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On derivations over trivial extensions

Brahim Boudine
Sidi Mohamed Ben Abdellah University, Morocco
and
Mohammed Zerra
Sidi Mohamed Ben Abdellah University, Morocco
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Abstract

In this paper, we provide a detailed analysis of the structure of derivations on trivial extensions, the centre of trivial extensions, and the conditions for a trivial extension to be prime. Additionally, we examine the structure of derivations on trivial extensions when the underlying ring, R, is a prime ring, under the conditions of Herstein's Theorem, Posner's Theorem, and Bell's theorem.

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1. Introduction

Let R and T be associative rings with unity, θ and ϕ be homomorphisms of unitary rings from T into R, and M be an R-bimodule. Then, a (θ, ϕ)-derivation d is an additive mapping $d : T \to M$ such that, for every $x, y \in T$, we have $d(x.y) = d(x).\phi(y) + \theta(x).d(y)$ (for more details see [6]). If $R = T$ and $\theta = \phi = id_R$, then we say only that d is a derivation. In particular, for a fixed element $r \in R$, the map $d : M \to M$ defined by $d(m) = [r, m] := r.m - m.r$ for all $m \in M$ will be called the inner derivation induced by r to keep the coherence with the case of inner derivations in rings. In this case, if $d(m) = 0$ for every $m \in M$, then we say that r is a centralizer of M in R, and we write $r \in Z_R(M)$, where $Z_R(M) := \{r \in R \mid r.m - m.r = 0, \forall m \in M\}$. As well as, we call the inner derivation on R into M induced by $m \in M$ the map $d : R \to M$ defined by $d(r) = [m, r] = m.r - r.m$ for every $r \in R$. In this case, if $d(r) = 0$ for every $r \in M$, then we say that m is a centralizer of R in M, and we write $m \in Z_M(R)$, where $Z_M(R) := \{m \in M \mid r.m - m.r = 0, \forall r \in R\}$. Considerable attention has been given to the study of additive mappings and their impact on the overall structure of a ring in recent decades, including derivations, homomorphisms, and related maps (see references [1], [7], [8], [9], [10], [11]).

The ring R is said to be prime if $xRy = 0$ implies that either $x = 0$ or $y = 0$, for any elements x and y in R. As well, R is called semiprime if $xRx = 0$ implies that $x = 0$, for any $x \in R$. Herstein theorem says that if R is a noncommutative prime ring with $char(R) \neq 2$ (namely, $x + x = 0$ implies $x = 0$, for any $x \in R$), and d is a derivation on R verifying $d(x).d(y) = d(y).d(x)$ for every $x, y \in R$, then $d = 0$ (see [7]). As well, Posner's theorem, a classic result, states that the non-commutativity of a prime ring R forces a centralizing derivation on R to be zero (see [12]). Also, Bell's theorem shows that if R is a prime ring having a derivation d which satisfies $[d(x), d(y)] = [x, y]$, for every $x, y \in R$, then R is commutative (see [4]).

Actually, we focus on trivial extensions. Recall that the trivial extension of R by M [2] is the ring $R \propto M := R \bigoplus M$ such that for every $(r, m), (r', m') \in R \propto M$, we have

$$\begin{cases} (r, m) + (r', m') = (r + r', m + m'), \\ (r, m) \propto (r', m') = (r.r', r.m' + m.r'). \end{cases}$$

In 2018, Bahmani et al. [3] studied Jordan generalized derivations on trivial extensions. Later, in 2019, Bennis et al. [5] studied Lie generalized derivations on trivial extensions. In this paper, we present a comprehensive investigation of derivations, trivial extensions, and their interplay. Our focus is on providing a thorough analysis of the structure of derivations over trivial extensions, and conditions for trivial extensions to be prime. Additionally, we provide an in-depth analysis of the structure of derivations in the case where R is a prime ring under the conditions of Herstein's theorem, Posner's theorem and Bell's theorem mentioned above.

