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Higher order mKdV breathers: nonlinear stability *

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Abstract

We are interested in stability results for breather solutions of the 5th, 7th and 9th order mKdV equations. We show that these higher order mKdV breathers are stable in $H^2(\mathbf{R})$, in the same way as classical mKdV breathers. We also show that breather solutions of the 5th, 7th and 9th order mKdV equations satisfy the same stationary fourth order nonlinear elliptic equation as the mKdV breather, independently of the order, 5th, 7th or 9th, considered.

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1. Introduction

In this note we extend previous results on the stability of breather solutions of the focusing modified Korteweg-de Vries (mKdV) equation (see [4]),

$$u_t + (u_{xx} + 2u^3)_x = 0, \quad u(t,x) \in \mathbf{R}, \tag{1.1}$$

to new breather solutions of higher order focusing versions of (1.1). Namely, we are going to deal with

the focusing 5th-order mKdV equation

$$\begin{aligned} &u_t + (u_{4x} + f_5(u))_x = 0, \\ &f_5(u) := 10uu_x^2 + 10u^2u_{xx} + 6u^5, \end{aligned} \tag{1.2}$$

the focusing 7th-order mKdV equation

$$\begin{aligned} &u_t + (u_{6x} + f_7(u))_x = 0, \\ &f_7(u) := 14u^2u_{4x} + 56uu_xu_{3x} + 42uu_{xx}^2 + 70u_x^2u_{xx} + 70u^4u_{xx} + 140u^3u_x^2 \\ &+20u^7, \end{aligned} \tag{1.3}$$

and the focusing 9th-order mKdV

$$\begin{aligned} &u_t + (u_{8x} + f_9(u))_x = 0, \\ &f_9(u) := 18u^2u_{6x} + 108uu_xu_{5x} + 228uu_{2x}u_{4x} + 210(u_x)^2u_{4x} + 126u^4u_{4x} \\ &+138u(u_{3x})^2 + 756u_xu_{2x}u_{3x} + 1008u^3u_xu_{3x} + 182(u_{2x})^3 + 756u^3(u_{2x})^2 \\ &+3108u^2(u_x)^2u_{2x} + 420u^6u_{2x} + 798u(u_x)^4 + 1260u^5(u_x)^2 + 70u^9, \end{aligned} \tag{1.4}$$

and which we will denote them as 5th, 7th and 9th-mKdV equations hereafter. All these higher order mKdV equations are members of an infinite family of equations, the so call focusing mKdV hierarchy of equations, as it was shown by Alejo [2] (see [7],[10] for defocusing mKdV versions of this hierarchy). Note that we are only interested in focusing mKdV versions since these models are the only mKdV equations bearing regular (not singular) and real breather solutions. Moreover, other higher order mKdV cases, (e.g. $(2n+1)th$-mKdV, $n \geq 5$) will not be treated here, since beside increasing the number of terms in each equation of the higher order hierarchy (see Appendix A), we have not at hand a global well posedness theory of them in a Sobolev space $H^s(\mathbf{R})$ with $s \leq 2$, as it was pointed out by

Grünrock [12, p.506, Cor.2.1]. Since our stability result is stated taking into account small perturbations in $H^2(\mathbf{R})$, only higher order mKdV equations with a Cauchy problem well defined in a Sobolev space $H^s(\mathbf{R}), s \leq 2$ are going to be considered here, namely the 5th, 7th and 9th-mKdV equations (see [17], [12] for further reading).

These higher order mKdV equations are a well-known completely integrable set of models [1, 8, 15], with infinitely many conservation laws. On the other hand, solutions of (1.2), (1.3) and (1.4) are invariant under space and time translations. In fact, for any given solution u and for any $t_0, x_0 \in \mathbf{R}$, $u(t-t_0, x-x_0)$ and, even more $-u$ are also solutions of (1.2), (1.3) and (1.4).

About the Cauchy problem of higher order versions of (1.1), Linares by using a contraction mapping argument showed in [17] that the initial value problem for the 5th-mKdV equation is locally well-posed at $H^2(\mathbf{R})$. Kwon, [14], obtained a better result: the 5th-mKdV equation is locally well-posed at $H^s(\mathbf{R}), s \geq \frac{3}{4}$. Finally, Grünrock, [12], deduced well-possedness results to other higher-order mKdV equations at Theorem 2.1. This same author established that 7th-mKdV equation is locally well-posed at $H^s(\mathbf{R}), s \geq \frac{5}{4}$. The Cauchy problem for the 5th-mKdV equation is globally well-posed at $H^s(\mathbf{R}), s \geq 1$ and in the case of the 7th and 9th-mKdV (1.3)-(1.4) equations at $H^s(\mathbf{R}), s \geq 2$. See e.g. Linares [17], Kwon [14] and Grünrock [12] for further details. Note moreover that we have the following inner relation between mKdV

$$u_t = -\partial_x(u_{xx} + 2u^3), \tag{1.5}$$

and its higher order versions, namely the 5th-mKdV,

$$u_t = -\partial_x\Big(\partial_x^2(u_{xx} + 2u^3) - (2uu_x^2 - 4u^2u_{xx} - 6u^5)\Big), \tag{1.6}$$

the 7th-mKdV

$$\begin{aligned} u_t = \; & -\partial_x\Big(\partial_x^4(u_{xx} + 2u^3) - \partial_x^2(2uu_x^2 - 4u^2u_{xx} - 6u^5) \\ & -(4uu_xu_{3x} - 4u^2u_{4x} - 2uu_{xx}^2 - 40u^4u_{xx} - 20u^3u_x^2 - 20u^7)\Big), \end{aligned} \tag{1.7}$$

and the 9th-mKdV

$$
\begin{aligned}
u_t \ = & -\partial_x\Big[\partial_x^6(u_{xx}+2u^3)-\partial_x^4(2uu_x^2-4u^2u_{xx}-6u^5)\\
& -\partial_x^2(4uu_xu_{3x}-4u^2u_{4x}-2uu_{xx}^2-40u^4u_{xx}-20u^3u_x^2-20u^7)\\
& -\Big(4uu_{6x}-8u^2u_{6x}-26uu_xu_{5x}+16u_xu_{5x}-52uu_{xx}u_{4x}+28u_{xx}u_{4x}\\
& -39u_x^2u_{4x}-39u_{xx}u_{4x}-56u^4u_{4x}-24uu_{3x}^2+16u_{3x}^2\\
& -84u_xu_{xx}u_{3x}-168u^3u_xu_{3x}-12u_{xx}^3-196u^3u_{xx}^2\\
& -168u^2u_x^2u_{xx}-280u^6u_{xx}+42uu_x^4-420u^5u_x^2-70u^9\Big)\Big].
\end{aligned} \tag{1.8}
$$

In the case of the 5th, 7th and 9th-mKdV equations (1.2)-(1.3)-(1.4), the profile of their soliton solutions is completely similar to the well known *sech* mKdV soliton profile, and it is explicitly given by the formula (we denote by v_5, v_7, v_9 the speeds of 5th, 7th and 9th order solitons)

$$
\begin{aligned}
& u(t,x) := Q_c(x-v_it)|_{i=5,7,9}, \quad v_5=c^2, v_7=c^3, v_9=c^4\\
& Q_c(s) := \sqrt{c}sech(\sqrt{c}s), c>0.
\end{aligned} \tag{1.9}
$$

