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Projective non-commuting graph of a group

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Abstract

Let G be a finite non-abelian group and let T be a transversal of the center of G in G.

The non-commuting graph of G on a transversal of the center is the graph whose vertices are the non-central elements of T and two vertices x and y are joined by an edge whenever $xy \neq yx$. In this paper, we classify the groups whose non-commuting graph on a transversal of the center is projective.

Keywords: *Non-commuting graph, projective graph, finite group.*

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1. Introduction

In this paper, we consider finite undirected graphs without loops or multiple edges. Given a graph $\mathcal{G}$, we write $V(\mathcal{G})$ and $E(\mathcal{G})$ to indicate, respectively, the vertex set and the edge set of $\mathcal{G}$. If V' is a subset of $V(\mathcal{G})$, the subgraph of $\mathcal{G}$ induced by V' is the graph whose vertex set is V' and the edge set is $\{\{x, y\} : x, y \in V' \, and \, \{x, y\} \in E(\mathcal{G})\}$.

Let G be a finite non-abelian group. The non-commuting graph of G is the graph denoted by $\nabla(G)$ and defined as follows: $V(\nabla(G)) = G \setminus Z(G)$, where $Z(G)$ is the center of G, and $\{x, y\} \in E(\nabla(G))$ if and only if $xy \neq yx$. Studies on the non-commuting graph $\nabla(G)$ can be seen in the papers [1, 2, 3, 4, 6, 7, 8]. As noted in [6, p. 911], if T is a transversal of the center $Z(G)$ in G, then the adjacency relations in $\nabla(G)$ can be obtained from adjacency relations between elements of $T \setminus Z(G)$, because two vertices x and y are adjacent in $\nabla(G)$ if and only if there are adjacent vertices x' and y' in $\nabla(G)$, with $\{x', y'\} \subset T$, such that $x \in x'Z(G)$ and $y \in y'Z(G)$. Here, the subgraph of $\nabla(G)$ induced by $T \setminus Z(G)$ is called non-commuting graph of G on a transversal of the center and denoted by $\mathbf{T}(G)$. So, $V(\mathbf{T}(G)) = T \setminus Z(G)$, with $|V(\mathbf{T}(G))| \geq 3$, and $E(\mathbf{T}(G)) = \{\{x, y\} : x, y \in T \setminus Z(G) \, and \, xy \neq yx\}$. Further, we note that if T' is another transversal of $Z(G)$ in G, then the non-commuting graphs obtained from T and T' are isomorphic.

In the study of graph $\nabla(G)$ in [3, 4, 6, 7, 8], properties of graph $\mathbf{T}(G)$ were highlighted. Results on $\mathbf{T}(G)$ can be seen in [6, 9, 10]. In [6, 8], the graph $\mathbf{T}(G)$ was called the underlying graph associated with $\nabla(G)$ and denoted by $\nabla^u(G)$. In [11], we also see results about the complemented graph of $\mathbf{T}(G)$.

We recall that a graph is said planar if it can be drawn in the plane in such a way that no two edges intersect except at a vertex which both are incident. Given a positive integer k, let N_k be a surface formed by a connected sum of k projective planes. The smallest positive integer k such that a graph $\mathcal{G}$ can be embedded in N_k is called crosscap of $\mathcal{G}$. A planar graph is considered with crosscap 0. A graph with crosscap 1 is a projective graph and, in this case, the graph is non-planar and it can be embedded in the projective plane. For details on the concept of embedding of graphs in surface, see [5].

In [10], we see a classification of the groups with a planar non-commuting graph on a transversal of the center. In this paper, we determine the structure of a finite group G in the case where $\mathbf{T}(G)$ is projective. We prove the following result.

Theorem 1.1. *Let G be a finite non-abelian group. The non-commuting graph $\mathbf{T}(G)$ is projective if and only if either $G/Z(G)$ is isomorphic to the dihedral group of order 8 or $G/Z(G)$ is isomorphic to $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$ and the derived subgroup of G is isomorphic to $\mathbf{Z}_2 \times \mathbf{Z}_2$.*

We observe that there is no finite non-abelian group G such that $\nabla(G)$ is projective (see [2, Theorem 3.3]). In Example 2.5, we see some groups with a projective non-commuting graph on a transversal of the center.

