Proyecciones Journal of Mathematics
Vol. 43, N^o 1, pp. 247-263, February 2024.
Universidad Católica del Norte
Antofagasta - Chile

10.22199/issn.0717-6279-6117

Tight bounds for the N_2-chromatic number of graphs

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Received : August 2023. Accepted : November 2023

Abstract

Let G be a connected graph. A vertex coloring of G is an N_2-vertex coloring if, for every vertex v, the number of different colors assigned to the vertices adjacent to v is at most two. The N_2-chromatic number of G is the maximum number of colors that can be used in an N_2-vertex coloring of G. In this paper, we establish tight bounds for the N_2-chromatic number of a graph in terms of its maximum degree and its diameter, and characterize those graphs that attain these bounds.

Keywords: *Vertex coloring, N_2-vertex coloring, N_2-chromatic number, maximum degree, diameter.*

MSC (2020): *05C15.*

1. Introduction

Let G be a simple, finite, connected, and undirected graph with vertex set $V(G)$ and edge set $E(G)$. For $v \in V(G)$, the notations $N_G(v)$ and $d_G(v)$ denote the open neighborhood of v in G and the degree of v in G, respectively, and we set $N_G[v] = N_G(v) \cup \{v\}$ as the closed neighborhood of v. The graph $G - v$ is the subgraph of G wherein the vertex v and the edges incident to it are removed from G. The maximum degree of G is denoted by $\Delta(G)$. For $S \subseteq V(G)$, we define the open neighborhood and closed neighborhood of S as $N_G(S) = \bigcup_{v \in S} N_G(v)$ and $N_G[S] = N_G(S) \cup S$, respectively, and we denote the induced subgraph of S in G by $G[S]$. For $x, y \in V(G)$, the distance between x and y in G is denoted by $d_G(x, y)$. The diameter of G is denoted by $\mathrm{diam}(G)$, and a shortest path between two vertices whose distance is $\mathrm{diam}(G)$ is also referred to as a diameter of G. A peripheral vertex of G is an endpoint of a diameter of G. For each k, $1 \leq k \leq \mathrm{diam}(G)$, we denote by $N_G^k(v)$ the set of vertices of distance k from vertex v in G.

Graph coloring techniques have been studied by graph theorists for many decades now. A graph coloring is an assignment of colors (usually integers for convenience) to the vertices or edges, or both, subject to certain conditions. In [5], Czap introduced a graph coloring called M_i-edge coloring. An edge coloring of a graph G is called M_i-edge coloring if at most i colors appear at any vertex of G. This edge coloring focuses on determining the maximum number of colors $K_i(G)$ used in an M_i-edge coloring of G. Budajová and Czap [4] proved that every graph G with maximum degree at least 2 has an M_2-edge coloring with at least $\alpha(G) + 1$ colors, where $\alpha(G)$ is the size of a maximum matching in G. Moreover, Czap, Ivančo and Šugerek [6] determined the $K_i(G)$ for trees, cacti, complete multipartite graphs, and graph joins.

In [1], Akbari, Alipourfard, Jandaghi and Mirtaheri introduced a vertex-coloring version of M_2-edge coloring. A vertex coloring f is called an N_2-vertex coloring of G if $|\{f(x) : x \in N_G(v)\}| \leq 2$ for each $v \in V(G)$. Let $t_2(G)$, which we call the N_2-chromatic number of G, be the maximum number of colors that can be used in an N_2-vertex coloring of G. Also in [1], some lower and upper bounds for $t_2(G)$ in terms of girth, diameter, and size of G were provided and the formula $t_2(T) = n - \ell + 2$ for any tree T of order n with ℓ leaves was obtained.

It is not difficult to verify that $2 \leq t_2(G) \leq n$ for any connected graph G of order $n \geq 2$. The complete graph K_n, $n \geq 2$ and $n \neq 3$, has N_2-

chromatic number of 2, and $t_2(G) = n$ if and only if $G = P_n$ (the path of order $n \geq 2$) or $G = C_n$ (the cycle of order $n \geq 3$).

In this paper, we provide some bounds for the N_2-chromatic number of graphs in terms of their maximum degree and their diameter. We also prove that these bounds are tight and characterize those graphs that attain these bounds.

For convenience, we call the numbers assigned to the vertices as colors and, instead of formally exhibiting a function as a vertex coloring of a particular graph, we may sometimes assign colors to the vertices to describe a vertex coloring. For graph-theoretic terms and notations that are not explicitly defined in this paper, we refer the readers to [3].

2. N_2-Chromaticity and the Maximum Degree

We again note that $t_2(G) \geq 2$ for any connected graph G with at least 2 vertices. Moreover, the only connected graphs of order at most 3 are the paths P_2 and P_3 and the cycle C_3, and we have remarked that $t_2(P_2) = 2$, $t_2(P_3) = 3$, and $t_2(C_3) = 3$.