Moreover, it is easy to see, by substitution that both 5th, 7th and 9th-mKdV soliton solutions Q_c (1.9) satisfy the same nonlinear stationary elliptic equation

$$Q_c''-c\,Q_c+2Q_c^3=0, \quad Q_c>0, \quad Q_c\in H^1(\mathbf{R}). \tag{1.10}$$

Note that this second order ODE is precisely the one satisfied by the mKdV classical soliton. Moreover, note that the soliton solution (1.9) of the 5th, 7th and 9th-mKdV equations also satisfy the 4th, 6th and 8th order elliptic ODEs coming naturally from integration in space of the 5th, 7th and 9th order mKdV equations (1.2)-(1.3)-(1.4) respectively. Namely, 5th, 7th and 9th higher order mKdV solitons satisfy the following nonlinear stationary elliptic equations:

$$Q_c^{(iv)}-c^2\,Q_c+f_5(Q_c)=0, \tag{1.11}$$

$$Q_c^{(vi)}-c^3Q_c+f_7(Q_c)=0, \tag{1.12}$$

and

$$Q_c^{(viii)} - c^4 Q_c + f_9(Q_c) = 0. \tag{1.13}$$

Instead integrating directly in space (1.2), (1.3) and (1.4), another way to check the validity of (1.11), (1.12) and (1.13) is by using the lowest order nonlinear stationary elliptic equation (1.10) satisfied by all higher order mKdV solitons. For instance, in the case of (1.11), we just substitute and obtain:

$$\begin{aligned}
& Q_c^{(iv)} - c^2\, Q_c + f_5(Q_c) \\
= \; & Q_c^{(iv)} - c^2\, Q_c + 10(Q_c^{'})^2 Q_c + 10 Q_c^2 Q_c^{''} + 6Q_c^5 \\
= \; & (cQ_c - 2Q_c^3)^{''} - c^2 Q_c + 10(Q_c^{'})^2 Q_c + 10Q_c^2(cQ_c - 2Q_c^3) + 6Q_c^5 \\
= \; & cQ_c^{''} - 12 Q_c (Q_c^{'})^2 - 6Q_c^2 Q_c^{''} - c^2 Q_c + 10(Q_c^{'})^2 Q_c + 10Q_c^2(cQ_c - 2Q_c^3) \\
& +6Q_c^5 \\
= \; & c(cQ_c - 2Q_c^3) - 2Q_c(cQ_c^2 - Q_c^4) - 6Q_c^2(cQ_c - 2Q_c^3) - c^2 Q_c \\
& +10Q_c^2(cQ_c - 2Q_c^3) + 6Q_c^5 = 0.
\end{aligned}$$

The proof for the other higher order nonlinear identities (1.12) and (1.13) follows in the same way. Note moreover that the second order elliptic equation (1.10) satisfied by all higher order mKdV solitons is deeply related to the variational meaning of the soliton solution. To be more precise, it is well-known that some of the (first) standard conservation laws of 5th, 7th and 9th-mKdV equations are the mass

$$M[u](t) := \frac{1}{2}\int_{\mathbf{R}} u^2(t,x)dx = M[u](0), \tag{1.14}$$

the energy

$$E[u](t) := \frac{1}{2}\int_{\mathbf{R}} \left(u_x^2 - u^4\right)(t,x)dx = E[u](0), \tag{1.15}$$

and the higher order energies, defined respectively in $H^2(\mathbf{R})$

$$E_5[u](t) := \int_{\mathbf{R}} \left(\frac{1}{2}u_{xx}^2 - 5u^2u_x^2 + u^6\right)(t,x)dx = E_5[u](0), \tag{1.16}$$

in $H^3(\mathbf{R})$

$$E_7[u](t) := \int_{\mathbf{R}} \left(\frac{1}{2}u_{3x}^2 + \frac{7}{2}u_x^4 - 7u^2u_{xx}^2 + 35u^4u_x^2 - \frac{5}{2}u^8\right)(t,x)dx = E_7[u](0), \tag{1.17}$$

and in $H^4(\mathbf{R})$

$$\begin{aligned} E_9[u](t) \quad &:= \textstyle\int_{\mathbf{R}} \Big(\frac{1}{2}u_{4x}^2 - 9u^2u_{3x}^2 + 20uu_{xx}^3 + 51u_x^2u_{xx}^2 + 63u^4u_{xx}^2 - 133u^2u_x^4 \\ &-210u^6u_x^2 + 7u^{10}(t,x)dx = E_9[u](0). \end{aligned} \tag{1.18}$$

Using the lowest order conserved quantities (i.e., mass and energy (1.15)-(1.16)), the variational structure of any higher order mKdV soliton (1.9) can be characterized as follows: there exists a well-defined Lyapunov functional, invariant in time and such that any higher order mKdV soliton Q_c (1.9) is an extremal point. Moreover, it is a global minimizer under fixed mass. For the 5th, 7th and 9th-mKdV cases, this functional is given by (see [6] for the mKdV case)

$$\mathcal{H}_3[u](t) = E[u](t) + c\,M[u](t), \tag{1.19}$$

where $c > 0$ is the scaling of the solitary wave (1.9), and $M[u]$, $E[u]$ are given in (1.14) and (1.15). Indeed, it is easy to see that for any small perturbation $z(t) \in H^1(\mathbf{R})$,

$$\mathcal{H}_3[Q_c + z](t) = \mathcal{H}_3[Q_c] - \int_{\mathbf{R}} z(Q_c'' - cQ_c + 2Q_c^3) + O(\|z(t)\|_{H^1}^2). \tag{1.20}$$

The zero order term above is independent of time, and the first order term in z is zero from (1.10), which it implies the critical character of Q_c.

Note that by using higher order conservation laws (1.16), (1.17) and (1.18), and therefore higher order Lyapunov functionals, we are also able to characterize 5th, 7th and 9th-mKdV solitons (1.9) as extremal points of these higher order functionals. More precisely, for instance, in the 5th-mKdV case, and using the quantities $M[u]$, $E_5[u]$ given in (1.14) and (1.16), this functional is explicitly given, for any $c > 0$, by

$$\mathcal{H}_5[u](t) = E_5[u](t) - c^2\,M[u](t). \tag{1.21}$$

For the 7th-mKdV case, using the quantities $M[u]$, $E_7[u]$ given in (1.14) and (1.17), we get

$$\mathcal{H}_7[u](t) = E_7[u](t) + c^3\,M[u](t), \tag{1.22}$$

and finally for the 9th-mKdV case, using the quantities $M[u]$, $E_9[u]$ given in (1.14) and (1.18), we get

$$\mathcal{H}_9[u](t) = E_9[u](t) - c^4\, M[u](t). \tag{1.23}$$

In fact, it is easy to see that for any small $z(t) \in H^2(\mathbf{R})$ (and $H^3(\mathbf{R}), H^4(\mathbf{R})$ respectively),

$$\mathcal{H}_5[Q_c + z](t) = \mathcal{H}_5[Q_c] + \int_{\mathbf{R}} z\Big(Q_c^{(iv)} - c^2\, Q_c + f_5(Q_c)\Big) + O(\|z(t)\|_{H^2}^2), \tag{1.24}$$

$$\mathcal{H}_7[Q_c + z](t) = \mathcal{H}_7[Q_c] - \textstyle\int_{\mathbf{R}} z\Big(Q_c^{(vi)} - c^3 Q_c + f_7(Q_c)\Big) + O(\|z(t)\|_{H^3}^2),$$

and

$$\mathcal{H}_9[Q_c + z](t) = \mathcal{H}_9[Q_c] + \textstyle\int_{\mathbf{R}} z\Big(Q_c^{(viii)} - c^4 Q_c + f_9(Q_c)\Big) + O(\|z(t)\|_{H^4}^2).$$

In all cases, the zero order term is independent of time, and the first order term in z is zero from (1.11), (1.12) and (1.13). Finally, and from the functionals (1.21)-(1.23) above, we conjecture that the following Lyapunov functional (here we identify $E_3 \equiv E$)

$$\mathcal{H}_{2n+1}[u](t) = E_{2n+1}[u](t) + (-1)^{n+1} c^n\, M[u](t), \quad n \in \mathbf{N}^+, \tag{1.25}$$

generates the associated nonlinear ODE

$$Q_c^{(2n)} - c^n\, Q_c + f_{2n+1}(Q_c) = 0, \quad n \in \mathbf{N}^+, \tag{1.26}$$

satisfied by any soliton solution of the corresponding member of the focusing mKdV hierarchy (see [2]).

1.1. Breathers in 5th, 7th and 9th order mKdV equations

Beside these soliton solutions of 5th, 7th and 9th-mKdV equations (1.2)-(1.3)-(1.4), it is possible to find another big set of explicit and oscillatory solutions, known in the physical and mathematical literature as the breather solution, and which is a spatially localized, and periodic in time, up to translations, real function.