2. Proof of Theorem 1.1

In this section, we prove the main result of this work. We start with some concepts and notation.

Let G be a finite group. Given $x, y \in G$, the commutator $[x, y]$ of x and y is given by $[x, y] = xyx^{-1}y^{-1}$ and the derived subgroup of G is denoted by G'. The order of x is indicated by $o(x)$ and the centralizer of x in G is denoted by $C_G(x)$. The dihedral group of order m (with $m \geq 3$) is denoted by D_{2m} and the cyclic group of order n ($n \geq 1$) is indicated by $\mathbf{Z}_n$.

Let $\mathcal{G}$ be a graph. The degree of a vertex u of $\mathcal{G}$ is indicated by $\deg(u)$. The complete graph on n vertices is denoted by K_n and the graph without edges on n vertices is indicated by $\overline{K_n}$. The complete multipartite graph with m partite sets of sizes $n_1, n_2, \ldots, n_m$, with $1 \leq n_1 \leq n_2 \leq \ldots \leq n_m$, is denoted by $K_{n_1,n_2,\ldots,n_m}$. Given graphs $\mathcal{G}_1$ and $\mathcal{G}_2$, with $V(\mathcal{G}_1) \cap V(\mathcal{G}_2) = \emptyset$, we write $\mathcal{G}_1 \vee \mathcal{G}_2$ to indicate the graph whose vertex set is $V(\mathcal{G}_1) \cup V(\mathcal{G}_2)$ and the edge set is $E(\mathcal{G}_1) \cup E(\mathcal{G}_2) \cup \{\{x, y\} : x \in V(\mathcal{G}_1)\, and\, y \in V(\mathcal{G}_2)\}$. For example, we note that $K_{1,1,1,1,3}$ is isomorphic to $\overline{K_3} \vee K_4$.

To prove Theorem 1.1, we need the following four lemmas.

Lemma 2.1. *Let $\mathcal{G}$ be a finite graph with $|V(\mathcal{G})| \geq 3$.*

(i) *If $\mathcal{G}$ is planar, then $|E(\mathcal{G})| \leq 3|V(\mathcal{G})| - 6$.*

(ii) *If $\mathcal{G}$ is a projective connected graph, then $|E(\mathcal{G})| \leq 3|V(\mathcal{G})| - 3$.*

Proof. Part (i) is [5, Corollary 10.21]. Now, by [5, Corollary 10.39], we get that if $\mathcal{G}$ is a connected graph which is embeddable on a surface S, then $|E(\mathcal{G})| \leq 3(|V(\mathcal{G})| - c(S))$, where $c(S)$ is the Euler characteristic of S, and we know that if S is the projective plane, then $c(S) = 1$ (see [5, p. 279]). This proves statement (ii). □

Lemma 2.2. *Let G be a finite non-abelian group.*

(i) *The non-commuting graph $\mathbf{T}(G)$ is connected.*

(ii) $\deg(x) = [G : Z(G)] - [C_G(x) : Z(G)]$*, for any $x \in V(\mathbf{T}(G))$.*

(iii) $4|E(\mathbf{T}(G))| \geq |V(\mathbf{T}(G))|^2 + |V(\mathbf{T}(G))|$.