2. Derivations over trivial extensions

Let d be an additive map on $R \propto M$. Let π_1 and π_2 be the two projections over $R \propto M$, defined by $\pi_1 : (r,m) \in R \propto M \mapsto r \in R$, and $\pi_2 : (r,m) \in R \propto M \mapsto m \in M$. We write $d_1 := \pi_1 \circ d$ and $d_2 := \pi_2 \circ d$; namely, $d = (d_1, d_2)$. Furthermore, we define the following maps

$$\begin{array}{rccc} s_1: & M & \to & M \\ & m & \mapsto & s_1(m) := d_2((0,m)). \end{array} \quad , \quad \begin{array}{rccc} s_2: & R & \to & M \\ & r & \mapsto & s_2(r) := d_2((r,0)). \end{array}$$

We give necessary and sufficient conditions for d to be a derivation.

Theorem 2.1. *With the above notations, the map d is a derivation on $R \propto M$ if and only if the following statements hold:*

1. *d_1 is a (π_1, π_1)-derivation,*
2. *s_2 is a derivation,*
3. *$s_1(r.m) = r.s_1(m) + d_1(r,0).m$ and $s_1(m.r) = s_1(m).r + m.d_1(r,0)$, for every $r \in R$ and $m \in M$,*

Proof. Let $(r,m), (r',m') \in R \propto M$, and suppose that d is a derivation over $R \propto M$. Then,

$$\begin{cases} d((r,m) \propto (r',m')) = (r,m) \propto d(r',m') + d(r,m) \propto (r',m'), \\ d((r,m) \propto (r',m')) = d((r.r', r.m' + m.r')). \end{cases}$$

It follows that $d((r.r', r.m' + m.r')) = (r,m) \propto d(r',m') + d(r,m) \propto (r',m')$. Then, we obtain the following equations:

$$\begin{cases} d_1((r.r', r.m' + m.r')) = r.d_1(r',m') + d_1(r,m).r', \\ d_2((r.r', r.m' + m.r')) = r.d_2(r',m') + m.d_1(r',m') + d_1(r,m).m' + d_2(r,m).r'. \end{cases}$$

In other words, we get the following equalities:

$$\begin{cases} d_1((r,m)\propto(r',m')) = & \pi_1(r,m).d_1(r',m')+d_1(r,m).\pi_1(r',m'), \\ d_2((r,m)\propto(r',m')) = & (\pi_1(r,m).d_2(r',m')+\pi_2(r,m).d_1(r',m'))+ \\ & (d_2(r,m).\pi_1(r',m')+d_1(r,m).\pi_2(r',m')). \end{cases}$$

Therefore, d_1 is a (π_1,π_1)-derivation into R. In particular, if $m=m'=0$, then we get that

$$d_2((r,0)\propto(r',0)) = s_2(r.r') = r.s_2(r')+s_2(r).r';$$

namely, s_2 is a derivation of R into $R\propto M$. On the other hand, if $r'=0$ and $m=0$, then we get that

$$d_2((r,0)\propto(0,m')) = d_2((0,r.m')) = s_1(r.m') = r.s_1(m')+d_1(r,0).m'.$$

As well as, if $r=0$ and $m'=0$, then we get that

$$d_2((0,m)\propto(r',0)) = d_2((0,m.r')) = s_1(m.r') = s_1(m).r'+m.d_1(r',0).$$

Conversely, let $d:(r,m)\in R\propto M\mapsto d((r,m)) := (d_1(r,m),d_2(r,m))$. Then, for every $(r,m),(r',m')\in R\propto M$, we have:

$$d((r,m)\propto(r',m')) = (d_1((r,m)\propto(r',m')),d_2((r,m)\propto(r',m')).$$

Let us compute the first projection:

$$\begin{aligned} d_1((r,m)\propto(r',m')) &= d_1((r.r',r.m'+m.r')), \\ &= r.d_1(r',m')+d_1(r,m).r'. \end{aligned}$$