For the 5th, 7th and 9th-mKdV equations (1.2)-(1.3)-(1.4), the breather solution in the line can be obtained by using different methods (e.g. Inverse

Scattering, Hirota method. See [18, 19, 20] for further details). Particularly we use here a matching method to find these breather solutions, i.e. proposing a well known ansatz, with speeds as free parameters to be determined in order to define a solution. Note that the same procedure can be used to obtain periodic breather solutions of the 5th, 7th and 9th-mKdV equations.

Definition 1.1 (5th, 7th and 9th-mKdV breathers). *Let $\alpha, \beta > 0$ and $x_1, x_2 \in \mathbf{R}$. The real-valued breather solution of the 5th, 7th and 9th-mKdV equations (1.2)-(1.3)-(1.4) is given explicitly by the formula*

$$(1.27) B \equiv B_{\alpha,\beta}(t,x;x_1,x_2) := \partial_x \tilde{B}_\mu := 2\partial_x \left[\arctan\left(\frac{\beta}{\alpha} \frac{\sin(\alpha y_1)}{\cosh(\beta y_2)} \right) \right],$$

with y_1 and y_2

$$(1.28) \qquad y_1 = x + \delta_i t + x_1, \quad y_2 = x + \gamma_i t + x_2, i = 5,7,9$$

and with velocities (δ_5, γ_5) in the 5th order case

$$(1.29) \qquad \delta_5 := -\alpha^4 + 10\alpha^2\beta^2 - 5\beta^4, \quad \gamma_5 := -\beta^4 + 10\alpha^2\beta^2 - 5\alpha^4,$$

(δ_7, γ_7) in the 7th order case

$$\delta_7 := \alpha^6 - 21\alpha^4\beta^2 + 35\alpha^2\beta^4 - 7\beta^6, \quad \gamma_7 := -\beta^6 + 21\alpha^2\beta^4 - 35\alpha^4\beta^2 + 7\alpha^6,$$
(1.30)

and (δ_9, γ_9) in the 9th order case

$$(1.31) \qquad \begin{aligned} \delta_9 &:= -\alpha^8 + 36\alpha^6\beta^2 - 126\alpha^4\beta^4 + 84\alpha^3\beta^6 - 9\beta^8, \\ \gamma_9 &:= -\beta^8 + 36\alpha^2\beta^6 - 126\alpha^4\beta^4 + 84\alpha^6\beta^2 - 9\alpha^8. \end{aligned}$$

Remark 1.1. *Observe that breather solutions for 5th, 7th and 9th order mKdV equations have the same functional expression as the classical mKdV breather solution [4, Def.1.1]*

$$(1.32) \qquad B \equiv B_{\alpha,\beta}(t,x;x_1,x_2) := 2\partial_x \left[\arctan\left(\frac{\beta}{\alpha} \frac{\sin(\alpha y_1)}{\cosh(\beta y_2)} \right) \right],$$

with $y_1 = x + \delta t + x_1, y_2 = x + \gamma t + x_2$, *and velocities* $\delta = \alpha^2 - 3\beta^2, \gamma = 3\alpha^2 - \beta^2$, *and in fact only differing in speeds (1.29)-(1.31).*

Remark 1.2. *Finally be aware that these 5th, 7th and 9th breather solutions (1.27) in* $\mathbf{R}$ *could be used to re-approach the ill-posedness of the Cauchy problem for 5th, 7th and 9th-mKdV equations (1.2)-(1.3) and (1.4), in the same way they were used by Kenig-Ponce and Vega [13] and Alejo [3], to show a failure of the flow map associated to some nonlinear dispersive equations to be uniformly continuous. This procedure could afford a complementary proof to the previous works on the ill-posedness of these higher order equations presented by Kwon [14] and Grünrock [12].*

One of the main results of this work will be to prove that, exactly as it happens with all 5th, 7th and 9th soliton solutions (1.9) which satisfy the same nonlinear elliptic equation (1.10), breather solutions (1.27) of the 5th, 7th and 9th mKdV equations satisfy the same nonlinear fourth order stationary elliptic equation. Namely

Theorem 1.2. *Any 5th, 7th or 9th mKdV breather B satisfies the same fourth order stationary elliptic equation than the classical mKdV breather, namely*

$$B_{4x}+10BB_x^2+10B^2B_{xx}+6B^5-2(\beta^2-\alpha^2)(B_{xx}+2B^3)+(\alpha^2+\beta^2)^2B=0.$$

This fact can be interpreted as if all mKdV breathers and higher order mKdV breathers are characterized by the same elliptic equation, in a similar way as it was showed for the KdV equation by Lax [16]. Moreover, and as second main result in this paper, we give a positive answer to the question of breathers stability for these higher order mKdV equations.

Theorem 1.3. *5th, 7th and 9th mKdV breathers are orbitally stable in the H^2-topology.*

A more detailed version of this result is given in Theorem 5.1. As we have already shown, we need the space H^2 by a regularity argument and through the variational characterization that we obtain of these breather solutions of higher order mKdV equations.

1.2. Organization of this paper

In Sect.2 we present some higher order nonlinear identities adapted to 5th, 7th and 9th-mKdV breathers. Furthermore, we prove that any 5th, 7th or 9th-mKdV breather solutions satisfy a fourth order nonlinear ODE, which characterizes them. Sect.3 is devoted to collect and list the properties of a linearized operator associated to these higher order breather solutions. In Sect.4 we introduce a suitable H^2-Lyapunov functional for higher order mKdV equations (1.2), (1.3) and (1.4). Finally, in Sect.5 we present a detailed version of Theorem 5.1.

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2. Higher order nonlinear identities

The aim of this section is to show explicit nonlinear identities satisfied by any 5th, 7th or 9th-mKdV breathers.

First of all, consider the two directions associated to spatial translations. Let $B_{\alpha,\beta}$ as introduced in (1.27). Then we define

$$\begin{aligned} B_1(t,x;x_1,x_2) &:= \partial_{x_1} B_{\alpha,\beta}(t,x;x_1,x_2) \text{ and} \\ B_2(t,x;x_1,x_2) &:= \partial_{x_2} B_{\alpha,\beta}(t,x;x_1,x_2). \end{aligned} \tag{2.1}$$

It is clear that, for all $t \in \mathbf{R}$, and α, β as in definition (1.27) and $x_1, x_2 \in \mathbf{R}$, both B_1 and B_2 are real-valued, exponentially decreasing in space, functions in the Schwartz class. Moreover, it is not difficult to see that they are linearly independent as functions of the x-variable, for all time t fixed. We also define the partial mass associated to any 5th, 7th or 9th-mKdV breather B (1.27) as ($G = \frac{\beta}{\alpha}\sin(\alpha y_1), F = \cosh(\beta y_2)$):

$$\begin{aligned} \mathcal{M}(t,x) \equiv \mathcal{M}_{\alpha,\beta}(t,x) &:= \tfrac{1}{2}\int_{-\infty}^{x} B^2(t,s;x_1,x_2)ds \\ &= \beta + \tfrac{1}{2}\partial_x \log(G^2+F^2)(t,x). \end{aligned} \tag{2.2}$$

In the last expression we have used that $B^2 = \frac{1}{2}\partial_x^2 \log(G^2+F^2)(t,x)$, just following [5, Lemma 2.1, Appendix A] in the case of a vanishing boundary condition. Finally, let consider $\tilde{B} = \tilde{B}_{\alpha,\beta}$ as the following L^∞-function associated to mKdV breathers:

$$\tilde{B}(t,x) := 2\arctan\left(\frac{\beta}{\alpha}\frac{\sin(\alpha y_1)}{\cosh(\beta y_2)}\right). \tag{2.3}$$

The following nonlinear identities are satisfied by 5th, 7th and 9th-mKdV breathers:

Lemma 2.1. *We have for all $t \in \mathbf{R}$, and $\alpha, \beta > 0$, the following identities. Let $B = B_{\alpha,\beta}$ be any 5th, 7th or 9th-mKdV breather solution of the form (1.27) as it corresponds. Then*