Proof. Part (i) is [7, Lemma 5.3]. To prove (ii), consider $x \in V(\mathbf{T}(G))$ and write $N(x) = \{u \in V(\mathbf{T}(G)) : \{x, u\} \in E(\mathbf{T}(G))\}$. Thus, $\deg(x) = |N(x)|$. Further, if $V(\mathbf{T}(G)) \setminus N(x) = \{x, x_1, \ldots, x_m\}$, then

$$C_G(x) = Z(G) \cup xZ(G) \cup \left(\bigcup_{i=1}^{m} x_i Z(G) \right).$$

So, $|\{x, x_1, \ldots, x_m\}| = [C_G(x) : Z(G)] - 1$. Since $|V(\mathbf{T}(G))| = [G : Z(G)] - 1$, we have

$$\deg(x) = |N(x)| = |V(\mathbf{T}(G))| - |\{x, x_1, \ldots, x_m\}| = [G : Z(G)] - [C_G(x) : Z(G)].$$

Let us prove (iii). Write $\nu = |V(\mathbf{T}(G))|$ and $\epsilon = |E(\mathbf{T}(G))|$. By [5, Theorem 1.1], we know that

$$2\epsilon = \sum_{x \in V(\mathbf{T}(G))} \deg(x) \tag{2.1}$$

Given $x \in V(\mathbf{T}(G))$, we observe that $[C_G(x) : Z(G)] \leq [G : Z(G)]/2$ and thus $\deg(x) = [G : Z(G)] - [C_G(x) : Z(G)] \geq [G : Z(G)] - [G : Z(G)]/2 = [G : Z(G)]/2$. Since $\nu + 1 = [G : Z(G)]$, we obtain

$$\deg(x) \geq \frac{1}{2}(\nu + 1) \tag{2.2}$$

Using (2.1) and (2.2) we get

$$2\epsilon = \sum_{x \in V(\mathbf{T}(G))} \deg(x) \geq \frac{1}{2}\nu(\nu + 1)$$

and, consequently, $4\epsilon \geq \nu^2 + \nu$. □

Lemma 2.3. *Let G be a finite non-abelian group such that $[G : Z(G)] = 2m$, where $m \geq 2$.*

(i) *The non-commuting graph $\mathbf{T}(G)$ is isomorphic to $\overline{K_{m-1}} \vee K_m$ if and only if G has an abelian normal subgroup A such that $[G : A] = 2$.*

(ii) *If $G/Z(G)$ has a cyclic subgroup of order m, then $\mathbf{T}(G)$ is isomorphic to $\overline{K_{m-1}} \vee K_m$.*

(iii) *If $\mathbf{T}(G)$ is isomorphic to $\overline{K_{m-1}} \vee K_m$ and x is a vertex of $\mathbf{T}(G)$ such that $\deg(x) = 2m - 2$, then $o(xZ(G)) = 2$.*

Proof. Let G be a finite non-abelian group such that $[G : Z(G)] = 2m \geq 4$.

(i) Suppose that $\mathbf{T}(G)$ is isomorphic to $\overline{K_{m-1}} \vee K_m$. If $x_1, x_2, \ldots, x_{m-1}$ are the vertices of $\mathbf{T}(G)$ such that the subgraph induced by $\{x_1, \ldots, , x_{m-1}\}$ is isomorphic to $\overline{K_{m-1}}$, then it is not difficult to prove that the subgroup of G generated by $Z(G) \cup \{x_1, \ldots, x_{m-1}\}$ is an abelian normal subgroup of G of index 2.

Conversely, suppose that G has an abelian normal subgroup A such that $[G : A] = 2$. So, the subgraph of $\mathbf{T}(G)$ induced by $V(\mathbf{T}(G)) \cap A$ is isomorphic to $\overline{K_{m-1}}$. Given $x \in V(\mathbf{T}(G)) \setminus A$, arguing as in the third paragraph of the proof of [10, Lemma 3.12], we can verify that x is adjacent to all other vertices of $\mathbf{T}(G)$. Therefore, $\mathbf{T}(G)$ is isomorphic to $\overline{K_{m-1}} \vee K_m$.

(ii) If $A/Z(G)$ is a cyclic subgroup of order m of $G/Z(G)$, then A is an abelian normal subgroup of G and $[G : A] = 2$. By part (i), we obtain that $\mathbf{T}(G)$ is isomorphic to $\overline{K_{m-1}} \vee K_m$.