Theorem 1. *Let G be a connected graph of order $n \geq 4$. If $\Delta(G) = n-1$, then $2 \leq t_2(G) \leq 3$.*

Proof. From the preceding paragraph, we are left to show that $t_2(G) \leq 3$. Let $v \in V(G)$ such that $d_G(v) = n-1$. Then the neighbors of v can be colored by at most two colors other than that of v. Thus, we have $2 \leq t_2(G) \leq 3$. □

Theorem 2. *Let G be a connected graph of order $n \geq 4$ with a vertex v of degree $n-1$. Then $t_2(G) = 3$ if and only if $G - v$ is disconnected or is bipartite.*

Proof. Suppose that $G-v$ is disconnected. Let f be a vertex coloring of G such that $f(v) = 1$, $f(u) = 2$ for all vertices u of one component of $G-v$, and $f(w) = 3$ for all other vertices w. This is an N_2-vertex coloring of G using three colors, which implies that $t_2(G) \geq 3$. Therefore, by Theorem 1, we have $t_2(G) = 3$.

Suppose that $G - v$ is bipartite, and let (X, Y) be its bipartition. Let f be a vertex coloring of G such that $f(v) = 1$, $f(x) = 2$ for all $x \in X$, and $f(y) = 3$ for all $y \in Y$. This color assignment is an N_2-vertex coloring of G

using three colors, which implies that $t_2(G) \geq 3$. By Theorem 1, we have $t_2(G) = 3$.

Suppose that $t_2(G) = 3$ and $G - v$ is connected. Then, in any N_2-vertex coloring of G, the vertices of $G - v$ must be colored by using two colors different from the color assigned to v, and adjacent vertices in $G - v$ must be colored differently. Thus, we have $G - v$ is bipartite. □

The following corollary follows from Theorems 1 and 2.

Corollary 3. *Let G be a connected graph of order $n \geq 4$ with a vertex v of degree $n - 1$. Then $t_2(G) = 2$ if and only if $G - v$ is a connected non-bipartite graph.*

Corollary 4. *Let G be a connected graph of order $n \geq 4$ with exactly two vertices u and v of degree $n - 1$. Then $t_2(G) = 3$ if and only if $G - \{u, v\}$ is an empty graph.*

Proof. Note that $G - u$ is a connected graph. From Theorem 2, we know that $t_2(G) = 3$ if and only if $G - u$ is bipartite. Observe $G - u$ is bipartite if and only if $G - \{u, v\}$ is an empty graph. □

Corollary 5. *Let G be a connected graph of order $n \geq 4$ with at least 3 vertices of degree $n - 1$. Then $t_2(G) = 2$.*

Proof. For any vertex x of G of degree $n - 1$, observe that $G - x$ is connected and non-bipartite. Thus, by Corollary 3, we obtain $t_2(G) = 2$. □

We observe that if G is a connected graph of order $n \geq 4$ with maximum degree $n - 2$ and v a vertex of degree $n - 2$, then there is a unique vertex u such that $uv \notin E(G)$.

Theorem 6. *Let G be a connected graph of order $n \geq 4$ with $\Delta(G) = n-2$. Then $2 \leq t_2(G) \leq 4$.*

Proof. Because $t_2(G) \geq 2$ for any graph G, we only need to show that $t_2(G) \leq 4$. Let v be a vertex with $d_G(v) = n-2$. Observe that any coloring of the vertices of G using at least five colors will force the vertices in $N_G(v)$ to be colored using at least three different colors. Thus, G cannot have an N_2-vertex coloring using at least five colors, and the inequality follows. □

Theorem 7. *Let G be a connected graph of order $n \geq 4$ with $\Delta(G) = n-2$, and let v and u be vertices such that $d_G(v) = n - 2$ and $u \notin N_G(v)$. Then $t_2(G) = 4$ if and only if $d_G(x) = 2$ for all $x \in N_G(u)$.*

Proof. Note that $t_2(G) = 4$ if and only if there exists an N_2-vertex coloring f of G such that $f(u) \neq f(v)$ and two different colors in $N_G(v)$ distinct from $f(u)$ and $f(v)$. This happens if and only if $d_G(x) = 2$ for all $x \in N_G(u)$. □

Theorem 8. *Let G be a connected graph of order $n \geq 4$ with $\Delta(G) = n-2$, and let v and u be vertices such that $d_G(v) = n-2$ and $u \notin N_G(v)$. Then $t_2(G) = 3$ if and only if $d_G(x) \geq 3$ for some $x \in N_G(u)$ and at least one of the following conditions is satisfied:*

(i) *$N_G(y) \cap N_G(u) = \emptyset$ for some $y \in N_G(v)$, or*

(ii) *$G[N_G(v)]$ is disconnected or bipartite.*

Proof. Suppose that $t_2(G) = 3$. Theorems 6 and 7 guarantee that $d_G(x) \geq 3$ for some $x \in N_G(u)$. Let f be an N_2-vertex coloring of G that uses three colors. We consider two cases:

Case 1. If $f(u) \neq f(v)$, then there is a vertex $y \in N_G(v)$ such that $f(y)$ is different from $f(u)$ and $f(v)$. Because f is an N_2-vertex coloring of G, it follows that $N_G(y) \cap N_G(u) = \emptyset$.