1. *For any fixed $t \in \mathbf{R}$, we have $\tilde{B}_t$ well-defined in the Schwartz class, satisfiying respectively for the 5th, 7th or 9th-mKdV equations that*

$$\tilde{B}_t + B_{2nx} + f_{2n+1}(B) = 0, \quad \textit{with} \quad n = 2, 3, 4. \tag{2.4}$$

2. *Let $\mathcal{M}$ be defined by (2.2). Then*

   1. *The 5th order case:*

$$B_{xx}^2 - 2B\tilde{B}_t + 2\mathcal{M}_t - 2B^6 - 2B_xB_{xxx} - 10B^2B_x^2 = 0. \tag{2.5}$$

   2. *The 7th order case:*

$$\begin{aligned} B_{3x}^2 \quad &+2B\tilde{B}_t - 2\mathcal{M}_t + 5B^8 + 2B_xB_{5x} - 2B_{xx}^2B_{4x} \\ &+28B^2B_xB_{3x} - 14B^2B_{xx}^2 + 56BB_x^2B_{xx} + 7B_x^4 + 70B^4B_x^2 = 0. \end{aligned} \tag{2.6}$$

   3. *The 9th order case:*

$$\begin{aligned} B_{4x}^2 \quad &-2B\tilde{B}_t + 2\mathcal{M}_t - 2B_{7x}B_x + 2B_{6x}B_{xx} - 2B_{5x}B_{3x} + F[B] = 0, \\ &F[B] := -2\int_{-\infty}^x f_9(B)(s)B_s ds. \end{aligned} \tag{2.7}$$

Proof. In the 5th case, the first item (2.4) is a consequence of (2.3) and a convenient integration in space (from $-\infty$ to x). To obtain (2.5) we multiply (2.4), when $n = 2$, by B_x and integrate in space in the same region. The proofs in the 7th and 9th order cases follow similar steps as in the 5th order case. □

We compute now the higher order energies (1.16), (1.17) and (1.18) of any higher order breather solution of (1.2), (1.3) and (1.4) equations.

Lemma 2.2. *Let $B = B_{\alpha,\beta}$ be any 5th, 7th or 9th order mKdV breather solutions respectively, for α, β as in definition (1.27). Then the higher order energies (1.16), (1.17) and (1.18) of a 5th, 7th and 9th-mKdV breather B are respectively*

$$(2.8)\quad E_5[B] := -\frac{2}{5}\beta\gamma_5, \quad E_7[B] := \frac{2}{7}\beta\gamma_7, \quad \text{and} \quad E_9[B] := -\frac{2}{9}\beta\gamma_9,$$

with $\gamma_5, \gamma_7, \gamma_9$ given in (1.29)-(1.30)-(1.31).

Remark 2.1. *Note that as it happens with the classical mKdV breather solution B, where $E[B] := \frac{2}{3}\beta\gamma$ (see [4, Lemma 2.4]), the sign of the higher order energies E_5, E_7, E_9 is driven by a nonlinear balance among the different terms depending on scalings α, β.*

Remark 2.2. *From the above Lemma, we conjecture that for any $(2n+1)$-order mKdV breather B, its $(2n+1)$-order energy is given by*

$$(2.9)\qquad E_{2n+1}[B](t) = (-1)^{n+1}\frac{2\beta}{2n+1}\gamma_{2n+1}, \quad n \in \mathbf{N},$$

and with

$$\gamma_{2n+1} := \sum_{j=0}^{n}(-1)^j\frac{(2n+1)!}{(2j)!(2n+1-2j)!}\alpha^{2j}\beta^{2(n-j)}, \quad n \in \mathbf{N}.$$

Proof. (of Lemma 2.2) We start with the 5th order case. First of all, let us prove the following reduction

$$(2.10)\qquad E_5[B](t) = -\frac{1}{5}\int_{\mathbf{R}}\mathcal{M}_t(t,x)dx.$$

Indeed, we multiply (2.4) by B and integrate in space: we get

$$\int_{\mathbf{R}} B_{xx}^2 = \int_{\mathbf{R}} 20B^2B_x^2 - 6B^6 - B\tilde{B}_t.$$

On the other hand, integrating (2.5),

$$\int_{\mathbf{R}} B_{xx}^2 = \frac{2}{3}\int_{\mathbf{R}} B^6 + \frac{2}{3}\int_{\mathbf{R}} B\tilde{B}_t - \frac{2}{3}\int_{\mathbf{R}}\mathcal{M}_t + \frac{10}{3}\int_{\mathbf{R}} B^2B_x^2.$$

From these two identities, we get

$$\int_{\mathbf{R}} B^6 = \frac{1}{10}\int_{\mathbf{R}}\mathcal{M}_t - \frac{1}{4}\int_{\mathbf{R}} B\tilde{B}_t + \frac{5}{2}\int_{\mathbf{R}} B^2B_x^2,$$

and therefore

$$\int_{\mathbf{R}} B_{xx}^2 = -\frac{3}{5}\int_{\mathbf{R}} \mathcal{M}_t + \frac{15}{3}\int_{\mathbf{R}} B^2B_x^2 + \frac{1}{2}\int_{\mathbf{R}} B\tilde{B}_t.$$

Finally, substituting the last two identities into (1.16), we get (2.10), as desired. Proceeding in the same way, in the 7th and 9th order cases we obtain the corresponding simplifications

$$E_7[B](t) = \frac{1}{7}\int_{\mathbf{R}} \mathcal{M}_t(t,x)dx, \quad E_9[B](t) = \frac{1}{9}\int_{\mathbf{R}} \mathcal{M}_t(t,x)dx. \tag{2.11}$$

Now we prove (2.8). From (2.2), we have that

$$\mathcal{M}_t(t,x) = \frac{1}{2}\partial_x\partial_t \log(G^2+F^2)(t,x).$$

Now substituting in the energy (2.10), remembering the identity (2.4) and the explicit expression for $\mathcal{M}[B]$ in (2.2), we get

$$\begin{aligned} E_5[B](t) &= -\tfrac{1}{5}\textstyle\int_{\mathbf{R}} \mathcal{M}_t(t,x)\ dx = -\tfrac{1}{5}\tfrac{1}{2}\int_{\mathbf{R}} \Big(\partial_x\partial_t \log(G^2+F^2)\Big)dx \\ &= -\Big(\tfrac{1}{5}\tfrac{1}{2}\partial_t \log(G^2+F^2)\Big)\Big|_{-\infty}^{+\infty} = -\tfrac{2}{5}\beta\gamma_5. \end{aligned}$$

For the 7th and 9th order cases, we proceed as above, but now using (2.4), (2.6) and (2.7), and we get

$$E_7[B] = \frac{2}{7}\beta\gamma_7, \quad \text{and} \quad E_9[B] = -\frac{2}{9}\beta\gamma_9.$$

□

Note that since the profiles of 5th, 7th and 9th order mKdV breathers (solitons) agree with the expression of the classical mKdV breather (soliton), and since the energy E (1.15) is a conserved quantity for the mKdV and 5th, 7th and 9th higher order equations, when the lowest energy E (1.15) is evaluated in these 5th, 7th and 9th higher order breathers we obtain in both cases the same value than the mKdV breather energy, $\frac{2}{3}\beta\gamma$. For the sake of simplicity and to understand that property, we remember here the relation [5, (4.2),(4.4)] in the case of low order conserved quantities evaluated at breather solutions B and at soliton solutions Q_c:

$$M[B] = 2Re\Big[M[Q_c]|_{\sqrt{c}=\beta+i\alpha}\Big] \text{ and } E[B] = 2Re\Big[E[Q_c]|_{\sqrt{c}=\beta+i\alpha}\Big]. \tag{2.12}$$

The next nontrivial identity for 5th-mKdV breathers (1.27) will be useful in the proof of the nonlinear stationary equation that they satisfy.