Now, we compute the second projection. Let $(r,m),(r',m')\in R\propto M$. Notice first that

$$(r,m)\propto(r',m') = (r.r',0)+(0,r.m')+(0,m.r').$$

Then, we obtain that

$$\begin{aligned} d_2((r,m)\propto(r',m')) &= d_2((r.r',0))+d_2((0,r.m'))+d_2((0,m.r')), \\ &= s_2(r.r')+s_1(r.m')+s_1(m.r'), \\ &= r.s_2(r')+s_2(r).r'+r.s_1(m')+d_1((r,0)).m'+s_1(m).r'+m.d_1((r',0)), \\ &= r.d_2((r',m'))+d_2((r,m)).r'+d_1((r,0)).m'+m.d_1((r',0)). \end{aligned}$$

It follows that

$$\begin{aligned}
d((r,m)\propto(r',m')) = \;& (r.d_1((r',m'))+d_1((r,m)).r', r.d_2((r',m'))+d_2((r,m)).r'+\\
& d_1((r,0)).m'+m.d_1((r',0))),\\
=\;& (r.d_1((r',m')), r.d_2((r',m'))+m.d_1((r',0)))+\\
& (d_1((r,m)).r', d_2((r,m)).r'+d_1((r,0)).m'),\\
=\;& (r,m)\propto(d_1((r',m')),\\
& d_2((r',m')))+(d_1((r,m)),d_2((r,m)))\propto(r',m'),\\
=\;& (r,m)\propto d((r',m'))+d((r,m))\propto(r',m').
\end{aligned}$$

Therefore, d is a derivation. □

Now, we prove that we can always construct a derivation on $R\propto M$, whenever we have a derivation f on R, a derivation g on R into $R\propto M$, and an additive map $s: M\to R\propto M$ satisfying some conditions.

Theorem 2.2. *Let R be a ring and M be an R-bimodule. Suppose that there exist*

1. *a derivation $f: R\to R$,*
2. *a derivation $g: R\to R\propto M$,*
3. *an additive map $s: M\to R\propto M$ such that $s(r.m)=r.s(m)+f(r).m$ and $s(m.r)=s(m).r+m.f(r)$; for every $r\in R$ and $m\in M$.*

Then, the map

$$\begin{array}{rccc}
d: & R\propto M & \to & R\propto M\\
 & (r,m) & \mapsto & d((r,m)) := (f(r), g(r)+s(m)),
\end{array}$$

is a derivation on $R\propto M$.

Proof. It is easy to see that d is an additive map. On the other hand, let $(r,m),(r',m')\in R\propto M$. Then, we have

$$\begin{aligned}
d((r,m)\propto(r',m')) \;& = d((r.r', r.m'+m.r')),\\
& = (f(r.r'), g(r.r')+s(r.m'+m.r')),\\
& = (f(r).r'+r.f(r'), g(r).r'+r.g(r')+s(r.m')+s(m.r')),\\
& = (f(r).r'+r.f(r'), g(r).r'+r.g(r')+r.s(m')+f(r).m'\\
& +s(m).r'+m.f(r')),\\
& = (f(r).r', f(r).m'+(g(r)+s(m)).r')+(r.f(r'), r.(g(r')\\
& +s(m'))+m.f(r')),\\
& = (f(r), g(r)+s(m))\propto(r',m')+(r,m)\propto(f(r'), g(r')\\
& +s(m')),\\
& = d((r,m))\propto(r',m')+(r,m)\propto d((r',m')).
\end{aligned}$$

Hence d is a derivation on $R \propto M$. $\square$

Let $i_1 : R \to R \propto M$ and $i_2 : M \to R \propto M$ be the two maps defined by $i_1(r) = (r, 0)$ and $i_2(m) = (0, m)$ for every $(r, m) \in R \times M$. Then, we characterize inner derivations on $R \propto M$.

