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# On universal realizability in the left half-plane

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**Abstract**

*A list* $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ *of complex numbers is said to be realizable if it is the spectrum of a nonnegative matrix.* $\Lambda$ *is said to be universally realizable (*$\mathcal{UR}$*) if it is realizable for each possible Jordan canonical form allowed by* $\Lambda$*. In this paper, using companion matrices and applying a procedure by Šmigoc, we provide sufficient conditions for the universal realizability of left half-plane spectra, that is, spectra* $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ *with* $\lambda_1 > 0$, $\operatorname{Re}\lambda_i \leq 0$, $i = 2, \ldots, n$. *It is also shown how the effect of adding a negative real number to a not* $\mathcal{UR}$ *left half-plane list of complex numbers, makes the new list* $\mathcal{UR}$*, and a family of left half-plane lists that are* $\mathcal{UR}$ *is characterized.*

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## 1. Introduction

A list $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ of complex numbers is said to be *realizable* if it is the spectrum of an $n$-by-$n$ nonnegative matrix $A$, and $A$ is said to be a *realizing matrix* for $\Lambda$. The problem of the realizability of spectra is called *nonnegative inverse eigenvalue problem* (NIEP). From the Perron-Frobenius Theorem we know that if $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ is the spectrum of an $n$-by-$n$ nonnegative matrix $A$, then the leading eigenvalue of $A$ equals to the spectral radius of $A$, $\rho(A) =: \max\limits_{1 \leq i \leq n} |\lambda_i|$. This eigenvalue is called the *Perron eigenvalue,* and we shall assume in this paper, that $\rho(A) = \lambda_1$.

A matrix is said to have *constant row sums,* if each one of its rows sums up to the same constant $\alpha$. The set of all matrices with constant row sums equal to $\alpha$, is denoted by $\mathcal{CS}_\alpha$. Then, any matrix $A \in \mathcal{CS}_\alpha$ has the eigenvector $\mathbf{e}^T = [1, 1, \ldots, 1]$, corresponding to the eigenvalue $\alpha$. The real matrices with constant row sums are important because it is known that the problem of finding a nonnegative matrix with spectrum $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$, is equivalent to the problem of finding a nonnegative matrix in $\mathcal{CS}_{\lambda_1}$ with spectrum $\Lambda$ (see [4]). We denote by $\mathbf{e}_k$, the n-dimensional vector, with 1 in the $k^{th}$ position and zeros elsewhere. If $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$, then $s_k(\Lambda) = \sum\limits_{i=1}^{n} \lambda_i^k$, $k \in \mathbf{N}$.

Since a list of complex numbers is always the spectrum of some matrix (a diagonal matrix for instance) we shall use the word spectrum or list interchangeably. A list $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ of complex numbers, is said to be *diagonalizably realizable* ($\mathcal{DR}$)*,* if there is a diagonalizable realizing matrix for $\Lambda$. The list $\Lambda$ is said to be *universally realizable* ($\mathcal{UR}$), if it is realizable for each possible Jordan canonical form (JCF) allowed by $\Lambda$. The problem of the universal realizability of spectra, is called *universal realizability problem* (URP). The URP contains the NIEP, and both problems are equivalent if the given numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ are distinct. In terms of $n$, both problems remain unsolved for $n \geq 5$. It is clear that if $\Lambda$ is $\mathcal{UR}$, then $\Lambda$ must be $\mathcal{DR}$. The first known results on the URP are due to Minc [8, 9]. In terms of the URP, Minc [8] showed that if a list $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ of complex numbers is the spectrum of a diagonalizable positive matrix, then $\Lambda$ is $\mathcal{UR}$. The positivity condition is necessary for Minc's proof, and the question set by Minc himself, whether the result holds for nonnegative realizations was open for almost 40 years. Recently, two extensions of Minc's result have been obtained in [2, 5]. In [2], Collao et al. showed that a nonnegative matrix $A \in \mathcal{CS}_{\lambda_1}$, with a positive column, is similar to a positive matrix. Note that if $A$ is nonnegative with a positive row and $A^T$ has a positive eigenvector, then $A^T$ is also similar to a positive matrix. Besides, if $\Lambda$ is diagonalizably realizable by a matrix $A \in \mathcal{CS}_{\lambda_1}$ having a positive column, then $\Lambda$ is $\mathcal{UR}$. In [5], Johnson et al. introduced the concept of ODP matrices, that is, nonnegative matrices with all positive off-diagonal entries (zero diagonal entries are permitted) and proved that if $\Lambda$ is diagonalizably ODP realizable, then $\Lambda$ is $\mathcal{UR}$. Note that both extensions contain, as a particular case, Minc's result in [8]. Both extensions allow us to significantly increase the set of spectra that can be proved to be $\mathcal{UR}$. In particular, the extension in [5] allows to show, for instance, that certain spectra $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ with $s_1(\Lambda) = 0$ are $\mathcal{UR}$, which is not possible from Minc's result. In particular, we shall use the extension in [2] to generate some of our results.