Lemma 2.3. *Let $B = B_{\alpha,\beta}$ be any 5th-mKdV breather (1.27). Then, for all $t \in \mathbf{R}$,*

$$\tilde{B}_t = (\alpha^2+\beta^2)^2 B - 2(\beta^2-\alpha^2)(B_{xx}+2B^3). \tag{2.13}$$

Proof. We will use the following notation:

$$B := 2\partial_x\Big[\arctan\Big(\frac{\beta}{\alpha}\frac{\sin(\alpha y_1)}{\cosh(\beta y_2)}\Big)\Big] = \frac{H(t,x)}{N(t,x)},$$

$$H := H(t,x) = 2\Big(\beta\alpha^2\cosh(\beta y_2)\cos(\alpha y_1) - \beta^2\alpha\sinh(\beta y_2)\sin(\alpha y_1)\Big),$$

$$N := N(t,x) = \alpha^2\cosh^2(\beta y_2) + \beta^2\sin^2(\alpha y_1),$$

and from $\tilde{B}$ (2,3)

$$\tilde{B}_t := 2\partial_t\Big[\arctan\Big(\frac{\beta}{\alpha}\frac{\sin(\alpha y_1)}{\cosh(\beta y_2)}\Big)\Big] = \frac{P(t,x)}{N(t,x)},$$

$$P := P(t,x) = 2\Big(\beta\alpha\delta_5\cosh(\beta y_2)\cos(\alpha y_1) - \beta\alpha\gamma_5\sinh(\beta y_2)\sin(\alpha y_1)\Big), \tag{2.14}$$

with δ_5, γ_5 as in (1.29). For the sake of simplicity, we are going to use the following notation:

$$N_1 := N_x = 2\alpha\beta^2\cos(\alpha y_1)\sin(\alpha y_1) + 2\alpha^2\beta\cosh(\beta y_2)\sinh(\beta y_2), \tag{2.15}$$

$$\begin{aligned} N_2 := N_{xx} = 2\alpha^2\beta^2(&\cos^2(\alpha y_1) - \sin^2(\alpha y_1) + \cosh^2(\beta y_2) \\ &+ \sinh^2(\beta y_2)), \end{aligned} \tag{2.16}$$

and

$$H_1 := H_x = -2\alpha\beta(\alpha^2+\beta^2)\cosh(\beta y_2)\sin(\alpha y_1), \tag{2.17}$$

$$\begin{aligned} H_2 := H_{xx} = -2\alpha\beta(\beta^2+\alpha^2)(&\alpha\cosh(\beta y_2)\cos(\alpha y_1) \\ &+\beta\sin(\alpha y_1)\sinh(\beta y_2)). \end{aligned} \tag{2.18}$$

First of all, we start rewriting the following terms of the l.h.s. of (2.13):

$$B_{xx} + 2B^3 = \frac{1}{N^3}\Big(2H^3 + H_2N^2 - 2H_1NN_1 + 2HN_1^2 - HNN_2\Big), \tag{2.19}$$

and hence, we have that

$$-\tilde{B}_t - 2(\beta^2-\alpha^2)(B_{xx}+2B^3) + (\alpha^2+\beta^2)^2 B = \tfrac{M_0}{N^3}, \tag{2.20}$$

with

$$\begin{aligned} M_0 &:= -PN^2 + (\alpha^2+\beta^2)^2HN^2 \\ &-2(\beta^2-\alpha^2)\Big(2H^3 - 2NH_1N_1 + 2HN_1^2 + N^2H_2 - HNN_2\Big). \end{aligned} \tag{2.21}$$

Indeed, we verify, after substituting P and $H's$ and $N's$ terms explicitly in M0 and having in mind basic trigonometric and hyperbolic identities, that

$$M_0 = 0, \tag{2.22}$$

and we conclude. $\square$

We are ready now to present one of the most important results of this work, namely, we are going to show that in fact, breather solutions (1.27) of 5th, 7th and 9th-mKdV equations satisfy the same fourth order ODE satisfied by the classical mKdV breather solution (2.23) and it characterizes them. This result means that this ODE identifies breather functions at different levels in the mKdV hierarchy, i.e. at the mKdV level and at 5th, 7th and 9th mKdV levels, as being solutions of the same stationary fourth order ODE.

Theorem 2.4. *Let $B = B_{\alpha,\beta}$ be any 5th, 7th or 9th-mKdV breather solution given in (1.27). Then, for any fixed $t \in \mathbf{R}$, B satisfies the same nonlinear stationary equation than the classical mKdV breather solution (2.23), namely*

$$\begin{aligned} G[B] &:= B_{4x} + 10BB_x^2 + 10B^2B_{xx} + 6B^5 - 2(\beta^2-\alpha^2)(B_{xx}+2B^3) \\ &+(\alpha^2+\beta^2)^2B = 0. \end{aligned} \tag{2.23}$$

Proof. [Proof of Proposition 2.4] In the case of the 5th order breather, since by (2.4) the first four terms in (2.23) equal $-\tilde{B}_t$ and using the above identity (2.13), we simply get

$$G[B] = -\tilde{B}_t - 2(\beta^2 - \alpha^2)(B_{xx} + 2B^3) + (\alpha^2 + \beta^2)^2 B = 0.$$

The 7th and 9th order cases are more involved since we do not have at hand any identity like (2.13). Therefore, we first recast the l.h.s. of (2.23). Taking into account the r.h.s. of (1.6), we rewrite the first four terms in (2.23) and simplify the l.h.s. of (2.23), as follows:

$$\begin{aligned}
&B_{4x} + 10BB_x^2 + 10B^2B_{xx} + 6B^5 - 2(\beta^2 - \alpha^2)(B_{xx} + 2B^3) + (\alpha^2 + \beta^2)^2 B \\
&= \partial_x^2(B_{xx} + 2B^3) - 2B(B_x^2 - 2BB_{xx} - 3B^4) - 2(\beta^2 \\
&-\alpha^2)(B_{xx} + 2B^3) + (\alpha^2 + \beta^2)^2 B \\
&= \partial_x^2(B_{xx} + 2B^3) - 2B([B_x^2 + B^4] - 2B[B_{xx} + 2B^3]) \\
&-2(\beta^2 - \alpha^2)(B_{xx} + 2B^3) + (\alpha^2 + \beta^2)^2 B \\
&= \partial_x^2(B_{xx} + 2B^3) + (4B^2 - 2(\beta^2 - \alpha^2))(B_{xx} + 2B^3) \\
&-2B[B_x^2 + B^4] + (\alpha^2 + \beta^2)^2 B.
\end{aligned} \tag{2.24}$$

Now, we prove directly that (2.24) vanishes. Having in mind notation (2.14) and (2.15)-(2.18), we extend it considering the following derivatives:

$$\begin{aligned}
N_3 &:= N_{xxx} = -8\alpha^3\beta^2 \cos(\alpha y_1)\sin(\alpha y_1) + 8\alpha^2\beta^3 \cosh(\beta y_2)\sinh(\beta y_2), \\
N_4 &:= N_{4x} = 8\alpha^2\beta^2(-\alpha^2\cos^2(\alpha y_1) + \alpha^2\sin^2(\alpha y_1) + \beta^2\cosh^2(\beta y_2) \\
&\quad +\beta^2\sinh^2(\beta y_2)),
\end{aligned}$$

and

$$\begin{aligned}
H_3 &:= H_{xxx} = 2\alpha\beta((\alpha^4 - \beta^4)\cosh(\beta y_2)\sin(\alpha y_1) \\
&\quad -2\alpha\beta(\alpha^2 + \beta^2)\cos(\alpha y_1)\sinh(\beta y_2)), \\
H_4 &:= H_{4x} = 2\alpha\beta((\alpha^5 - 2\alpha^3\beta^2 - 3\alpha\beta^4)\cosh(\beta y_2)\cos(\alpha y_1) \\
&\quad +(3\alpha^4\beta + 2\alpha^2\beta^3 - \beta^5)\sin(\alpha y_1)\sinh(\beta y_2)).
\end{aligned}$$