Case 2. If $f(u) = f(v)$, then $f(x)$ is different from $f(u)$ and $f(v)$ for all $x \in N_G(v)$. As in the proof of Theorem 2, we see that $G[N_G(v)]$ is disconnected or bipartite.

Conversely, suppose that $d_G(x) \geq 3$ for some $x \in N_G(u)$ and at least one of the two conditions is satisfied. By Theorems 6 and 7, we know that $t_2(G) \leq 3$. We show that each condition yields an N_2-vertex coloring of G.

Suppose that condition (i) holds. We color vertex u with 1, vertex y with 2, and the remaining vertices with 3. We can check that the vertices u and v have neighbors with at most two colors. Because of (i), each vertex in $N_G(v)$ has at most two colors in its neighborhood. Thus, this color assignment is an N_2-vertex coloring of G.

Finally, suppose that condition (ii) holds. If $G[N_G(v)]$ is disconnected, then we can obtain an N_2-vertex coloring of G by assigning color 1 to both u and v, color 2 to all vertices of one component of $G[N_G(v)]$, and color 3 to all remaining vertices. On the other hand, if $G[N_G(v)]$ is bipartite, then we color the vertices u and v with 1, the vertices in one bipartition of $G[N_G(v)]$ with 2, and the vertices in other bipartition of $G[N_G(v)]$ with 3. The latter color assignment is again an N_2-vertex coloring of G. This completes the proof of the theorem. □

The following corollary is a quick consequence of Theorems 7 and 8.

Corollary 9. *Let G be a connected graph of order $n \geq 4$ with $\Delta(G) = n-2$. Then $t_2(G) = 2$ if and only if the following conditions are satisfied:*

(i) *for every $y \in N_G(v)$, there is a $z \in N_G(u)$ such that $yz \in E(G)$, and*

(ii) *$G[N_G(v)]$ is connected and non-bipartite.*

We generalize the upper bounds set in Theorems 1 and 6 in the following theorem.

Theorem 10. *Let n and k be integers such that $n \geq 2$ and $1 \leq k \leq n-1$, and let G be a connected graph of order n with $\Delta(G) = k$. Then $t_2(G) \leq n-k+2$.*

Proof. Let $v \in V(G)$ such that $d_G(v) = \Delta(G) = k$. The maximum number of colors an N_2-vertex coloring of G can use is $1+2+(n-k-1) = n-k+2$, which is obtained by using 1 color for v, 2 colors for the vertices in $N_G(v)$, and one different color for each of the other $n-k-1$ vertices that are not in $N_G(v)$. □

The properties claimed in the following lemma follow immediately by considering a vertex of maximum degree.

Lemma 11. *Let G be a connected graph of order $n \geq 6$ and with $3 \leq \Delta(G) \leq n-3$, and let $v \in V(G)$ with $d_G(v) = \Delta(G)$. If $t_2(G) = n-\Delta(G)+2$ and f is an N_2-vertex coloring that uses $t_2(G)$ colors, then*

(i) *f uses exactly three colors to the vertices in $N_G[v]$, and*

(ii) *f assigns different colors (other than the three colors used in $N_G[v]$) to the vertices in $V(G) \setminus N_G[v]$.*

Theorem 12. *Let G be a connected graph of order $n \geq 6$ and with $3 \leq \Delta(G) \leq n-3$, and let $v \in V(G)$ with $d_G(v) = \Delta(G)$. Then $t_2(G) = n - \Delta(G) + 2$ if and only if G satisfies the following conditions:*

(i) *$d_G(u) = 2$ for every $u \in N_G(v) \cap N_G(N_G^2(v))$,*

(ii) *$1 \leq d_G(u) \leq 2$ for every $u \in V(G) \setminus N_G[N_G(v)]$, and*

(iii) *$|N_G(u) \cap (V(G) \setminus N_G[v])| \leq 1$ for every $u \in N_G^2(v)$.*

Before proving the preceding theorem, we need some notations. For $S \subset V(G)$, we define the graph $G(S)$ as that subgraph of G with $V(G(S)) = N_G[S]$ and $E(G(S)) = \{xy \in E(G) : x \in S \text{ or } y \in S\}$.

Proof of Theorem 12. For convenience, we let $A = \{u \in N_G(v) : N_G(u) \subseteq N_G[v]\}$, $B = N_G(v) \setminus A = N_G(v) \cap N_G(N_G^2(v))$, and $C = V(G) \setminus N_G[v]$.

Suppose that $t_2(G) = n - \Delta(G) + 2$ and f is an N_2-vertex coloring of G that uses $t_2(G)$ colors.