(iii) Take $x \in V(\mathbf{T}(G))$ such that $\deg(x) = 2m-2$, that is, x is adjacent to all vertices of $V(\mathbf{T}(G)) \setminus \{x\}$. If $o(xZ(G)) > 2$, then $x^2 \notin Z(G)$ and, consequently, there is $y \in V(\mathbf{T}(G)) \cap x^2 Z(G)$ such that $y \neq x$ and $[x, y] = 1$, that is, y is not adjacent to x, a contradiction. Hence, $o(xZ(G)) = 2$. $\square$

Lemma 2.4. *Let G be a finite non-abelian group such that $G/Z(G)$ is isomorphic to $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$. The non-commuting graph $\mathbf{T}(G)$ is isomorphic to $K_{1,1,1,1,3}$ if and only if G' is isomorphic to $\mathbf{Z}_2 \times \mathbf{Z}_2$.*

Proof. Let G be a finite non-abelian group and suppose that $G/Z(G)$ is isomorphic to $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$. First, we claim that G' is an elementary abelian 2-group. In fact, given $u, v \in G$, we have that $u^2 \in Z(G)$ and $G' \leq Z(G)$; thus, $[v, u]u = u[v, u] = uvuv^{-1}u^{-1} = uvu^{-1}u^2v^{-1}u^{-2}u = [u, v]u$, which implies that $[u, v]^2 = 1$. Since G' is abelian, we get that G' is an elementary abelian 2-group.

Suppose that $\mathbf{T}(G)$ is isomorphic to $K_{1,1,1,1,3}$. Thus, there are vertices u and v in $\mathbf{T}(G)$ such that $uv = vu$. Since $o(uZ(G)) = o(vZ(G)) = 2$, we have $uv \notin Z(G) \cup uZ(G) \cup vZ(G)$. So, given $w \in V(\mathbf{T}(G)) \setminus (\{u, v\} \cup uvZ(G))$, we can suppose that $V(\mathbf{T}(G)) = \{u, v, w, uv, uw, vw, uvw\}$ and the subgraph of $\mathbf{T}(G)$ induced by $\{u, v, uv\}$ is isomorphic to $\overline{K_3}$. Putting $[u, w] = a$ and $[v, w] = b$, we have that $a \neq 1$, $b \neq 1$ and $a \neq b$ (because if $a = b$, then $[uv, w] = [u, w][v, w] = ab = a^2 = 1$, that is, $\{w, uv\} \notin E(\mathbf{T}(G))$, a contradiction). Now, given $x, y \in G$, we can write $x = u^\alpha v^\beta w^\gamma z$ and $y = u^\delta v^\epsilon w^\kappa t$, where $\alpha, \beta, \gamma, \delta, \epsilon, \kappa \in \{0, 1\}$ and $z, t \in Z(G)$. Since $[u, v] = 1$

and G' is an elementary abelian 2-group such that $G' \leq Z(G)$, it is routine to check that

$$[x,y] = [u^\alpha v^\beta w^\gamma z, u^\delta v^\epsilon w^\kappa t] = [u,w]^{\alpha\kappa-\gamma\delta}[v,w]^{\beta\kappa-\gamma\epsilon} = a^{\alpha\kappa-\gamma\delta}b^{\beta\kappa-\gamma\epsilon}.$$

Therefore, G' is isomorphic to $\mathbf{Z}_2 \times \mathbf{Z}_2$.