First of all, remembering from (2.19) that

$$B_{xx} + 2B^3 = \frac{1}{N^3}\Big(2H^3 + H_2N^2 - 2H_1NN_1 + 2HN_1^2 - HNN_2\Big), \tag{2.25}$$

we get

$$\begin{aligned}\partial_x^2(B_{xx}+2B^3) \quad &= \tfrac{1}{N^5}\Big(6H^2N(H_2N-6H_1N_1)+6H^3(4N_1^2-NN_2)\\ &+N(N(H_4N^2-4H_3NN_1+12H_2N_1^2-6H_2NN_2)\\ &-4H_1(6N_1^3-6NN_1N_2+N^2N_3))+H(24N_1^4-36NN_1^2N_2\\ &+2N^2(6H_1^2+3N_2^2+4N_1N_3)-N^3N_4)\Big).\end{aligned} \tag{2.26}$$

Hence, we have that

$$\partial_x^2(B_{xx}+2B^3)+(4B^2-2(\beta^2-\alpha^2))(B_{xx}+2B^3)=\tfrac{M_1}{N^5}, \tag{2.27}$$

with

$$\begin{aligned}&M_1 := \Big(8H^5+2H^2N(5H_2N-22H_1N_1)+2H^3(16N_1^2-5NN_2\\ &+2(\alpha^2-\beta^2)N^2)+H\Big[24N_1^4-36NN_1^2N_2+2N^2(6H_1^2+3N_2^2+4N_1N_3\\ &+2(\alpha^2-\beta^2)N_1^2)-N^3(N_4+2(\alpha^2-\beta^2)N_2)\Big]\\ &+N[-24H_1N_1^3+12NN_1(H_2N_1+2H_1N_2)+N^3(H_4+2(\alpha^2-\beta^2)H_2)\\ &-2N^2(2H_3N_1+3H_2N_2+2H_1(N_3+(\alpha^2-\beta^2)N_1))]\Big).\end{aligned} \tag{2.28}$$

Moreover, we have that

$$-2B[B_x^2+B^4]=\tfrac{-2H}{N^5}\Big(H^4+(H_1N-HN_1)^2\Big), \tag{2.29}$$

and therefore,

$$-2B[B_x^2+B^4]+(\alpha^2+\beta^2)^2B=\tfrac{M_2}{N^5}, \tag{2.30}$$

with

$$M_2 := \Big(H(-2(H^4+(H_1N-HN_1)^2)+(\alpha^2+\beta^2)^2N^4)\Big). \tag{2.31}$$

Hence, we get the following simplification of (2.24):

$$\begin{aligned}G[B] \quad &= \partial_x^2(B_{xx}+2B^3)+(4B^2-2(\beta^2-\alpha^2))(B_{xx}+2B^3)\\ &-2B[B_x^2+B^4]+(\alpha^2+\beta^2)^2B\\ &=\tfrac{M_1+M_2}{N^5},\end{aligned} \tag{2.32}$$

with M_1, M_2 in (2.28) and (2.31) respectively. In fact, we verify, using the symbolic software Mathematica, that after substituting $H's$ and $N's$ terms explicitly in (2.32) and lengthy rearrangements, we get

$$\begin{aligned}&M_1 + M_2 = \textstyle\sum_{i=1}^{3} p_{ij} \sin(\alpha y_1)^{2i} + \sum_{i=1}^{4} q_{ij} \sin(\alpha y_1)^{2i-1},\\ &p_{ij} = \textstyle\sum_{j=0}^{L_i} a_{ij} \cos(\alpha y_1) \cosh(\beta y_2)^{2j+1},\\ &q_{ij} = \textstyle\sum_{j=0}^{L_i^{'}} b_{ij} \sinh(\beta y_2) \cosh(\beta y_2)^{2j}, \quad L_i, L_i^{'} \in \mathbf{N}.\end{aligned} \tag{2.33}$$

It is easy to see that $a_{ij} = b_{ij} = 0, \forall i = 1, \ldots, 4, j = 0, \ldots, L_i, L_i^{'}$. Therefore we get that

$$M_1 + M_2 = 0, \tag{2.34}$$

and we conclude. □

A direct consequence from Theorem 2.4 and identity (2.4), implies that for the 7th and 9th order cases, we are able to obtain a new identity relating $\tilde{B}_t$ and lower order spatial derivatives of the 7th and 9th-mKdV breathers (see (2.4) for comparison):

Corollary 2.5. *Let $B = B_{\alpha,\beta}$ be any 7th or 9th-mKdV breather solutions (1.27) as it corresponds. Then, for any fixed $t \in \mathbf{R}$, the associated profile $\tilde{B}$ (2.3) to any 7th or 9th-mKdV breather satisfies the following nonlinear identities:*

1. *7th order case:*

$$\begin{aligned}&\tilde{B}_t - 2(\beta^2 - \alpha^2)(\alpha^2 + \beta^2)^2 B + 4(\alpha^4 - 6\alpha^2\beta^2 + \beta^4)B^3\\ &+4(\beta^2 - \alpha^2)B^5 - 4B^7 + (3\alpha^4 - 10\alpha^2\beta^2 + 3\beta^4)B_{xx} + 4(\beta^2 - \alpha^2)BB_x^2\\ &-20B^3B_x^2 + 2BB_{xx}^2 - 4BB_xB_{3x} = 0.\end{aligned} \tag{2.35}$$

2. *9th order case:*

$$\begin{aligned}&\tilde{B}_t + a_0B + a_1B^3 + a_2B^5 + 16\left(\beta^2 - \alpha^2\right)B^7 - 26B^9 + a_3B_x^2B\\ &+32\left(\alpha^2 - \beta^2\right)B_x^2B^3 - 100B_x^2B^5 - 2B_x^4B + a_4B_{xx} - 6\left(\alpha^2 + \beta^2\right)^2 B_{xx}B^2\\ &+20\left(\beta^2 - \alpha^2\right)B_{xx}B^4 - 28B_{xx}B^6 + 4\left(\beta^2 - \alpha^2\right)B_x^2B_{xx} - 12B_x^2B_{xx}B^2\\ &+8\left(\beta^2 - \alpha^2\right)B_{xx}^2B - 4B_{xx}^2B^3 + 2B_{xx}^3 + 8\left(\alpha^2 - \beta^2\right)B_xB_{3x}B\\ &-32B_xB_{3x}B^3 - 4B_xB_{xx}B_{3x} - 2B_{3x}^2B = 0,\end{aligned} \tag{2.36}$$

for

$$\begin{aligned}
a_0 &= -\left(\alpha^2+\beta^2\right)^2\left(3\alpha^4-10\alpha^2\beta^2+3\beta^4\right),\\
a_1 &= -4(\alpha^2-\beta^2)(\alpha^4-14\alpha^2\beta^2+\beta^4),\\
a_2 &= -2\left(\alpha^4+18\alpha^2\beta^2+\beta^4\right),\\
a_3 &= 2\left(5\alpha^4-6\alpha^2\beta^2+5\beta^4\right),\\
a_4 &= -4(\alpha^2-\beta^2)(\alpha^4-6\alpha^2\beta^2+\beta^4).
\end{aligned}$$

Proof. For both 7th and 9th order cases, using B_{4x} in (2.23), and computing from it the expressions of B_{6x}, B_{8x}, and substituting recursively B_{4x}, we get (2.35) and (2.36). □

3. Spectral analysis

For any 5th, 7th or 9th-mKdV breather solution $B = B_{\alpha,\beta}$, we define the following fourth order linear operator:

$$\begin{aligned}
\mathcal{L}[z](x;t) &:= z_{(4x)}(x) - 2(\beta^2-\alpha^2)z_{xx}(x) + (\alpha^2+\beta^2)^2 z(x)\\
&\quad +10B^2 z_{xx}(x) + 20BB_x z_x(x)\\
&\quad + \Big[10B_x^2 + 20BB_{xx} + 30B^4 - 12(\beta^2-\alpha^2)B^2\Big] z(x).
\end{aligned} \tag{3.1}$$

As a direct consequence of the already studied spectral properties of the linearized operator $\mathcal{L}[z]$ associated to the mKdV breather solution B in [4] and after a proper rescaling of B, we obtain the same results for the 5th, 7th or 9th-mKdV breather solutions. In the following lines and for the sake of completeness, we only summarize and list the main features of (3.1): consider first the functions B_1, B_2 (2.1) associated to 5th, 7th and 9th-mKdV breather solutions B (as it corresponds) and denote as scaling directions, the derivatives

$$\Lambda_\alpha B = \partial_\alpha B, \quad \Lambda_\beta B = \partial_\beta B. \tag{3.2}$$

We get the following

Lemma 3.1. *For any 5th, 7th or 9th-mKdV breather solution $B = B_{\alpha,\beta}$, we get that*

1. *(Continuous spectrum) $\mathcal{L}$ is a linear, unbounded operator in $L^2(\mathbf{R})$, with dense domain $H^4(\mathbf{R})$. Moreover, $\mathcal{L}$ is self-adjoint, and is a compact perturbation of the constant coefficients operator*

$$\mathcal{L}_0[z] := z_{(4x)} - 2(\beta^2-\alpha^2)z_{xx} + (\alpha^2+\beta^2)^2 z.$$

In particular, the continuous spectrum of $\mathcal{L}$ is the closed interval $[(\alpha^2+\beta^2)^2,+\infty)$ in the case $\beta \geq \alpha$, and $[4\alpha^2\beta^2,+\infty)$ in the case $\beta < \alpha$, with no embedded eigenvalues are contained in this region.