The vertices in B are those vertices in $N_G(v)$ with at least one adjacent vertex in $N_G^2(v)$. Since f is an N_2-vertex coloring, every vertex $u \in B$ needs to have at most two colors around it, that is, one color from v and another color from a vertex in $N_G(v)$ or $N_G^2(v)$. But because u is adjacent to at least one vertex in $N_G^2(v)$ and vertices in $N_G^2(v)$ are assigned with different colors other than the colors used in $N_G[v]$ by Lemma 11(i) and (ii), it follows that u is adjacent only to v and to exactly one vertex in $N_G^2(v)$, and that u is not adjacent to any vertex in $N_G(v)$. Thus, we have $d_G(u) = 2$ for every $u \in B$, which establishes (i).

Lemma 11(ii) implies condition (ii).

Because f is an N_2-vertex coloring of G, every vertex $u \in N_G(N_G(v)) \cap (V(G) \setminus N_G(v))$ is adjacent to at most one vertex in $V(G) \setminus N_G[v]$. Condition (iii) follows.

Suppose that a connected graph G satisfies conditions (i), (ii), and (iii). Since $d_G(v) = \Delta(G) \leq n-3$, we have $|V(G) \setminus N_G[v]| \geq 2$ and $G(N_G(v)) - v$ is disconnected.

Let f be a vertex coloring of G defined as follows: $f(v) = 1$, $f(u) = 2$ for all vertices $u \in N_G(v)$ in one component of $G(N_G(v)) - v$, $f(u) = 3$ for all vertices $u \in N_G(v)$ in the other components of $G(N_G(v)) - v$, and $f(u_i) = i + 3$ for $u_i \in V(G) \setminus N_G[v] = \{u_1, u_2, \ldots, u_{n-\Delta(G)-1}\}$. It is not difficult to check that f is an N_2-vertex coloring of G using $n - \Delta(G) + 2$ colors. By Theorem 10, it follows that $t_2(G) = n - \Delta(G) + 2$. ∎

To exhibit a family of graphs that satisfies the conditions in Theorem 12, we use the vertex amalgamation of graphs. For each k, $1 \leq k \leq n$, let G_k be a connected graph and v_k a vertex of G_k. The vertex amalgamation of the graphs G_k at v_k, denoted by

$$\bigodot_{k=1}^{n}(G_k, v_k) = (G_1, v_1) \odot (G_2, v_2) \odot \cdots \odot (G_n, v_n),$$

is the graph obtained by identifying (or amalgamating) the vertices $v_1, v_2, \ldots, v_n$.

Corollary 13. *For each k, $1 \leq k \leq n$, let G_k be either a path of order at least 2 or a cycle of order at least 3. Let $G = \bigodot_{k=1}^{n}(G_k, v_k)$ for any vertex v_k of G_k. Then $t_2(G) = |V(G)| - \Delta(G) + 2$.*

3. N_2-Chromaticity and the Diameter

In [1, Lemma 2.1(i)], it was presented that if the diameter of a connected graph G is d, then a lower bound for $t_2(G)$ is $\lfloor d/2 \rfloor$ (although, in its proof, it showed $\lceil d/2 \rceil$). We give an improvement of this lower bound.

Theorem 1. *Let G be a connected graph with at least two vertices. If* $\operatorname{diam}(G) = d$, *then $t_2(G) \geq \lceil d/2 \rceil + 1$.*

Proof. Let v be a peripheral vertex of G, and let f be a vertex coloring of G defined as follows: $f(v) = 1$ and $f(u) = k+1$ for $u \in N_G^{2k-1}(v) \cup N_G^{2k}(v)$, $1 \leq k \leq \lceil d/2 \rceil$. It is not difficult to verify that f is an N_2-vertex coloring of G that uses $\lceil d/2 \rceil + 1$ colors. □

To establish the tightness of the inequality in the preceding theorem, we need some graph operations. Let G_1 and G_2 be two graphs with disjoint vertex sets V_1 and V_2 and edge sets E_1 and E_2, respectively. Their union, denoted by $G_1 \cup G_2$, is the graph with $V(G_1 \cup G_2) = V_1 \cup V_2$ and $E(G_1 \cup G_2) = E_1 \cup E_2$. This "union" operation can be easily extended to several graphs. The join of G_1 and G_2, denoted by $G_1 + G_2$, is the graph with $V(G_1 + G_2) = V_1 \cup V_2$ and $E(G_1 + G_2) = E_1 \cup E_2 \cup \{uv : u \in V_1, v \in V_2\}$.

