, denoted by $Wb_{1,n}$, is the graph obtained by attaching a pendant edge to each vertex of the outer cycle of the closed helm graph $CH_{1,n}$.

**Proposition 3.12.** For $n \geq 3$, $\varphi(Wb_{1,n}) = 2$.

**Proof.** The proof follows from arguments similar to those in Proposition 3.9.

A **double wheel** $DW_n$ is obtained by taking two copies of a wheel $W_n$ $n \geq 3$ and merging the two central vertices.

**Proposition 3.13.** For $n \geq 3$, $\varphi(DW_n) = 1$. 


Proof. Follows directly from Theorem 2.5.

4. Deg-centrication of Graph Operations

We have already seen that \( \varphi(K_{a,b}) = 1 \), and this fact immediately leads us to observe that the join of two graphs, with at least one of them is non-trivial, also has the deg-centrication number 1. Another elementary graph operation is the disjoint union of graphs denoted by \( G \cup H \). Clearly, \( \varphi(G \cup H) = \max\{ \varphi(G), \varphi(H) \} \). Recall that the corona between \( G \) of order \( n \) and \( H \) is denoted by \( G \circ H \). It is obtained by taking \( n \) copies of \( H \) and joining a copy of \( H \) to each vertex of \( G \). Also, a split graph is one in which the vertices of the graph can be partitioned into a clique and an independent vertex set.

**Theorem 4.1.** For a non trivial graph \( G \) and a graph \( H \), it follows that

1. If \( G \) is complete and \( H \cong K_1 \) or \( H \cong K_2 \) then \( \varphi(G \circ H) = 2 \).

2. If \( G \) is complete and \( H \cong K_n \geq 3 \) then \( \varphi(G \circ H) = 1 \).

3. If \( G \) is complete and \( H \) is more general than those in cases (i) and (ii), then \( \varphi(G \circ H) \leq 2 \).

4. If \( G \) and \( H \) are more general than those in cases (i), (ii) and (iii), then \( \varphi(G \circ H) \leq \varphi(G) + 1 \).

Proof.

1. Since \( G \) is non-trivial, it has an order of at least 2. If \( H \cong K_1 \) then \( J = (G \circ H)_d \) is a split graph and has \( \delta(J) \geq 2 \) and \( \text{diam}(J) = 2 \). Hence, \( \varphi(G \circ H) = 2 \).

   If \( H \cong K_2 \) then \( J = (G \circ H)_d \) is not complete and has \( \delta(J) \geq 3 \) and \( \text{diam}(J) = 3 \). Hence, \( \varphi(G \circ H) = 2 \).

2. Clearly, \( G \circ H \) has \( \delta(G \circ H) \geq 3 \) and \( \text{diam}(G \circ H) = 3 \). Therefore, \( \varphi(G \circ H) = 1 \).

3. Clearly, \( J = G \circ H \) has \( \delta(J_d) \geq 3 \) and \( \text{diam}(J_d) \leq 3 \). Therefore, \( \varphi(G \circ H) \leq 2 \).
4. If $G$ is a star, then $\varphi(G) = \infty$. Since $\varphi(S_{1,n} \circ H)$ is finite the result $\varphi(G \circ H) \leq \varphi(G)$ holds. Assume $G$ is not a star. After the first iteration, each copy of $H$ is complete. Furthermore, the induced subgraph $(N_G[v_i] \cup V(H_i))$ is complete. At worst, a total of $k \leq \varphi(G)$ is needed to yield $G$ in itself, complete. Then, a split graph resulted. Hence, $\varphi(G \circ H) \leq k + 1 \leq \varphi(G) + 1$.

5. Conclusion

The graph transformation called deg-centrication has been introduced. Various exploratory results have been presented to establish some foundation for further research. The adjacency regime can be subdivided as follows:

Observation 5.1. An edge $v_i v_j$ establishes if and only if

1. $d_G(v_i, v_j) < \text{deg}_G(v_i)$.
2. $d_G(v_i, v_j) \leq \text{deg}_G(v_i)$.
3. $d_G(v_i, v_j) = \text{deg}_G(v_i)$.
4. $d_G(v_i, v_j) \geq \text{deg}_G(v_i)$.
5. $d_G(v_i, v_j) > \text{deg}_G(v_i)$.

This paper initiated a study of case (b). Case (c) is currently being addressed. All other cases are also very promising for further research.

Recall that the density index of a graph $G$ of order $n$ and denoted by $\xi(G)$ is defined as, $\xi(G) = \frac{2\varepsilon(G)}{n(n-1)}$, $n \neq 1$. For the empty graph $N$ the density index is $\xi(N) = 0$ and for a complete graph $K_n$, $n \geq 2$, $\xi(K_n) = 1$. Conventionally a sparse graph $G$ is a graph for which $0 \leq \xi(G) < \frac{1}{2}$. A dense graph $H$ is a graph for which $\frac{1}{2} < \xi(H) \leq 1$. The introductory research suggests that determining a closed formula for $\varphi$ for sparse graphs is generally hard, if not intractable. Sparse graphs are worthy of further research.

Conjecture 5.2. For a non-star graph $G$, $0 \leq \varphi(G) \leq \max\{\left\lceil \frac{e(v_i)}{\text{deg}_G(v_i)} \right\rceil : v_i \in V(G)\}$. 

\[ \square \]
In Conjecture 5.2, the lower bound is necessary to acknowledge that a complete graph is a member of the family of non-star graphs.

Section 4 deals with only three elementary graph operations. The variety of graph products is worthy of research. Graph operators that yield graphs from graphs, such as line graphs, complement graphs, Boolean graphs, set graphs, and so on, open a wide avenue for research.

**Theorem 5.3.** For any graph (including a star), \( \varphi(G^k) \leq \varphi(G) \), \( k \geq 2 \) where \( G^k \) is the \( k^{th} \) power graph of \( G \).

**Proof.** Since \( G \) is always a spanning subgraph of \( G^k \) it implies that \( \delta(G) \leq \delta(G^k) \) and \( \text{diam}(G) \geq \text{diam}(G^k) \). Hence the result. \( \square \)

The facts mentioned above highlight the wide scope for further studies in this area.

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