Theorem 2.3. *With the above notations, the derivation d is the inner derivation on $R \propto M$ induced by $(r, m) \in R \propto M$, if and only if the following statements hold:*

1. $d_1 \circ i_2 = 0$,
2. $d_1 \circ i_1$ *is the inner derivation on R induced by r,*
3. s_1 *is the inner derivation on M induced by r,*
4. s_2 *is the inner derivation on R into M, induced by m.*

Proof. Let $(a, x) \in R \propto M$, then we obtain:

$$d((a, x)) = (r.a - a.r, (r.x - x.r) + (m.a - a.m)).$$

It follows that $d_1 \circ i_2 = 0$, and $d_1 \circ i_1(a) = r.a - a.r = [r, a]$, for every $a \in R$; namely, d_1 is the inner derivation on R induced by r. As well, $s_1(x) = r.x - x.r = [r, x]$, for every $x \in M$; namely, s_1 is the inner derivation on M induced by r. So that $s_2(a) = m.a - a.m = [m, a]$, for every $a \in R$; namely, s_2 is the inner derivation on R into M induced by m. Convesely, we have:

$$\begin{aligned} d(a, x) &= (d_1 \circ i_1(a) + d_1 \circ i_2(x), s_1(x) + s_2(r)), \\ &= (r.a - a.r, (r.x - x.r) + (m.a - a.m)), \\ &= (r, m) \propto (a, x) - (a, x) \propto (r, m), \\ &= [(r, m), (a, x)]. \end{aligned}$$

Therefore, d is the inner derivation on $R \propto M$ induced by (r, m). $\square$

Recall that a derivation δ on a ring A is called centralizing, if for every $x \in A$, we have $[\delta(x), x] \in Z_A(A)$. Then, in order to study the centralizing derivations on $R \propto M$, we need first to study the structure of $Z_{R \propto M}(R \propto M)$; the center of $R \propto M$.

Proposition 2.4. *Let R be a ring, M be an R-bimodule, and $(r, m) \in A := R \propto M$. Then, (r, m) is a centralizer in $R \propto M$ if and only if the following statements hold:*

1. $r \in Z_R(R)$,
2. $m \in Z_M(R)$,
3. $Z_R(R) \subseteq Z_R(M)$.

Then, we write $Z_{R\propto M}(R \propto M) = Z_R(R) \propto Z_M(R)$ *with* $Z_R(R) \subseteq Z_R(M)$.

Proof. Suppose that $[(a, x), (r, m)] = (0, 0)$, for every element (a, x) in $R \propto M$; equivalently,

$$\begin{cases} a.r - r.a = 0, \\ a.m - m.a + x.r - r.x = 0, \end{cases}$$

From the first equation, we see that $r \in Z_R(R)$. Suppose that $x = 0$. Then, the second equation implies that $a.m - m.a = 0$, for every $a \in R$, which yields that $m \in Z_M(R)$. In this case, we get that $x.r - r.x = 0$, where $r \in Z_R(R)$. Therefore, $Z_R(R) \subseteq Z_R(M)$. Conversely, since $r \in Z_R(R)$ and $m \in Z_M(R)$, we get that $a.r - r.a = 0$ and $a.m - m.a = 0$. It follows that $[(a, x), (r, m)] = (0, x.r - r.x)$. But we have $Z_R(R) \subseteq Z_R(M)$, then $x.r - r.x = 0$. Thus, (r, m) is a centralizer of $R \propto M$. □

Corollary 2.5. *Let R be a ring and M an R-bimodule. The trivial extension $R \propto M$ is commutative if and only if R is commutative and $r.m = m.r$ for every $(r, m) \in R \times M$.*

Proof. By Proposition 2.4, $R \propto M$ is commutative if and only if $Z_R(R) = R$ and $Z_M(R) = M$; namely, R is commutative and $r.m = m.r$ for every $(r, m) \in R \times M$. □

So we give a characterization of centralizing derivations on $R \propto M$.