In [1, 10] the authors proved, respectively, that lists of complex numbers $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$, of Suleimanova type, that is,

$$\lambda_1 > 0, \;\; \operatorname{Re}\lambda_i \leq 0, \;\; |\operatorname{Re}\lambda_i| \geq |\operatorname{Im}\lambda_i|\,, \;\; i = 2, 3, \ldots, n,$$

and of Šmigoc type, that is,

$$\lambda_1 > 0, \;\; \operatorname{Re}\lambda_i \leq 0, \;\; \sqrt{3}\,|\operatorname{Re}\lambda_i| \geq |\operatorname{Im}\lambda_i|\,, \;\; i = 2, 3, \ldots, n,$$

are realizable if and only if $\sum_{i=1}^{n} \lambda_i \geq 0$, while in [11, 3] the authors proved, respectively, that both lists, Suleimanova type and Šmigoc type, are $\mathcal{UR}$ if and only if they are realizable if and only if $\sum_{i=1}^{n} \lambda_i \geq 0$.

Outline of the paper: The paper is organized as follows: In Section 2, we present the mathematical tools that will be used to generate our results. In Section 3, we study the URP for a left half-plane list and we give sufficient conditions for it to be $\mathcal{UR}$. In Section 4, we discuss the effect of adding a negative real number $-c$ to a left half-plane list $\Lambda = \{\lambda_1, -a \pm bi, \ldots, -a \pm bi\}$, which is not $\mathcal{UR}$ (or even not realizable), or we do not know whether it is, and we show how $\Lambda \cup \{-c\}$ becomes $\mathcal{UR}$. We also characterize a family of left half-plane lists that are $\mathcal{UR}$. In Section 5, we show that the merge of two lists diagonalizably realizable $\Gamma_1 \in CS_{\lambda_1}$ and $\Gamma_2 \in CS_{\mu_1}$ is $\mathcal{UR}$. Examples are shown to illustrate the results.

## 2. Preliminaries

Throughout this paper we use the following results: The first one, by Šmigoc [10], gives a procedure that we call Šmigoc's glue technique, to obtain from two matrices $A$ and $B$ of size $n$-by-$n$ and $m$-by-$m$, respectively, a new $(n+m-1)$-by-$(n+m-1)$ matrix $C$, preserving in certain way, the corresponding Jordan forms of $A$ and $B$. The second one, by Laffey and Šmigoc [7], is one of the most important results on the NIEP. It completely solves the NIEP for left half-plane spectra, that is, lists with $\lambda_1 > 0$, $\operatorname{Re}\lambda_i \leq 0$, $i = 2, \ldots, n$. Moreover, we also use Lemma 5 in [7].

**Theorem 2.1.** *[10] Suppose $B$ is an $m$-by-$m$ matrix with a JCF that contains at least one 1-by-1 Jordan block corresponding to the eigenvalue $c$:*

$$J(B) = \begin{bmatrix} c & 0 \\ 0 & I(B) \end{bmatrix}.$$

*Let $\mathbf{t}$ and $\mathbf{s}$, respectively, be the left and the right eigenvectors of $B$ associated with the 1-by-1 Jordan block in the above canonical form. Furthermore, we normalize vectors $\mathbf{t}$ and $\mathbf{s}$ so that $\mathbf{t}^T\mathbf{s} = 1$. Let $J(A)$ be a JCF for the $n$-by-$n$ matrix*

$$A = \begin{bmatrix} A_1 & \mathbf{a} \\ \mathbf{b}^T & c \end{bmatrix},$$

*where $A_1$ is an $(n-1)$-by-$(n-1)$ matrix and $\mathbf{a}$ and $\mathbf{b}$ are vectors in $\mathbf{C}^{n-1}$. Then the matrix*

$$C = \begin{bmatrix} A_1 & \mathbf{a}\mathbf{t}^T \\ \mathbf{s}\mathbf{b}^T & B \end{bmatrix}$$

*has JCF*

$$J(C) = \begin{bmatrix} J(A) & 0 \\ 0 & I(B) \end{bmatrix}.$$

**Theorem 2.2.** *[7] Let $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ be a list of complex numbers with $\lambda_1 \geq |\lambda_i|$ and $\operatorname{Re}\lambda_i \leq 0$, $i = 2, \ldots, n$. Then $\Lambda$ is realizable if and only if*

$$s_1 = s_1(\Lambda) \geq 0, \quad s_2 = s_2(\Lambda) \geq 0, \quad s_1^2 \leq n s_2.$$

**Lemma 2.1.** *[7] Let $t$ be a nonnegative real number and let $\lambda_2, \lambda_3, \ldots, \lambda_n$ be complex numbers with real parts less than or equal to zero, such that the list $\{\lambda_2, \lambda_3, \ldots, \lambda_n\}$ is closed under complex conjugation. Set $\rho = 2t - \lambda_2 - \cdots - \lambda_n$ and*

$$(2.1) \quad f(x) = (x-\rho)\prod_{j=2}^{n}(x-\lambda_j) = x^n - 2tx^{n-1} + b_2x^{n-2} + \cdots + b_n.$$