Introduced in [2] and named in [3], the sequential join of the graphs $G_1, G_2, \ldots, G_n$, denoted by $G_1 \uplus G_2 \uplus \cdots \uplus G_n$ or $\biguplus_{k=1}^{n} G_k$, is the graph

$$\biguplus_{k=1}^{n} G_k = \bigcup_{k=1}^{n-1} (G_k + G_{k+1}) = (G_1 + G_2) \cup (G_2 + G_3) \cup \cdots \cup (G_{n-1} + G_n).$$

Corollary 2. *For integer $d \geq 2$, let $a_1, a_2, \ldots, a_{d+1}$ be positive integers for which $a_k \geq 2$ for $2 \leq k \leq d$. Then, for any collection $\{G_k\}$ of connected graphs with $|V(G_k)| = a_k$ for $k = 1, 2, \ldots, d+1$, we have*

$$\operatorname{diam}\left(\biguplus_{k=1}^{d+1} G_k\right) = d \quad \text{and} \quad t_2\left(\biguplus_{k=1}^{d+1} G_k\right) = \left\lceil \frac{d}{2} \right\rceil + 1.$$

Proof. It is not difficult to verify that $\operatorname{diam}\left(\biguplus_{k=1}^{d+1} G_k\right) = d$ for any collection $\{G_k\}$ of connected graphs with $|V(G_k)| = a_k$. By Theorem 1, we have $t_2\left(\biguplus_{k=1}^{d+1} G_k\right) \geq \lceil d/2 \rceil + 1$.

Suppose that there is an N_2-vertex coloring of $\biguplus_{k=1}^{d+1} G_k$ that uses more than $\lceil d/2 \rceil + 1$ colors. Then there is an integer k, $2 \leq k \leq d$, such that this N_2-vertex coloring assigns at least 3 different colors to the vertices of G_{k-1}, G_k, and G_{k+1}. Since $a_k \geq 2$, there is a vertex in G_{k-1}, G_k, or G_{k+1} that has 3 colors in its neighborhood, a contradiction. Thus, we have $t_2\left(\biguplus_{k=1}^{d+1} G_k\right) = \lceil d/2 \rceil + 1$. □

By the contrapositive of Theorem 1 and after noting that the only connected graph G with $\operatorname{diam}(G) = 1$ and $t_2(G) = 2$ is the complete graph, we obtain the following corollary.

Corollary 3. *Let G be a connected graph of order $n \geq 4$. If $t_2(G) = 2$, then* $\operatorname{diam}(G) \leq 2$. *Moreover, if G is not a complete graph with $t_2(G) = 2$, then* $\operatorname{diam}(G) = 2$.

The following lemma follows from Theorems 1, 6, and 10.

Lemma 4. *Let G be a connected graph of order $n \geq 7$ with $t_2(G) \geq 6$. Then $\Delta(G) \leq n - 4$.*

Theorem 5. *Let G be a connected graph of order $n \geq 5$ with* $\operatorname{diam}(G) = 2$. *Then $2 \leq t_2(G) \leq 5$.*

Proof. By Theorem 1, we know that $t_2(G) \geq 2$. It suffices to show that there exists no N_2-vertex coloring of G using six colors. Clearly, there is no way we can color G with 6 colors if $n = 5$. Moreover, if $n = 6$ and G admits an N_2-vertex coloring using 6 colors, then G must be the path P_6 or the cycle C_6, contradicting our assumption that $\operatorname{diam}(G) = 2$. We are left to consider the case when $n \geq 7$.

Let G be a connected graph of order $n \geq 7$ with $\operatorname{diam}(G) = 2$. Suppose that $t_2(G) \geq 6$. Let $v \in V(G)$ with $\deg(v) = \Delta(G)$. For each $w \in N_G^2(v)$, there is a vertex $u \in N_G(v)$ such that u is adjacent to w. Note that, by Lemma 4, we have $|N_G^2(v)| \geq 3$.

Let f be an N_2-vertex coloring of G that uses at least 6 colors. Then there exist 3 vertices in $N_G^2(v)$ with different colors other than the colors used in $N_G[v]$, say w_1, w_2, and w_3. For $1 \leq i \leq 3$, observe that the sets $N_G(w_i) \cap N_G(v)$ are pairwise disjoint. Because $\operatorname{diam}(G) = 2$, there exists a vertex w' in $N_G^2(v)$ that is adjacent to two of these w_i's. This vertex w' has at least 3 colors in its neighborhood, a contradiction. Therefore, no N_2-vertex coloring of G uses 6 or more colors. □

Corollary 6. *For every ordered pair (n, k) of integers, where $n \geq 5$ and $2 \leq k \leq 5$, there exists a connected graph G of order n with $\operatorname{diam}(G) = 2$ and $t_2(G) = k$.*

Proof. The graph $K_n - e$ for any edge e of the complete graph K_n, the graph $K_2 + \overline{K_{n-2}}$ (the join of K_2 and the empty graph $\overline{K_{n-2}}$ with $n-2$ vertices), and the complete bipartite graph $K_{2,n-2}$ have order n and diameter 2. Moreover, it is not difficult to compute that $t_2(K_n - e) = 2$, $t_2(K_2 + \overline{K_{n-2}}) = 3$, and $t_2(K_{2,n-2}) = 4$.