Theorem 2.6. *Let R be a ring, M be an R-bimodule, and d be a derivation on $R \propto M$. Then, the derivation d centralizing if and only if, for every $(r, m) \in R \propto M$, the following statements hold:*

1. $d_1((r, m)).r - r.d_1((r, m)) \in Z_R(R)$,
2. $s_2(r).r - r.s_2(r) \in Z_M(R)$,
3. $d_1((0, m)).m - m.d_1((0, m)) \in Z_M(R)$,
4. $s_1(m).r - r.s_1(m) \in Z_M(R)$.

Proof. Let $(r,m) \in R \propto M$, and suppose that $[d((r,m)),(r,m)] \in Z_{R\propto M}(R \propto M)$. Then, we obtain

$$(d_1((r,m)).r - r.d_1((r,m)), d_1((r,m)).m - m.d_1((r,m)) + d_2((r,m)).r$$

$$-r.d_2((r,m))) \in Z_{R\propto M}(R \propto M).$$

By Proposition 2.4, we get that

$$\begin{cases} d_1((r,m)).r - r.d_1((r,m)) \in Z_R(R), \\ d_1((r,m)).m - m.d_1((r,m)) + d_2((r,m)).r - r.d_2((r,m)) \in Z_M(R). \end{cases}$$

If $m = 0$, then the first statement shows that $d_1((r,0)).r - r.d_1((r,0)) \in Z_R(R)$, and the second statement shows that $s_2(r).r - r.s_2(r) \in Z_M(R)$. It follows that $d_1((0,m)).r - r.d_1((0,m)) \in Z_R(R)$. Therefore, $d_1((r,m)).r - r.d_1((r,m)) \in Z_R(R)$. On the other hand, if $r = 0$, then we get from the second statement that $d_1((0,m)).m - m.d_1((0,m)) \in Z_M(R)$. Finally, we conclude that $s_1(m).r - r.s_1(m) \in Z_M(R)$. The converse is obvious. □

Recall that a derivation δ on a ring A is called commuting, if for every $x \in A$, we have $[\delta(x), x] = 0$. So, we characterize the commuting derivations on $R \propto M$.

Theorem 2.7. *Let R be a ring, M be an R-bimodule, and d be a derivation on $R \propto M$. Then, the derivation d commuting if and only if, for every $(r,m) \in R \propto M$, the following statements hold:*

1. $d_1((r,m)).r - r.d_1((r,m)) = 0$,
2. $s_2(r).r - r.s_2(r) = 0$,
3. $d_1((0,m)).m - m.d_1((0,m)) = 0$,
4. $s_1(m).r - r.s_1(m) = 0$.

Proof. The same method used in the proof of Theorem 2.6. □

3. Derivations over $R \propto M$ when R is a prime ring

First, we study the primeness and semiprimeness of trivial extensions.

Proposition 3.1. *Let R be a ring and M be an R-bimodule. Then, $R \propto M$ is a prime ring if and only if R is prime and $M = \{0\}$.*

Proof. Let $m \in M$. Then, for any $(a, x) \in R \propto M$, we have $(0, m) \propto (a, x) \propto (0, m) = (0, 0)$; namely, $(0, m) \propto R \propto (0, m) = (0, 0)$. Since $R \propto M$ is prime, $(0, m) = (0, 0)$. Thus, $M = \{0\}$. Now, Let r and r' be two elements in R such that $rRr' = 0$. Then, for any $(a, 0) \in R \propto M$, we have $(r, 0) \propto (a, 0) \propto (r', 0) = (rar', 0) = (0, 0)$. Since $R \propto M$ is prime, we get that either $(r, 0) = (0, 0)$ or $(r', 0) = (0, 0)$; namely, $r = 0$ or $r' = 0$. Thus, R is prime. Conversely, we have $R \propto \{0\} \cong R$. Therefore, $R \propto \{0\}$ is prime. □