*Then $b_2 \leq 0$ implies $b_j \leq 0$ for $j = 3, 4, \ldots, n$.*

## 3. Companion matrices and the Šmigoc's glue.

We say that a list $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ of complex numbers is on the left half-plane if $\lambda_1 > 0$, $\operatorname{Re}\lambda_i \leq 0$, $i = 2, 3, \ldots, n$. In this section we give sufficient conditions for a realizable left half-plane list of complex numbers to be $\mathcal{UR}$. Of course, it is our interest to consider lists of complex numbers containing elements out of realizability region of lists of Šmigoc type. Our strategy consists in decomposing the given list $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ into sublists

$$\Lambda_k = \{\lambda_{k1}, \lambda_{k2}, \ldots, \lambda_{kp_k}\}, \ \lambda_{11} = \lambda_1, \ k = 1, 2, \ldots, t,$$

with auxiliary lists

$$\begin{aligned} \Gamma_1 &= \Lambda_1 \\ \Gamma_k &= \{s_1(\Gamma_{k-1}), \lambda_{k1}, \lambda_{k2}, \ldots, \lambda_{kp_k}\}, \quad k = 2, , \ldots, t, \end{aligned}$$

each one of them being the spectrum of a nonnegative companion matrix $A_k$, in such a way that it be possible to apply Šmigoc's glue technique to the matrices $A_k$, to obtain an $n$-by-$n$ nonnegative matrix with spectrum $\Lambda$ for each possible JCF allowed by $\Lambda$. In the case $s_1(\Lambda) > 0$, with $\lambda_i \neq 0$, $i = 2, \ldots, n$, we may choose, if they exist, sublists $\Gamma_k$ being the spectrum of a diagonalizable nonnegative companion matrix $A_k$ with a positive column. Then, after Šmigoc's glue, we obtain a diagonalizable nonnegative $n$-by-$n$ matrix $A$ with spectrum $\Lambda$ and a positive column. Thus, $A$ is similar to a diagonalizable positive matrix, and then, from the extension in [2], $\Lambda$ is $\mathcal{UR}$. Next we have the following corollary from Theorem 2.1:

**Corollary 3.1.** *Let $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ be a realizable left half-plane list of complex numbers. Suppose that for each JCF $\mathbf{J}$ allowed by $\Lambda$, there exists a decomposition of $\Lambda$ as*

$$\Lambda = \Lambda_1 \cup \Lambda_2 \cup \cdots \cup \Lambda_t, \text{ where}$$

$$\Lambda_k = \{\lambda_{k1}, \lambda_{k2}, \ldots, \lambda_{kp_k}\},\ k = 1, 2, \ldots, t,\ \lambda_{11} = \lambda_1,$$

*with auxiliary lists*

$$\begin{aligned} \Gamma_1 &= \Lambda_1, \\ \Gamma_k &= \{s_1(\Gamma_{k-1}), \lambda_{k1}, \lambda_{k2} \ldots, \lambda_{kp_k}\},\ k = 2, \ldots, t, \end{aligned}$$

*being the spectrum of a nonnegative companion matrix $A_k$ with JCF $J(A_k)$ being a submatrix of $\mathbf{J}$, $k = 1, 2, \ldots, t$.*

*Then $\Lambda$ is universally realizable.*

**Proof.** Let

$$\mathbf{J} = \begin{bmatrix} J_{n_1}(\lambda_1) & & & & \\ & J_{n_2}(\lambda_2) & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & J_{n_t}(\lambda_t) \end{bmatrix}$$

be a JCF allowed by $\Lambda$. Since each matrix $A_k$, $k = 1, 2, \ldots, t$, is nonnegative companion with JCF

$$J(A_k) = \begin{bmatrix} J_{n_p}(\lambda_p) & & & \\ & J_{n_q}(\lambda_q) & & \\ & & \ddots & \\ & & & J_{n_s}(\lambda_s) \end{bmatrix},\ 1 \leq p, q, s \leq t$$

being a submatrix of $\mathbf{J}$, then, from Šmigoc's glue applied to matrices $A_k$, we obtain an $n$-by-$n$ nonnegative matrix with spectrum $\Lambda$ and JCF $\mathbf{J}$. As $\mathbf{J}$ is any JCF allowed by $\Lambda$, then $\Lambda$ is $\mathcal{UR}$. $\square$

The following example illustrates Corollary 3.1:

**Example 3.1.** *Consider the list*

$$\Lambda = \{10, -1, -1, -1 \pm 3i, -1 \pm 3i, -1 \pm 3i\},$$

*and suppose we want to construct a realizing matrix for* $\Lambda$*, with JCF*

$$\mathbf{J} = diag\{J_1(10),\ J_2(-1),\ J_2(-1 \pm 3i),\ J_1(-1 \pm 3i)\}.$$

*Then we take*

$$\begin{aligned} \Gamma_1 &= \{10, -1 \pm 3i, -1 \pm 3i\} \\ \Gamma_2 &= \{6, -1, -1, -1 \pm 3i\} \end{aligned}$$

*with companion realizing matrices*

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1000 \\ 1 & 0 & 0 & 0 & 300 \\ 0 & 1 & 0 & 0 & 200 \\ 0 & 0 & 1 & 0 & 16 \\ 0 & 0 & 0 & 1 & 6 \end{bmatrix},\ A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 60 \\ 1 & 0 & 0 & 0 & 122 \\ 0 & 1 & 0 & 0 & 68 \\ 0 & 0 & 1 & 0 & 9 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix},$$