For $k = 5$, we define a graph G as follows:

$$V(G) = \{a, b, c, d, u_1, u_2, \ldots, u_{n-4}\}$$

and

$$E(G) = \{ab, bc, cd, au_1, au_2, \ldots, au_{n-4}, du_1, du_2, \ldots, du_{n-4}\}.$$

It can easily be verified that G has order n and diameter 2. By assigning color 1 to a, 2 to b, 3 to c, 4 to d, and 5 to all u_i's, we obtain an N_2-vertex coloring of G using 5 colors. By Theorem 5, it follows that $t_2(G) = 5$. □

Theorem 7. *Let G be a connected graph of order $n \geq 5$ with $\operatorname{diam}(G) = 2$, and v a peripheral vertex. Then $t_2(G) = 5$ if and only if the induced subgraph $G[N_G(v)]$ is an empty graph and any of the following two conditions holds:*

(i) *there exists a partition $\{A, B\}$ of $N_G^2(v)$ such that*

  (a) *$G[N_G^2(v)]$ is a complete bipartite graph with bipartition $\{A, B\}$, and*

  (b) *$G[A \cup A']$ and $G[B \cup B']$ are complete bipartite graphs with bipartitions $\{A', A\}$ and $\{B', B\}$, respectively, where $A' = N_G(A) \cap N_G(v)$ and $B' = N_G(B) \cap N_G(v)$;*

(ii) *there exists a partition $\{A, B, C\}$ of $N_G^2(v)$ such that*

  (a) *$G[A \cup B]$ is a complete bipartite graph with bipartition $\{A, B\}$,*

  (b) *$N_G(C) \cap (A \cup B) = \emptyset$, and*

(c) *$G[N_G(v) \cup C]$ is a complete bipartite graph with bipartition $\{N_G(v), C\}$.*

Before we prove the preceding theorem, we need the following lemmas.

Lemma 8. *Let G be a connected graph of order $n \geq 5$ with $\operatorname{diam}(G) = 2$, and v a peripheral vertex. Then, in any N_2-vertex coloring of G that uses 5 colors, there are two nonempty disjoint subsets A and B of $N_G^2(v)$ such that*

(i) *all vertices in A and B use two colors, one unique color for each set, that are different from the colors assigned to the vertices in $N_G[v]$, and*

(ii) *the induced subgraph $G[A \cup B]$ is a complete bipartite graph with bipartition $\{A, B\}$.*

Moreover, the sets A and B are maximal in the sense that each set contains all vertices in $N_G^2(v)$ that receive the corresponding unique color.

Proof. Let f be an N_2-vertex coloring of G that uses 5 colors. It is not difficult to see that there exists two vertices $a, b \in N_G^2(v)$ whose colors are different from each other and from the colors of the vertices in $N_G[v]$. Define sets $A = \{x \in N_G^2(v) : f(x) = f(a)\}$ and $B = \{y \in N_G^2(v) : f(y) = f(b)\}$.

Since f is an N_2-vertex coloring, observe that no vertex in $N_G(v) \cup N_G^2(v)$ is adjacent to a vertex in A and to a vertex in B. Thus, because $\operatorname{diam}(G) = 2$, every vertex in A must be adjacent to every vertex in B. Since every vertex in $A \cup B$ is adjacent to a vertex in $N_G(v)$ whose color is different from the color of the vertices in $A \cup B$, it follows that no vertex in A is adjacent to another vertex in A. Similarly, no vertex in B is adjacent to another vertex in B. Hence, $A \cup B$ induces a complete bipartite graph. $\square$

Lemma 9. *Let G be a connected graph of order $n \geq 5$ with $\operatorname{diam}(G) = 2$, and v a peripheral vertex. Then any N_2-vertex coloring of G that uses 5 colors assigns 2 or 3 colors to the vertices in $N_G^2(v)$. Moreover, if an N_2-vertex coloring uses 3 colors in $N_G^2(v)$, then one of these colors is the same as that color assigned to v.*

Proof. Let f be an N_2-vertex coloring of G that uses 5 colors. By Lemma 8, there are subsets A and B of $N_G^2(v)$ whose vertices have corresponding unique colors different from those used in $N_G[v]$, and $G[A \cup B]$ is a complete bipartite graph. Thus, f uses at least 2 colors in $N_G^2(v)$.

Suppose that f uses at least 3 colors in $N_G^2(v)$. Let c be a vertex in $N_G^2(v)$ such that $f(c)$ is different from the colors assigned to A and B. We show that $f(c) = f(v)$.

Suppose, on the contrary, that $f(c) \neq f(v)$. Because f is an N_2-vertex coloring, $\operatorname{diam}(G) = 2$, and of Lemma 8(i), the vertex c must be adjacent either to a vertex in A or to a vertex in B (but not both). Without loss of generality, suppose that c is adjacent to a vertex in A.

Let $A' = N_G(A) \cap N_G(v)$ and $B' = N_G(B) \cap N_G(v)$. Then $A' \cap B' = \emptyset$ (otherwise, a vertex in $A' \cap B'$ would have three different colors around it, that is, one from v, another one from A, and another one from B), and there is a vertex in A' that receives the color $f(c)$ (otherwise, a vertex in A would have three colors around it). Because $f(v)$, $f(c)$, and the unique color used in B are all different, every vertex in B' must not be adjacent to c. Since $\operatorname{diam}(G) = 2$, for each vertex $u \in B'$, there is a vertex $x \in N_G(v)$ such that x is adjacent to both c and u. This forces $f(x) = f(v)$ so that u would have two colors around it. This means that each vertex in $N_G(v)$ is colored with either $f(c)$ or $f(v)$.