2. *(Kernel) For each $t \in \mathbf{R}$, one has*

$$\ker \mathcal{L} = span\Big\{B_1(t;x_1,x_2), B_2(t;x_1,x_2)\Big\}.$$

3. *Consider the scaling directions $\Lambda_\alpha B$ and $\Lambda_\beta B$ introduced in (3.2). Then*

$$\int_{\mathbf{R}} \Lambda_\alpha B\, \mathcal{L}[\Lambda_\alpha B] = 16\alpha^2\beta > 0, \tag{3.3}$$

and

$$\int_{\mathbf{R}} \Lambda_\beta B\, \mathcal{L}[\Lambda_\beta B] = -16\alpha^2\beta < 0. \tag{3.4}$$

4. *Let*

$$B_0 := \frac{\alpha\Lambda_\beta B + \beta\Lambda_\alpha B}{8\alpha\beta(\alpha^2+\beta^2)}. \tag{3.5}$$

Then B_0 is Schwartz and satisfies $\mathcal{L}[B_0] = -B$,

$$\int_{\mathbf{R}} B_0 B = \frac{1}{4\beta(\alpha^2+\beta^2)} > 0, \quad \text{and} \quad \frac{1}{2}\int_{\mathbf{R}} B_0\mathcal{L}[B_0] = -\frac{1}{8\beta(\alpha^2+\beta^2)} < 0. \tag{3.6}$$

5. *Let B_1, B_2 the kernel elements defined in (3.1) and W the Wronskian matrix of the functions B_1 and B_2,*

$$W[B_1,B_2](t;x) := \begin{bmatrix} B_1 & B_2 \\ (B_1)_x & (B_2)_x \end{bmatrix}(t,x). \tag{3.7}$$

Then

$$\det W[B_1,B_2](t;x) = -\frac{8\alpha^3\beta^3(\alpha^2+\beta^2)[\alpha\sinh(2\beta y_2) - \beta\sin(2\alpha y_1)]}{(\alpha^2+\beta^2+\alpha^2\cosh(2\beta y_2) - \beta^2\cos(2\alpha y_1))^2}. \tag{3.8}$$

6. *The operator $\mathcal{L}$ defined in (3.1) (associated with 5th, 7th and 9th mKdV equations) has a unique negative eigenvalue $-\lambda_0^2 < 0$, of multiplicity one, and $\lambda_0 = \lambda_0(\alpha,\beta,x_1,x_2,t)$.*

7. *(Coercivity) Let us consider the quadratic from associated to $\mathcal{L}$ (3.1):*

$$\begin{aligned}\mathcal{Q}[z] \quad &:= \textstyle\int_{\mathbf{R}} z\mathcal{L}[z] = \int_{\mathbf{R}} z_{xx}^2 + 2(\beta^2-\alpha^2)\int_{\mathbf{R}} z_x^2 + (\alpha^2+\beta^2)^2\int_{\mathbf{R}} z^2\\ &\quad -10\textstyle\int_{\mathbf{R}} B^2 z_x^2 - 10\int_{\mathbf{R}} B_x^2 z^2 - 40\int_{\mathbf{R}} BB_x z z_x\\ &\quad +30\textstyle\int_{\mathbf{R}} B^4 z^2 - 12(\beta^2-\alpha^2)\int_{\mathbf{R}} B^2 z^2. \end{aligned} \tag{3.9}$$

There exists a continuous function $\nu_0 = \nu_0(\alpha,\beta)$, well-defined and positive for all $\alpha,\beta > 0$ and such that, for all $z_0 \in H^2(\mathbf{R})$ satisfying (here B_{-1} is the eigenfunction associated to the negative eigenvalue)

$$\int_{\mathbf{R}} z_0 B_{-1} = \int_{\mathbf{R}} z_0 B_1 = \int_{\mathbf{R}} z_0 B_2 = 0, \tag{3.10}$$

then

$$\mathcal{Q}[z_0] \geq \nu_0 \|z_0\|_{H^2(\mathbf{R})}^2. \tag{3.11}$$

For the proof of this Lemma, we refer the interested reader to [4, Sect.4].

4. Variational characterization of higher order mKdV breathers

In this section we define a H^2-Lyapunov functional for both 5th, 7th and 9th-mKdV equations (1.2), (1.3) and (1.4) and associated to any of the higher order breather solutions. This approach is completely similar to the one depicted in [4] for the classical mKdV breather solution.

Let $B = B_{\alpha,\beta}$ be any 5th, 7th or 9th-mKdV breather solution and $t \in \mathbf{R}$. Using a linear combination of the functionals $E_5[u]$, $E[u]$ and $M[u]$ given in (1.16), (1.15) and (1.14), we define

$$\mathcal{H}[u(t)] := E_5[u](t) + 2(\beta^2-\alpha^2)E[u](t) + (\alpha^2+\beta^2)^2 M[u](t). \tag{4.1}$$

Therefore, $\mathcal{H}[u]$ is a real-valued conserved quantity, well-defined for H^2-solutions of (1.2), (1.3) and (1.4).

Moreover, one has the following:

Lemma 4.1. *5th, 7th and 9th-mKdV breathers (1.27) are critical points of the Lyapunov functional $\mathcal{H}$ (4.1). In fact, for any $z \in H^2(\mathbf{R})$ with sufficiently small H^2-norm, and $B = B_{\alpha,\beta}$ any 5th, 7th and 9th-mKdV breather solutions, then, for all $t \in \mathbf{R}$, one has*

$$\mathcal{H}[B+z] - \mathcal{H}[B] = \frac{1}{2}\mathcal{Q}[z] + \mathcal{N}[z], \tag{4.2}$$

with

$\mathcal{Q}$ *being the quadratic form defined in (3.7), and* $\mathcal{N}[z]$ *satisfying* $|\mathcal{N}[z]| \leq K\|z\|^3_{H^2(\mathbf{R})}$.