*respectively. The left and the right eigenvectors of* $A_2$*,* $t^T$ *and* $s$*, are respectively,*

$$\begin{aligned} t^T &= \begin{bmatrix} \frac{1}{2842} & \frac{3}{1421} & \frac{18}{1421} & \frac{108}{1421} & \frac{648}{1421} \end{bmatrix} \\ s^T &= \begin{bmatrix} 10 & 22 & 15 & 4 & 1 \end{bmatrix}. \end{aligned}$$

*Then, the Šmigoc's glue of* $A_1$ *with* $A_2$ *is*

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{500}{1421} & \frac{3000}{1421} & \frac{18\,000}{1421} & \frac{108\,000}{1421} & \frac{648\,000}{1421} \\ 1 & 0 & 0 & 0 & \frac{150}{1421} & \frac{900}{1421} & \frac{5400}{1421} & \frac{32\,400}{1421} & \frac{194\,400}{1421} \\ 0 & 1 & 0 & 0 & \frac{100}{1421} & \frac{600}{1421} & \frac{3600}{1421} & \frac{21\,600}{1421} & \frac{129\,600}{1421} \\ 0 & 0 & 1 & 0 & \frac{8}{1421} & \frac{48}{1421} & \frac{288}{1421} & \frac{1728}{1421} & \frac{10\,368}{1421} \\ 0 & 0 & 0 & 10 & 0 & 0 & 0 & 0 & 60 \\ 0 & 0 & 0 & 22 & 1 & 0 & 0 & 0 & 122 \\ 0 & 0 & 0 & 15 & 0 & 1 & 0 & 0 & 68 \\ 0 & 0 & 0 & 4 & 0 & 0 & 1 & 0 & 9 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

*with the desired JCF* $\mathbf{J}$.

**Remark 3.1.** *From the result of Laffey and Smigoc [7] it is known when a realization by a companion matrix is possible. However, since companion matrices have Jordan forms with blocks of maximum size, to obtain a diagonalizable realizing companion matrix with spectrum $\Lambda$, the list $\Lambda$ must have distinct eigenvalues.*

The following result is well known and useful.

**Lemma 3.1.** *Let $A$ be a diagonalizable irreducible nonnegative matrix with spectrum $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ and a positive row or column. Then $A$ is similar to a diagonalizable nonnegative matrix $B \in CS_{\lambda_1}$, with the same positive row or column.*

**Proof.** If $A$ is irreducible nonnegative, it has a positive eigenvector $\mathbf{x}^T = [x_1, \ldots, x_n]$. Then if $D = dig\{x_1, \ldots, x_n\}$, the matrix

$$\mathrm{B} = \mathrm{D}^{-1}\mathrm{AD} = \left[\frac{x_j}{x_i} a_{i,j}\right] \in \mathrm{CS}_{\lambda_1}$$

is nonnegative with the same positive row or column. □

Now, suppose all lists $\Gamma_k$ in Corollary 3.1, can be taken as the spectrum of a diagonalizable nonnegative companion matrix $A_k$ with a positive column (the last one). Then, since the glue of matrices $A_k$ gives an $n$-by-$n$ diagonalizable irreducible nonnegative matrix $A$ with a positive column and spectrum $\Lambda$, $A$ is similar to a diagonalizable positive matrix with spectrum $\Lambda$ and therefore $\Lambda$ is $\mathcal{UR}$. This is what the next result shows.

**Corollary 3.2.** *Let $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, $\lambda_i \neq 0$, $i = 2, \ldots, n$, $s_1(\Lambda) > 0$, be a realizable left half-plane list of complex numbers. If there is a decomposition of $\Lambda$ as in Corollary 3.1, with all lists $\Gamma_k$ being the spectrum of a diagonalizable nonnegative companion matrix $A_k$, with a positive column, then $\Lambda$ is universally realizable.*

**Proof.** Let $\Gamma_1$ and $\Gamma_2$, be the spectrum, respectively, of matrices $A_1$ and $A_2$, which are diagonalizable nonnegative companion with a positive column (the last one). Then $A_1$ and $A_2$ are irreducible. In particular, $A_2$ has a positive eigenvector $\mathbf{s}$ and, since $A_2^T$ is also irreducible, $A_2$ has also a positive left eigenvector $\mathbf{t}^T$ with $\mathbf{t}^T\mathbf{s} = 1$. Now, let

$$A_1 = \begin{bmatrix} A_1' & \mathbf{a} \\ \mathbf{b}^T & s_1(\Gamma_1) \end{bmatrix}.$$

Since the last column of $A_1$ is positive, the vector $\mathbf{a}$ is also positive and $\mathbf{a}\mathbf{t}^T$ is a positive submatrix. Therefore, the glue of $A_1$ with $A_2$,

$$C_2 = \begin{bmatrix} A_1' & \mathbf{a}\mathbf{t}^T \\ \mathbf{s}\mathbf{b}^T & A_2 \end{bmatrix},$$

is a diagonalizable nonnegative matrix with its last column being positive. Note that $C_2$ is also irreducible. Thus $C_2$ has a positive eigenvector, and then it is similar to a nonnegative matrix with constant row sums and with its last column being positive. Then, Šmigoc's glue applied to $C_2$ with $A_3$ gives a matrix $C_3$ diagonalizable nonnegative with a positive column, and so on, until we obtain an $n$-by-$n$ diagonalizable irreducible nonnegative matrix $A$ with a positive column and spectrum $\Lambda$. Therefore, from the extension in [2] $\Lambda$ is $\mathcal{UR}$. □