Because f uses 5 colors and only 4 of them have been used so far (2 in $N_G[v]$ and 2 in $A \cup B$), f must assign the fifth color to a vertex in $N_G^2(v)$. However, this vertex cannot be of distance 1 or 2 to a vertex in $A \cup B$. This is a contradiction. Therefore, we have $f(c) = f(v)$, and so f assigns at most 3 colors to the vertices in $N_G^2(v)$. □

Proof of Theorem 7. Suppose that $t_2(G) = 5$ and let f be an N_2-vertex coloring of G that uses 5 colors. By Lemma 8, there exist two nonempty disjoint subsets A and B of $N_G^2(v)$ such that f assigns a unique color to each set that is different from those assigned to the vertices in $N_G[v]$ and $G[A \cup B]$ is a complete bipartite graph with bipartition $\{A, B\}$. With Lemma 9, we consider two cases.

Case 1. Suppose that there are exactly 2 colors used in $N_G^2(v)$. This implies that these colors are the ones used to color A and B and that $\{A, B\}$ partitions $N_G^2(v)$.

Because $\operatorname{diam}(G) = 2$ and $\{A, B\}$ partitions $N_G^2(v)$, the pair $\{A', B'\}$ partitions $N_G(v)$, the set $N_G(v)$ induces an empty graph, and the induced subgraphs $G[A \cup A']$ and $G[B \cup B']$ are also complete bipartite graphs. This establishes condition (i).

Case 2. Suppose that there are exactly 3 colors used in $N_G^2(v)$. By Lemmas 8 and 9, these 3 colors are $f(a)$, $f(b)$, and $f(v)$, where $a \in A$ and $b \in B$. The other two colors of f are used to color the vertices in $N_G(v)$.

Let $C = \{x \in N_G^2(v) : f(x) = f(v)\}$. Then $\{A, B, C\}$ partitions $N_G^2(v)$ and $N_G(C) \cap (A \cup B) = \emptyset$. Because $\mathrm{diam}(G) = 2$, it follows that every vertex in C is adjacent to every vertex in $N_G(v)$, and so C induces an empty graph.

Let A' and B' be the same sets defined in condition (i), and let $C' = N_G(v) \setminus (A' \cup B')$. Then $A' \cup B'$ induces an empty graph.

Suppose $C' \neq \emptyset$, and let x be a vertex in C'. Since $N_G(C) \cap (A \cup B) = \emptyset$ and $\mathrm{diam}(G) = 2$, for each vertex $a \in A$, there is a vertex $a' \in A'$ such that a' is adjacent to both x and a. However, the vertex a' has already 3 colors around it, a contradiction. This implies that $C' = \emptyset$, and so $N_G(v) = A' \cup B'$ induces an empty graph and $N_G(v) \cup C$ induces a complete bipartite graph with bipartition $\{N_G(v), C\}$. This establishes condition (ii) and completes the proof of the "only if" part of the theorem.

Conversely, suppose that the induced subgraph $G[N_G(v)]$ is an empty graph and any of the two conditions holds. We exhibit an N_2-vertex coloring of G that uses 5 colors for each condition.

With condition (i), we define a vertex coloring f_1 of G as follows: $f_1(v) = 1$, $f_1(u) = 2$ for $u \in A'$, $f_1(u) = 3$ for $u \in B'$, $f_1(u) = 4$ for $u \in A$, and $f_1(u) = 5$ for $u \in B$. It is not difficult to verify that f_1 is an N_2-vertex coloring of G.

With condition (ii), we define a vertex coloring f_2 of G as follows: $f_2(u) = 1$ for $u = v$ or $u \in C$, $f_2(u) = 2$ for $u \in A'$, $f_2(u) = 3$ for $u \in B'$, $f_2(u) = 4$ for $u \in A$, and $f_2(u) = 5$ for $u \in B$. It is not also difficult to verify that f_2 is an N_2-vertex coloring of G. This completes the proof of Theorem 7. ∎

4. Concluding Remarks

We have established the bounds for the N_2-chromatic number of a graph in terms of its maximum degree and its diameter. In Theorem 1, we showed that $2 \leq t_2(G) \leq 3$ when G is a connected graph of degree $n \geq 4$ with $\Delta(G) = n - 1$, and characterized those graphs with $t_2(G) = 2$ and those with $t_2(G) = 3$. We presented in Theorem 6 that $2 \leq t_2(G) \leq 4$ when G is a connected graph of order $n \geq 4$ with $\Delta(G) = n - 2$, and characterized those graphs that satisfy each value of $t_2(G)$. Generally, in Theorems 10 and 1, if G is a connected graph with at least two vertices, then

$$\lceil \mathrm{diam}(G)/2 \rceil + 1 \le t_2(G) \le |V(G)| - \Delta(G) + 2. \qquad (*)$$

For graphs with diameter 2, Theorem 5 guarantees that $2 \le t_2(G) \le 5$. These values of $t_2(G)$ were realized in Corollary 6. Finally, Theorem 7 characterized those graphs G with $\mathrm{diam}(G) = 2$ and $t_2(G) = 5$.