Conversely, suppose that G' is isomorphic to $\mathbf{Z}_2 \times \mathbf{Z}_2$, with $G' = \{1, a, b, ab\}$. Take $u, v, w \in G \setminus Z(G)$ such that $G/Z(G)$ is generated by $\{uZ(G), vZ(G), wZ(G)\}$ (we observe that $G/Z(G)$ is isomorphic to $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$). Thus, we can consider $V(\mathbf{T}(G)) = \{u, v, w, uv, uw, vw, uvw\}$. Here, we claim that there are $x, y \in V(\mathbf{T}(G))$ such that $x \neq y$ and $[x,y] = 1$. In fact, if $[u,v] = 1$ (or $[u,w] = 1$ or $[v,w] = 1$), then it is done. Suppose $[u,v] \neq 1$, $[u,w] \neq 1$ and $[v,w] \neq 1$. If $[u,v] = [u,w]$, then $[u,vw] = [u,v][v,w] = 1$; analogously, if $[u,v] = [v,w]$ or $[u,w] = [v,w]$, we get the desired. Finally, if we suppose $[u,v] = a$, $[u,w] = b$ and $[v,w] = ab$, then we have that $[uv,vw] = [u,v][u,w][v,w] = abab = 1$. Hence, there are $x, y \in V(\mathbf{T}(G))$ such that $x \neq y$ and $[x,y] = 1$. So, there is $r \in V(\mathbf{T}(G)) \cap xyZ(G)$ such that $[x,r] = [y,r] = 1$. Since G is non-abelian, we get that $C_G(x) \cap V(\mathbf{T}(G)) = \{x, y, r\}$. Thus, $C_G(x)$ is abelian and $[G : C_G(x)] = 2$. By Lemma 2.3(i), $\mathbf{T}(G)$ is isomorphic to $K_4 \vee \overline{K_3}$ and we know that $K_4 \vee \overline{K_3}$ isomorphic to $K_{1,1,1,1,3}$. □

We are ready to prove Theorem 1.1.

[**Proof of Theorem 1**] Let G be a finite non-abelian group. If $G/Z(G)$ is isomorphic to D_8 or if $G/Z(G)$ is isomorphic to $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$ and G' is isomorphic to $\mathbf{Z}_2 \times \mathbf{Z}_2$, then $\mathbf{T}(G)$ is isomorphic to $K_{1,1,1,1,3}$ (see Lemmas 2.3(ii) and 2.4). By Lemma 2.1, we have that $K_{1,1,1,1,3}$ is non-planar. By Figure 2.1, we conclude that $K_{1,1,1,1,3}$ is projective.

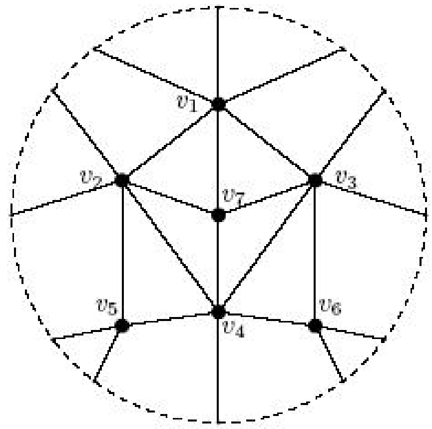

Figure 2.1: Drawing of $K_{1,1,1,1,3}$ in the projective plane

Conversely, let G be a finite non-abelian group such that $\mathbf{T}(G)$ is a projective graph. We write $|V(\mathbf{T}(G))| = \nu$ and $|E(\mathbf{T}(G))| = \epsilon$. Using part (ii) of Lemma 2.1 and parts (i) and (iii) of Lemma 2.2 we get that

$$\frac{1}{4}(\nu+1)\nu \leq \epsilon \leq 3\nu - 3$$

So $(\nu+1)\nu \leq 12\nu - 12$ and, consequently, $\nu \leq 9$. Thus, $[G : Z(G)] \leq 10$. Since G is non-abelian, we have that $G/Z(G)$ is non-cyclic. By [12, p. 85], up to isomorphism, there are 8 non-cyclic groups of order at most 10: $\mathbf{Z}_2 \times \mathbf{Z}_2$, D_6, $\mathbf{Z}_2 \times \mathbf{Z}_4$, $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$, D_8, the quaternion group of order 8, $\mathbf{Z}_3 \times \mathbf{Z}_3$ and D_{10}.