The following example illustrates Corollary 3.2

**Example 3.2.** *Consider the list*

$$\begin{aligned} \Lambda &= \{23, -2, -2, -1 \pm 5i, -1 \pm 5i, -1 \pm 5i, -2 \pm 7i, -2 \pm 7i\}, \textit{with} \\ \Gamma_1 &= \{23, -1 \pm 5i\},\ \Gamma_2 = \{21, -2, -1 \pm 5i, -2 \pm 7i\}, \\ \Gamma_3 &= \{13, -2, -1 \pm 5i, -2 \pm 7i\}. \end{aligned}$$

*The diagonalizable companion matrices*

$$\begin{aligned} A_1 &= \begin{bmatrix} 0 & 0 & 598 \\ 1 & 0 & 20 \\ 0 & 1 & 21 \end{bmatrix},\ A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 57\,876 \\ 1 & 0 & 0 & 0 & 0 & 35\,002 \\ 0 & 1 & 0 & 0 & 0 & 6266 \\ 0 & 0 & 1 & 0 & 0 & 1695 \\ 0 & 0 & 0 & 1 & 0 & 69 \\ 0 & 0 & 0 & 0 & 1 & 13 \end{bmatrix}, \\ A_3 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 35\,828 \\ 1 & 0 & 0 & 0 & 0 & 20\,618 \\ 0 & 1 & 0 & 0 & 0 & 3194 \\ 0 & 0 & 1 & 0 & 0 & 903 \\ 0 & 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 5 \end{bmatrix} \end{aligned}$$

*realize lists $\Gamma_1, \Gamma_2$ and $\Gamma_3$, respectively. Šmigoc's glue technique applied to matrices $A_1, A_2$ and $A_3$ gives a 13-by-13 diagonalizable irreducible non-negative matrix with a positive column and spectrum $\Lambda$. Therefore, from extension in [2], $\Lambda$ is UR.*

## 4. The effect of adding a negative real number to a not UR list

In this section we show how to add a negative real number $-c$ to a list of complex numbers

$$\Lambda = \{\lambda, \underbrace{-a \pm bi, \ldots, -a \pm bi}_{(n-1) \text{ complex numbers}}\}, \ \lambda, a, b > 0, \ \text{with } s_1(\Lambda) > 0,$$

which is not $\mathcal{UR}$ or we do not know whether it is, to make

$$\Lambda_c = \{\lambda, -c, \underbrace{-a \pm bi, \ldots, -a \pm bi}_{(n-2) \text{ complex numbers}}\}$$

$\mathcal{UR}$. For instance, the list $\Lambda_1 = \{6, -1 \pm 3i, -1 \pm 3i\}$ is realizable, but we do not know whether it is $\mathcal{UR}$, while $\Lambda_2 = \{17, -3 \pm 9i, -3 \pm 9i\}$ is not realizable. However, both lists become $\mathcal{UR}$ if we add an appropriate negative real number $-c$ to each of them.

**Example 4.1.** *Consider*

$$\Lambda_c = \{\frac{77}{4}, -3, \underbrace{-2 \pm 5i, \ldots, -2 \pm 5i}_{8 \text{ complex numbers}}\}.$$

*Suppose we want to obtain a nonnegative matrix with JCF*

$$\mathbf{J} = diag\{J_1(\frac{77}{4}), J_1(-3), J_2(-2+5i), (J_2(-2-5i)\}.$$

*Then,*

$$\begin{aligned} \Gamma_1 &= \{\frac{77}{4}, -2 \pm 5i, -2 \pm 5i\} \\ \Gamma_2 &= \{\frac{45}{4}, -3, -2 \pm 5i, -2 \pm 5i\}. \end{aligned}$$

*If $A_1, A_2$ are companion realizing matrices for $\Gamma_1$ and $\Gamma_2$, respectively, then from Lemma 2.1, $b_2(A_1) = 80$, $b_2(A_2) = \frac{103}{4}$ guarantee that $A_1$ and $A_2$ are nonnegative. Next, the glue of $A_1$ with $A_2$ gives a nonnegative matrix with JCF* $\mathbf{J}$.