Proposition 3.2. *Let R be a ring and M be an R-bimodule. Then, $R \propto M$ is a semiprime ring if and only if R is semiprime and $M = \{0\}$.*

Proof. By the same method used in the proof of Proposition 3.1. □

Lemma 3.3. *Let R be a ring, M be an R-bimodule, and d a derivation on $R \propto M$. Then, $d_1 \circ i_1$ is a derivation on R.*

Proof. It is easy to see that $d_1 \circ i_1$ is additive. Let $r, r' \in R$. We have $d_1 \circ i_1(r.r') = d_1(r.r', 0) = \pi_1(d((r, 0) \propto (r', 0)))$. Since d is a derivation, we get $d((r, 0) \propto (r', 0)) = (r, 0) \propto d((r', 0)) + d((r, 0)) \propto (r', 0)$. Therefore, $d_1 \circ i_1(r.r') = r.d_1((r', 0)) + d_1((r, 0)).r' = r.d_1 \circ i_1(r') + d_1 \circ i_1(r).r'$. Thus, $d_1 \circ i_1$ is a derivation on R. □

Remark 3.4. *It is well known that $R \propto \{0\}$ is isomorphic to R. Then, all known results on derivations on prime rings can be extended easily on the prime ring $R \propto \{0\}$ by taking $d_1 \circ i_1$ to be the derivation on R.*

Let us study the behaviour of derivations on $R \propto M$, where R is a prime ring, under the conditions Herstein's theorem [7].

Theorem 3.5. *Let R be a noncommutative prime ring with $char(R) \neq 2$, M be an R-bimodule, and d be a derivation on $R \propto M$. If $d((r, m)) \propto d((r', m')) = d((r', m')) \propto d((r, m))$ for every $(r, m), (r', m') \in R \propto M$, then $d_1 \circ i_1 = 0$ and $s_1 \in End_R(M)$. Furthermore, for every $m, m' \in M$, we get that $[s_2(m), d_1((0, m'))] = [d_1((0, m)), s_2(m')]$.*

Proof. By Lemma 3.3, we have $\pi_1 \circ d \circ i_1$ is a derivation on R. Since $d((r, 0)) \propto d((r', 0)) = d((r', 0)) \propto d((r, 0))$ for every $r, r' \in R$, we get that $d_1((r, 0)).d_1((r', 0)) = d_1((r', 0)).d_1((r, 0))$; namely, $d_1 \circ i_1(r).d_1 \circ i_1(r') = d_1 \circ i_1(r').d_1 \circ i_1(r)$. By Herstein's Theorem [7], we get that $d_1 \circ i_1 = 0$. It follows from Theorem 2.1 that $s_1(r.m) = r.s_1(m)$ and $s_1(m.r) = s_1(m).r$, for every $(r, m) \in R \times M$; namley, $s_1 \in Hom_R(M, R \propto M)$. On the other hand, the equation $d((r, m)) \propto d((r', m')) = d((r', m')) \propto d((r, m))$ is equivalent to the equation

$$d_2((r,m)).d_1((r',m')) + d_1((r,m)).d_2((r',m')) = d_2((r',m')).d_1((r,m))$$

$$+d_1((r',m')).d_2((r,m));$$

namely, we get that

$$[d_2((r,m)), d_1((r',m'))] = [d_1((r,m)), d_2((r',m'))], \quad (\forall (r,m),(r',m') \in R \propto M).$$

Since $d_1 \circ i_1 = 0$, the equation becomes as follows:

$$[d_2((r,m)), d_1((0,m'))] = [d_1((0,m)), d_2((r',m'))], \quad (\forall (r,m),(r',m') \in R \propto M).$$

Recall that $d_2((r, m)) = s_1(r) + s_2(m)$. In particular, if $m = 0$, then we obtain

$$[s_1(r), d_1((0,m'))] = [d_1((0,0)), d_2((r',m'))] = 0, \quad (\forall (r,0),(r',m') \in R \propto M).$$

Therefore, we conclude that

$$[s_2(m), d_1((0,m'))] = [d_1((0,m)), s_2(m')], \quad (\forall m, m' \in M).$$

□

Next, we make the same thing under the conditions of Posner's theorem [12].