Proof. Considering any 5th, 7th or 9th-mKdV breather B, we compute

$$\begin{aligned}
\mathcal{H}[B+z] &= \tfrac{1}{2}\int_{\mathbf{R}}(B_{xx}+z_{xx})^2 - 5\int_{\mathbf{R}}(B+z)^2(B_x+z_x)^2 + \int_{\mathbf{R}}(B+z)^6 \\
&+(\beta^2-\alpha^2)\int_{\mathbf{R}}(B_x+z_x)^2 - (\beta^2-\alpha^2)\int_{\mathbf{R}}(B+z)^4 \\
&+\tfrac{1}{2}(\alpha^2+\beta^2)^2\int_{\mathbf{R}}(B+z)^2 \\
&= \tfrac{1}{2}\int_{\mathbf{R}} B_{xx}^2 - 5\int_{\mathbf{R}} B^2B_x^2 + \int_{\mathbf{R}} B^6 + (\beta^2-\alpha^2)\int_{\mathbf{R}} B_x^2 \\
&-(\beta^2-\alpha^2)\int_{\mathbf{R}} B^4 + \tfrac{1}{2}(\alpha^2+\beta^2)^2\int_{\mathbf{R}} B^2 + \int_{\mathbf{R}} z\Big[B_{4x} + 10BB_x^2 \\
&+10B^2B_{xx} + 6B^5 - 2(\beta^2-\alpha^2)(B_{xx}+2B^3) + (\alpha^2+\beta^2)^2B\Big] \\
&+\tfrac{1}{2}\Big[\int_{\mathbf{R}} z_{xx}^2 + 2(\beta^2-\alpha^2)\int_{\mathbf{R}} z_x^2 + (\alpha^2+\beta^2)^2\int_{\mathbf{R}} z^2 + 10\int_{\mathbf{R}} B^2z_{xx}z \\
&-20\int_{\mathbf{R}} BB_xz_xz + \int_{\mathbf{R}}(30B^4 - 10B_x^2 - 12(\beta^2-\alpha^2)B^2)z^2\Big] \\
&-5\int_{\mathbf{R}}(z^2z_x^2 + 2B_xz^2z_x + 2Bzz_x^2) + \int_{\mathbf{R}} 20B^3z^3 + 15B^2z^4 \\
&+6\int_{\mathbf{R}} Bz^5 + \int_{\mathbf{R}} z^6 - 4(\beta^2-\alpha^2)\int_{\mathbf{R}} Bz^3 - (\beta^2-\alpha^2)\int_{\mathbf{R}} z^4.
\end{aligned}$$

We finally obtain:

$$\mathcal{H}[B+z] = \mathcal{H}[B] + \int_{\mathbf{R}} G[B]z(t)\ dx + \frac{1}{2}\mathcal{Q}[z] + \mathcal{N}[z],$$

where $\mathcal{Q}$ is defined in (3.7) and

$$G[B] := B_{4x} + 10BB_x^2 + 10B^2B_{xx} + 6B^5 - 2(\beta^2-\alpha^2)(B_{xx}+2B^3) + (\alpha^2+\beta^2)^2B.$$

From Theorem (2.4), one has $G[B]\equiv 0$. Finally, the term $\mathcal{N}[z]$ is given by

$$\begin{aligned}\mathcal{N}[z] \quad &:= -10\textstyle\int_{\mathbf{R}} Bzz_x^2 + \int_{\mathbf{R}}[\frac{10}{3}B_{xx}z^3 - 5z^2z_x^2 + 20B^3z^3 + 15B^2z^4 + 6Bz^5 + z^6]\\ &\quad -4(\beta^2-\alpha^2)\textstyle\int_{\mathbf{R}} Bz^3 - (\beta^2-\alpha^2)\int_{\mathbf{R}} z^4. \qquad (4.3)\end{aligned}$$

Therefore, from direct estimates one has $|\mathcal{N}[z]| \leq \mathcal{O}(\|z\|^3_{H^2(\mathbf{R})})$ as desired. $\square$

Using the previous Lemma, we are able to prove the main result of the paper.

5. Main Theorem

Theorem 5.1 (H^2-stability of 5th, 7th and 9th order mKdV breathers). *Let $\alpha,\beta\in\mathbf{R}\backslash\{0\}$ and $B=B_{\alpha,\beta}$ any 5th, 7th or 9th order mKdV breather. There exist positive parameters η_0, A_0, depending on α and β, such that the following holds. Consider $u_0\in H^2(\mathbf{R})$, and assume that there exists $\eta\in(0,\eta_0)$ such that*

$$\|u_0 - B(t=0;0,0)\|_{H^2(\mathbf{R})} \leq \eta. \qquad (5.1)$$

Then there exist $x_1(t), x_2(t)\in\mathbf{R}$ such that the solution $u(t)$ of the Cauchy problem for the 5th (1.2), 7th (1.3) or for the 9th (1.4) equations, with initial data $u_0\in H^2(\mathbf{R})$, satisfies

$$\sup_{t\in\mathbf{R}} \Big\| u(t) - B(t;x_1(t),x_2(t)) \Big\|_{H^2(\mathbf{R})} \leq A_0\eta, \qquad (5.2)$$

with

$$\sup_{t\in\mathbf{R}} |x_1'(t)| + |x_2'(t)| \leq KA_0\eta, \qquad (5.3)$$

for a constant $K>0$.

Remark 5.1. *Note that the same result is true for the negative breather $-B_{\alpha,\beta}$ which is also a solution of (1.2) or (1.3).*

Proof. [Proof of Theorem 5.1] We take $u = u(t) \in H^2(\mathbf{R})$ as the corresponding local in time solution of the Cauchy problem associated to (1.2), (1.3) or (1.4), with initial condition $u(0) = u_0 \in H^2(\mathbf{R})$ (cf. [17], [14], [12]).

Therefore once we guaranteed for the case of 5th, 7th and 9th-mKdV breathers, that they satisfy the same 4th order stationary ODE (2.23) as the classical mKdV breather, that a suitable coercivity property holds for the bilinear form $\mathcal{Q}$ associated to any of these higher order breathers (see (3.9), and the existence of a unique negative eigenvalue of the linearized operator $\mathcal{L}$ (3.1) associated again to these higher order breathers, the stability proof follows the same steps as the H^2-stability of classical mKdV breathers [4, Theorem 6.1]. Namely, we proceed assuming that the maximal time of stability T is finite and we arrive to a contradiction. $\square$

A. 11th-mKdV equation

For the sake of completeness, we show the 11th order mKdV equation. It is written as follows:

$$\begin{aligned}
u_t \quad &+\partial_x\Big(u_{10x} + 22u^2u_{8x} + 198u^4u_{6x} + 924u^6u_{4x} + 506u\,(u_{4x})^2 + 3036u^3\,(u_{3x})^2 \\
&+2310u^8u_{xx} + 8316u^5\,(u_{xx})^2 + 9372u^2\,(u_{xx})^3 + 9240u^7\,(u_x)^2 + 26796u^3\,(u_x)^4 \\
&+176uu_xu_{7x} + 484uu_{xx}u_{6x} + 462\,(u_x)^2\,u_{6x} + 836uu_{3x}u_{5x} + 2376u^3u_xu_{5x} \\
&+5016u^3u_{xx}u_{4x} + 2706\,(u_{xx})^2\,u_{4x} + 11220u^2\,(u_x)^2\,u_{4x} + 3498u_{xx}\,(u_{3x})^2 \\
&+11088u^5u_xu_{3x} + 21120u\,(u_x)^3\,u_{3x} + 54516u^4\,(u_x)^2\,u_{xx} \\
&+44748u\,(u_x)^2\,(u_{xx})^2 + 13398\,(u_x)^4\,u_{xx} + 2376u_xu_{xx}u_{5x} + 3696u_xu_{3x}u_{4x} \\
&+39336u^2u_xu_{xx}u_{3x} + 252u^{11}\Big) = 0.
\end{aligned} \tag{A.1}$$

Moreover, we are able to obtain the 11th order mKdV breather solution, in the same way we used to get (1.27):

Definition A.1 (11th-mKdV breather). *Let $\alpha, \beta > 0$ and $x_1, x_2 \in \mathbf{R}$. The real-valued breather solution of the 11th-mKdV equation (A1) is given explicitly by the formula*

$$B \equiv B_{\alpha,\beta}(t,x;x_1,x_2) := 2\partial_x\left[\arctan\left(\frac{\beta}{\alpha}\frac{\sin(\alpha y_1)}{\cosh(\beta y_2)}\right)\right], \tag{A.2}$$

with y_1 *and* y_2

$$y_1 = x + \delta_{11}t + x_1, \quad y_2 = x + \gamma_{11}t + x_2, \tag{A.3}$$

and with velocities

$$\begin{aligned} \delta_{11} &= \alpha^{10} - 55\alpha^8\beta^2 + 330\alpha^6\beta^4 - 462\alpha^4\beta^6 + 165\alpha^2\beta^8 - 11\beta^{10}, \\ \gamma_{11} &= 11\alpha^{10} - 165\alpha^8\beta^2 + 462\alpha^6\beta^4 - 330\alpha^4\beta^6 + 55\alpha^2\beta^8 - \beta^{10}. \end{aligned} \tag{A.4}$$

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