**Theorem 4.1.** *Let $\Lambda = \{\lambda, -a \pm bi, \ldots, -a \pm bi\}$, fixed $\lambda > 0$, $a, b > 0$, be a list of complex numbers with $s_1(\Lambda) > 0$.*

*If*

$$\frac{(2n-11)a^2 + b^2}{2a} \leq \lambda, \tag{4.1}$$

*and there is a real number $c > 0$ such that*

$$\frac{2a(na-\lambda) + b^2 - 7a^2}{\lambda - (n-2)a} \leq c \leq \lambda - (n-2)a, \tag{4.2}$$

*then*

$$\Lambda_c = \{\lambda, -c, \underbrace{-a \pm bi, \ldots, -a \pm bi}_{(n-2) \text{ complex numbers}}\}$$

*becomes universally realizable.*

**Proof.** Consider the diagonalizable decomposition $\Lambda_c = \Lambda_1 \cup \Lambda_2 \cup \cdots \cup \Lambda_{\frac{n-2}{2}}$, with

$$\begin{aligned} \Lambda_1 &= \{\lambda, -a \pm bi\}, \\ \Lambda_k &= \{-a \pm bi\}, \ k = 2, \ldots, \frac{n-4}{2}, \\ \Lambda_{\frac{n-2}{2}} &= \{-c, -a \pm bi\}. \end{aligned}$$

We take the auxiliary sub-lists

$$\begin{aligned} \Gamma_1 &= \Lambda_1 = \{\lambda, -a \pm bi\} \\ \Gamma_2 &= \{\lambda - 2a, -a \pm bi\} \\ \Gamma_3 &= \{\lambda - 4a, -a \pm bi\} \end{aligned}$$

$$\begin{aligned} &\vdots \\ \Gamma_{\frac{n-4}{2}} &= \{\lambda - (n-6)a, -a \pm bi\}, \\ \Gamma_{\frac{n-2}{2}} &= \{\lambda - (n-4)a, -c, -a \pm bi\}, \end{aligned}$$

where $\Gamma_{\frac{n-4}{2}}$ and $\Gamma_{\frac{n-2}{2}}$ are the spectrum of the diagonalizable companion matrices

$$A_{\frac{n-4}{2}} = \begin{bmatrix} 0 & 0 & (a^2+b^2)(\lambda-(n-6)a) \\ 1 & 0 & 2a\lambda - a^2(2n-11) - b^2 \\ 0 & 1 & \lambda-(n-4)a \end{bmatrix}$$

and

$$A_{\frac{n-2}{2}} = \begin{bmatrix} 0 & 0 & 0 & (a^2+b^2)(\lambda-(n-4)a)c \\ 1 & 0 & 0 & (a^2+b^2)(\lambda-(n-4)a)+(7a^2-b^2+2a\lambda-2a^2n)c \\ 0 & 1 & 0 & (\lambda-(n-2)a)c+(7a^2-b^2+2a\lambda-2a^2n) \\ 0 & 0 & 1 & \lambda-(n-2)a-c \end{bmatrix},$$

respectively. Observe that sublists $\Gamma_{\frac{n-6}{2}}, \ldots, \Gamma_2, \Gamma_1$ have the same pair of complex numbers that the list $\Gamma_{\frac{n-4}{2}}$, but with a bigger Perron eigenvalue. Then, if $\Gamma_{\frac{n-4}{2}}$ is diagonalizably companion realizable, $\Gamma_{\frac{n-6}{2}}, \ldots, \Gamma_2, \Gamma_1$ also are. Thus, from Lemma 2.1 we only need to consider the entries in position $(2,3)$ in $A_{\frac{n-4}{2}}$ and in position $(3,4)$ in $A_{\frac{n-2}{2}}$. From (4.1) and (4.2) these entries are nonnegative and therefore $A_{\frac{n-4}{2}}$ and $A_{\frac{n-2}{2}}$ are diagonalizable companion realizing matrices. Thus, after applying $\frac{n-4}{2}$ times Šmigoc's glue to the matrices $A_1, \ldots, A_{\frac{n-2}{2}}$, we obtain an $n$-by-$n$ diagonalizable nonnegative matrix $A$ with spectrum $\Lambda_c$. Thus $\Lambda_c$ is $\mathcal{DR}$.

To obtain an $n$-by-$n$ nonnegative matrix $A$ with spectrum $\Lambda_c$ and with a non-diagonal JCF $\mathbf{J}$, we take $\Lambda_c = \Lambda_1 \cup \cdots \cup \Lambda_t$ with auxiliary lists $\Gamma_k$ being the spectrum of a companion matrix $A_k$ with JCF as a submatrix of $\mathbf{J}$. Next we need to prove that all $A_k$ are nonnegative. To do this, we compute $b_2(A_k)$ from the formula in (2.1) for each matrix $A_k$ with spectrum $\Gamma_k$, $k = 1, \ldots, t$, with $\Gamma_t$ containing $-c$ in the decomposition of $\Lambda_c$. From (4.1) and (4.2) $b_2(A_k) \geq 0$, $k = 1, \ldots, t$. Therefore the glue of matrices $A_k$ gives an $n$-by-$n$ nonnegative matrix $A$ with the desired JCF $\mathbf{J}$. $\square$

**Example 4.2.** *i)* $\Lambda = \{6, -1 \pm 3i, -1 \pm 3i\}$ *is realizable by the companion matrix*

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & 600 \\ 1 & 0 & 0 & 0 & 140 \\ 0 & 1 & 0 & 0 & 104 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix},$$