While the extreme values in $(*)$ can be attained by some graphs (see Theorem 12, Corollary 13, and Corollary 2), there are families of graphs whose N_2-chromatic numbers lie strictly between these extreme bounds.

The double-headed kite $DK(K_m, K_n, P_k)$, where $m, n \ge 4$ and $k \ge 2$, is the graph

$$DK(K_m, K_n, P_k) = (((K_m, v) \odot (K_n, v_1)), v_2) \odot (P_n, u),$$

where v is a vertex of K_m, v_1 and v_2 are distinct vertices of K_n, and u is an endvertex of P_n. It can be computed that

$$t_2(DK(K_m, K_n, P_k)) = \mathrm{diam}(DK(K_m, K_n, P_k)) = k + 1,$$

which is one less than the upper bound set in $(*)$.

On the other hand, we slightly modify the graph and technique described in Corollary 2. For integer $d \ge 2$, let $a_1, a_2, \ldots, a_{d+1}$ be positive integers for which $a_k \ge 2$ for $2 \le k \le d-1$ and $a_d = a_{d+1} = 1$ if d is even, and $a_k \ge 2$ for $2 \le k \le d-2$ and $a_{d-1} = a_d = a_{d+1} = 1$ if d is odd. Then, for any collection $\{G_k\}$ of connected graphs with $|V(G_k)| = a_k$ for $k = 1, 2, \ldots, d+1$, we have

$$\mathrm{diam}\left(\biguplus_{k=1}^{d+1} G_k\right) = d \quad \text{and} \quad t_2\left(\biguplus_{k=1}^{d+1} G_k\right) = \left\lceil \frac{d}{2} \right\rceil + 2,$$

which is one more than the lower bound set in $(*)$.

With the technique again used in Corollary 2, it is worth observing that, given integers a and b, where $a \ge 4$ and $b \ge \lceil a/2 \rceil + 1$, there exists a connected graph G for which $\mathrm{diam}(G) = a$ and $t_2(G) = b$.

In Table 4.1, using Theorem 7 and with the help of a Python program created by Professor Jon Fernandez (a colleague of the second author) to verify some N_2-vertex colorings and some values, we are able to compute the N_2-chromatic number of all known graphs of largest possible order n with diameter 2, maximum degree Δ, defect k, and the number N of such graphs up to isomorphism. Some of these graphs attain the Moore bound (when $k = 0$, which are the Moore graphs) and the other graphs are simply optimal (that is, the largest possible order). We refer the readers to [8] for a survey of Moore graphs and the degree/diameter problem.

D	Δ	n	k	N	Known Graphs G	$t_2(G)$
2	2	4	1	1	Cycle C_4	4
2	2	5	0	1	Cycle C_5	5
2	3	10	0	1	Petersen graph	4
2	3	8	2	2	Refer to [8]	4
2	4	15	2	1	Found by Elspas [7]	4
2	5	24	2	1	Found by Elspas [7]	4
2	6	32	5	6	G_1, G_2, G_3 (see [9])	4
					G_4, G_5, G_6 (see [9])	3
2	7	50	0	1	Hoffman-Singleton graph	4

Table 4.1: Values of $t_2(G)$ for known Moore graphs and graphs of maximum order with $\mathrm{diam}(G) = 2$ and maximum degree Δ

The existence of the Moore graph of order 3250 with diameter 2 and maximum degree 57 is still unknown. However, because of the layer-type structure of Moore graphs, we can compute their N_2-chromatic numbers. Assuming that this unknown Moore graph does exist, Theorem 7 guarantees that its N_2-chromatic number is at most 4. Now, let v be a peripheral vertex of this graph. By assigning the color 1 to v, color 2 to one vertex in $N_G(v)$, color 3 to the remaining vertices in $N_G(v)$, and color 4 to the vertices in $N_G^2(v)$, it can be verified that this color assignment is an N_2-vertex coloring of the graph. Therefore, the N_2-chromatic number of this yet-to-be-found Moore graph would be 4.

While we are able to establish some tight bounds for the N_2-chromatic number of graphs, there are still some open questions on this vertex-coloring concept. We know that $t_2(G) \geq 2$ for any connected graph G of order at least 2. The complete graph of order at least 4 and the join of two connected graphs each of order at least 2 have N_2-chromatic number of 2. A particular case (when $d = 2$) of Corollary 2 yields another family of graphs with N_2-chromatic number of 2.

Problem. Characterize those graphs G with $t_2(G) = \mathrm{diam}(G) = 2$.

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