If $G/Z(G)$ is isomorphic to D_{10} then, by Lemma 2.3(iii), $\mathbf{T}(G)$ is isomorphic to $K_{1,1,1,1,1,4}$, a non-projective graph (by Lemma 2.1(ii)). Suppose that $G/Z(G)$ is isomorphic to $\mathbf{Z}_3 \times \mathbf{Z}_3$. In this case, given $x \in G \setminus Z(G)$, we observe that $y \notin C_G(x)$, for any $y \in G \setminus (Z(G) \cup xZ(G) \cup x^2Z(G))$, because G is non-abelian. Hence, we obtain $C_G(x) = Z(G) \cup xZ(G) \cup x^2Z(G)$ and, consequently, $\mathbf{T}(G)$ is isomorphic to $K_{2,2,2,2}$, that is, $\mathbf{T}(G)$ is non-projective (by Lemma 2.1(ii)). If $G/Z(G)$ is isomorphic to D_6, then $\mathbf{T}(G)$ is isomorphic to $K_{1,1,1,2}$ (Lemma 2.3(iii)) and so $\mathbf{T}(G)$ is planar. If $G/Z(G)$ is isomorphic to $\mathbf{Z}_2 \times \mathbf{Z}_2$, then $\mathbf{T}(G)$ is planar, because $|V(\mathbf{T}(G))| = 3$.

Finally, we suppose that $[G : Z(G)] = 8$. Since the complete graph on 7 vertices is non-projective (see Lemma 2.1(ii)), there are $u, v \in V(\mathbf{T}(G))$, $u \neq v$, such that $[u, v] = 1$. If $uv \in Z(G)$, then there is $r \in u^2Z(G) \cap V(\mathbf{T}(G))$ such that $r \neq u$, $r \neq v$ and $[u, r] = [v, r] = 1$. If $uv \notin Z(G)$, then there is $r \in uvZ(G) \cap V(\mathbf{T}(G))$ such that $r \neq u$, $r \neq v$ and $[u, r] =$

$[v, r] = 1$. Hence, we conclude that there is $r \in V(\mathbf{T}(G)) \setminus \{u, v\}$ such that the subgraph of $\mathbf{T}(G)$ induced by $\{u, v, r\}$ is isomorphic to $\overline{K_3}$. Further, we have that $\{u, v, r\} = V(\mathbf{T}(G)) \cap C_G(u)$ (because G is non-abelian). Thus, $C_G(u)$ is an abelian normal subgroup of G and $[G : C_G(u)] = 2$. By Lemma 2.3(i), we obtain that $\mathbf{T}(G)$ is isomorphic to $K_{1,1,1,1,3}$. Now, by Lemma 2.3(iii), all the elements of $V(\mathbf{T}(G)) \setminus \{u, v, r\}$ have order 2. So, if $C_G(u)/Z(G)$ is isomorphic to $\mathbf{Z}_2 \times \mathbf{Z}_2$, then $G/Z(G)$ is isomorphic to $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$ and, in this case, using Lemma 2.4, we get that G' is isomorphic to $\mathbf{Z}_2 \times \mathbf{Z}_2$. If $C_G(u)/Z(G)$ is isomorphic to $\mathbf{Z}_4$, then $G/Z(G)$ has two elements of order 4 and five elements of order 2. In this case, observing the list of groups of order 8, we obtain that $G/Z(G)$ is isomorphic to D_8. The proof is complete. $\square$

Example 2.5. *We observe that $D_{16}/Z(D_{16})$ is isomorphic to D_8. So, $\mathbf{T}(D_{16})$ is a projective graph. Now, let G be the group defined by*

$$G := \langle u, v, a \mid u^4 = v^4 = a^2 = 1, uv = vu, aua^{-1} = u^{-1}, ava^{-1} = v^{-1} \rangle.$$

Note that G is isomorphic to $(\mathbf{Z}_4 \times \mathbf{Z}_4)\mathbf{Z}_2$, where $\mathbf{Z}_2$ acts on $\mathbf{Z}_4 \times \mathbf{Z}_4$ by inversion. We can verify that G' is generated by $\{u^2, v^2\}$ and G' is isomorphic to $\mathbf{Z}_2 \times \mathbf{Z}_2$. Further, $|G| = 32$, $G' = Z(G)$ and $G/Z(G)$ is isomorphic to $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$. Hence, G is a group of one of the types given in Theorem 1.1.

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