*with a non-diagonal JCF. We do not know whether* $\Lambda$ *has a diagonalizable realization. Then, consider the list*

$$\Lambda_c = \{6, -c, -1 \pm 3i, -1 \pm 3i\}.$$

*Condition (4.1) is satisfied and from (4.2) we have* $1 \leq c \leq 2$. *Then for* $c = 1$, *we have that*

$$\Gamma_1 = \{6, -1 \pm 3i\}, \quad \Gamma_2 = \{4, -1, -1 \pm 3i\}$$

*are the spectrum of diagonalizable nonnegative companion matrices*

$$A_1 = \begin{bmatrix} 0 & 0 & 60 \\ 1 & 0 & 2 \\ 0 & 1 & 4 \end{bmatrix}, \text{ and } A_2 = \begin{bmatrix} 0 & 0 & 0 & 40 \\ 1 & 0 & 0 & 38 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix},$$

*respectively. Then, from Šmigoc's glue we obtain a diagonalizable nonnegative matrix with spectrum* $\Lambda_c$. *It is clear that, from the characteristic polynomial associated to* $\Lambda_c$, $\Lambda_c$ *has also a companion realization,*

$$A_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 600 \\ 1 & 0 & 0 & 0 & 0 & 740 \\ 0 & 1 & 0 & 0 & 0 & 244 \\ 0 & 0 & 1 & 0 & 0 & 104 \\ 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix},$$

*with a JCF with blocks of maximun size.*

*ii) Consider the list* $\Lambda = \{17, -3 \pm 9i, -3 \pm 9i\}$. *Since* $s_1(\Lambda) = 5$ *and* $s_2(\Lambda) = 1$, *from Theorem 2.2* $\Lambda$ *is not realizable. From condition (4.2),* $\frac{24}{5} \leq c \leq 5$. *Then for* $c = 5$,

$$\Lambda_c = \{17, -5, -3 \pm 9i, -3 \pm 9i\}$$

*is $\mathcal{UR}$. Observe that there are only two Jordan forms allowed by $\Lambda_c$. We take*

$$\Gamma_1 = \{17, -3 \pm 9i\} \text{ and } \Gamma_2 = \{11, -5, -3 \pm 9i\}$$

*which are the spectrum of the diagonalizable nonnegative companion matrices*

$$A_1 = \begin{bmatrix} 0 & 0 & 1530 \\ 1 & 0 & 12 \\ 0 & 1 & 11 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 0 & 4950 \\ 1 & 0 & 0 & 870 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

*The glue of Šmigoc gives rise to a diagonalizable nonnegative matrix with spectrum $\Lambda_c$. From the characteristic polynomial associated to $\Lambda_c$*

$$p(x = x^6 - 13x^4 - 2532x^3 - 23\,220x^2 - 189\,000x - 688\,500,$$

*we obtain a nonnegative companion matrix with spectrum $\Lambda_c$ and non-diagonal JCF. Therefore, $\Lambda_c$ is $\mathcal{UR}$.*

Observe that in Theorem 4.1, in spite that $s_1(\Lambda) > 0$, if $s_1(\Lambda)$ is small enough, there may be lists $\Lambda_c$ that are not $\mathcal{UR}$ or we cannot to prove they are from our procedure. However, we may compute a Perron eigenvalue $\lambda$, which guarantees that for a family of lists $\Lambda_c$, with $c > 0$ and $n \geq 6$, $\Lambda_c$ will be $\mathcal{UR}$. Then, the following result characterizes a family of left half-plane lists, which are $\mathcal{UR}$.

**Corollary 4.1.** *The left half-plane lists of the family*

$$\Lambda_c = \{\frac{1}{2a}((2n-7)a^2 + b^2), -c, \underbrace{-a \pm bi, \ldots, -a \pm bi}_{(n-2) \text{ complex numbers}}\},$$

*with $0 < \sqrt{3}a < b$, $0 < c \leq \frac{b^2 - 3a^2}{2a}$, are universally realizable.*

**Proof.** It is clear that for $\lambda = \frac{1}{2a}\left((2n-7)a^2 + b^2\right)$, conditions (4.1) and (4.2) in Theorem 4.1 are satisfied. Moreover, from $0 < \sqrt{3}a < b$, we have $\lambda - (n-2)a = \frac{b^2-3a^2}{2a} > 0$. □

Then, from Corollary 4.1 some families of left half-plane lists that are UR are:

$$i)\ \Lambda_c = \{\tfrac{2n-3}{2}a, -c, \underbrace{-a \pm 2ai, \ldots, -a \pm 2ai}_{(n-2)\text{complex numbers}}\}, \text{with } 0 < c \leq \tfrac{a}{2}$$

$$ii)\ \Lambda_c = \{(n+1)a, -c, \underbrace{-a \pm 3ai, \ldots, -a \pm 3ai}_{(n-2)\text{ complex numbers}}\},\ \text{with } 0 < c \leq 3a$$

$$iii)\ \Lambda_c = \{\tfrac{2n+9}{2}a, -c, \underbrace{-a \pm 4ai, \ldots, -a \pm 4ai}_{(n-2)\text{ complex numbers}}\},\ \text{with } 0 < c \leq \tfrac{13}{2}a$$

$$iv)\ \Lambda_c = \{\tfrac{8n-3}{8}a, -c, \underbrace{-a \pm \frac{5}{2}ai, \ldots, -a \pm \frac{5}{2}ai}_{(n-2)\text{complex numbers}}\},\ \text{with } 0 < c \leq \tfrac{13}{8}a,$$

and so on.

Observe that in Corollary 4.1, if $c$ is strictly less than its upper bound, then $\Lambda_c$, as we have seen, can be realized by a diagonalizable matrix with its last column being positive. Then, from the extension in [2], $\Lambda_c$ is $\mathcal{UR}$.

**Remark 4.1.** *In this section, for a list of the form* $\Lambda = \{\lambda, \underbrace{-a \pm bi, \ldots, -a \pm bi}_{(n-2)\text{complex numbers}}\}$, $\lambda, a, b > 0$, *with* $s_1(\Lambda) > 0$, *we have given an answer to the question: can a spectrum be modified so that it becomes UR?. We think it is interesting to study whether the answer can be extended to more general spectra, although this may become a more technical situation. We believe that such a result may be possible if we can decompose* $\Lambda$ *into sublists of the form* $\Lambda_c = \{\lambda, -c, -a \pm bi, \ldots, -a \pm bi\}$.

## 5. The merge of spectra

Let $\Gamma_1 = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ and $\Gamma_2 = \{\mu_1, \mu_2, \ldots, \mu_m\}$ be lists of complex numbers. In [6] the authors define the concept of the *merge of the spectra* $\Gamma_1$ with $\Gamma_2$ as

$$\Gamma = \{\lambda_1 + \mu_1, \lambda_2, \ldots, \lambda_n, \mu_2, \ldots, \mu_m\},$$

and prove that if $\Gamma_1$ and $\Gamma_2$ are diagonalizably ODP realizable, then the merge $\Gamma_1$ *with* $\Gamma_2$, is also diagonalizably ODP realizable, and therefore from the extension in [5], $\Gamma$ is $\mathcal{UR}$. Here we set a similar result as follows:

**Theorem 5.1.** *Let* $\Gamma_1 = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, $\lambda_1 > |\lambda_i|$, $i = 2, \ldots, n$, *be the spectrum of a diagonalizable nonnegative* $n$*-by-*$n$ *matrix* $A \in \mathcal{CS}_{\lambda_1}$ *with a positive column. Let* $\Gamma_2 = \{\mu_1, \mu_2, \ldots, \mu_m\}$, $\mu_1 > |\mu_i|$, $i = 2, \ldots, m$, *be the spectrum of a diagonalizable nonnegative* $m$*-by-*$m$ *matrix* $B \in \mathcal{CS}_{\mu_1}$ *with a positive column. Then*

$$\Gamma = \{\lambda_1 + \mu_1, \lambda_2, \ldots, \lambda_n, \mu_2, \ldots, \mu_m\}$$

*is universally realizable..*

**Proof.** Let $A \in \mathcal{CS}_{\lambda_1}$ be a diagonalizable nonnegative matrix with spectrum $\Gamma_1$. Without loss of generality we assume that $A$ has its last column being positive. Then $A$ is similar to a diagonalizable positive matrix $A'$. If $\alpha_1, \ldots, \alpha_n$ are the diagonal entires of $A'$, then

$$\mathrm{A}_1 = A' + \mathbf{e}[0, 0, \ldots, \mu_1] = \begin{bmatrix} A'_{11} & \mathbf{a} \\ \mathbf{b}^T & \alpha_n + \mu_1 \end{bmatrix} \in \mathcal{CS}_{\lambda_1 + \mu_1}$$

is diagonalizable positive with spectrum $\{\lambda_1 + \mu_1, \lambda_2, \ldots, \lambda_n\}$ and diagonal entries $\alpha_1, \alpha_2, \ldots, \alpha_n + \mu_1$. Let $B \in \mathcal{CS}_{\mu_1}$ be a diagonalizable nonnegative matrix with spectrum $\Gamma_2$. Without loss of generality we assume that $B$ has its last column being positive. Then $B$ is similar to a diagonalizable positive matrix $B'$ and

$$B_1 = B' + \mathbf{e}[\alpha_n, 0, \ldots, 0]$$

is diagonalizable positive with spectrum $\{\mu_1 + \alpha_n, \mu_2, \ldots, \mu_m\}$. Now, by applying the Šmigoc's glue to matrices $A_1$ and $B_1$, we obtain a diagonalizable positive matrix $C$ with spectrum $\Gamma$. Hence, $\Gamma$ is $\mathcal{UR}$ $\square$

Theorem 5.1 is useful to decide, in many cases, about the universal realizability of left half-plane list of complex numbers, as for instance:

**Example 5.1.** *Is the list*

$\Gamma = \{30, -1, -5, -1 \pm 3i, -1 \pm 3i, -1 \pm 3i, -3 \pm 9i, -3 \pm 9i\}$ $\mathcal{UR}$?

*Observe that from the results in Section* 4,

$$\begin{aligned} \Gamma_1 &= \{21, -5, -3 \pm 9i, -3 \pm 9i\}. \\ \Gamma_2 &= \{9, -1, -1 \pm 3i, -1 \pm 3i, -1 \pm 3i\} \end{aligned}$$

*are the spectrum of a diagonalizable nonnegative matrix with constant row sums and a positive column. Then, they are similar to diagonalizable positive matrices and from Theorem 5.1, the merge* $\Gamma$ *is also the spectrum of a diagonalizable positive matrix. Therefore,* $\Gamma$ *is* $\mathcal{UR}$.

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