Proposition 3.6. *Let R be a noncommutative prime ring, M be an R-bimodule such that $Z_M(R) = M$, and d be a derivation on $R \propto M$. Then, the derivation d centralizing if and only if, for every $(r, m) \in R \propto M$, the following statements hold:*

1. $d_1 \circ i_1 = 0$,
2. $d_1((0, m)).r - r.d_1((0, m)) \in Z_R(R)$.

Proof. By Lemma 3.3, we have $d_1 \circ i_1$ is a derivation on R. Further, by Theorem 2.6, we have $d_1((r,m)).r - r.d_1((r,m)) \in Z_R(R)$. In particular, if $m = 0$, then we get $d_1 \circ i_1(r).r - r.d_1 \circ i_1(r) \in Z_R(R)$; namely, $d_1 \circ i_1$ is a centralizing derivation on R. Since R is a noncommutative prime ring, Posner's Theorem [12] shows that $d_1 \circ i_1 = 0$. Therefore, we get that $d_1((0,m)).r - r.d_1((0,m)) \in Z_R(R)$. The converse is obtained by applying Theorem 2.6 since $Z_M(R) = M$. □

Proposition 3.7. *Let R be a noncommutative prime ring, M be an R-bimodule such that $Z_M(R) = M$, and d be a derivation on $R \propto M$. Then, the derivation d commuting if and only if, for every $(r,m) \in R \propto M$, the following statements hold:*

1. $d_1 \circ i_1 = 0$,
2. $d_1((0,m)).r - r.d_1((0,m)) = 0$.

Proof. By the same method used in the proof of Proposition 3.6. □

Next, we study the derivations on $R \propto M$ verifying the conditions of Bell's theorem [4].

Theorem 3.8. *Let R be a prime ring, M be an R-bimodule, and d be a derivation on $R \propto M$ such that $d_1 \circ i_2 = 0$. Suppose that, for every $(r,m),(r',m') \in R \propto M$, the derivation d verifies $[d((r,m)), d((r',m'))] = [(r,m),(r',m')]$. Then, $R \propto M$ is commutative.*

Proof. Let us compute $[d((r,m)), d((r',m'))]$, for every $(r,m),(r',m') \in R \propto M$.

$$\begin{aligned}
[d((r,m)), d((r',m'))] &= d((r,m)) \propto d((r',m')) - d((r',m')) \propto d((r,m)),\\
&= (d_1((r,m)).d_1((r',m')) - d_1((r',m')).d_1((r,m)),\\
&\quad d_1((r,m)).d_2((r',m')) + d_2((r,m)).d_1((r',m'))\\
&\quad -d_1((r',m')).d_2((r,m)) - d_2((r',m')).d_1((r,m))).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
[(r,m),(r',m')] &= (r,m) \propto (r',m') - (r',m') \propto (r,m),\\
&= (r.r' - r'.r, r.m' + m.r' - r'.m - m'.r).
\end{aligned}$$

If $m = m' = 0$, then we get that $[d_1 \circ i_1(r), d_1 \circ i_1(r')] = [r,r']$. By Lemma 3.3, we have $d \circ i_1$ is a derivation on R. Therefore, Bell's theorem [4] proves that R is commutative. Further, we get that

$$d_1 \circ i_1(r).s_1(r') - s_1(r').d_1 \circ i_1(r) = d_1 \circ i_1(r').s_1(r) - s_1(r).d_1 \circ i_1(r').$$

Since $d_1 \circ i_2 = 0$, the second component of the equality $[d((r,m)), d((r',m'))] = [(r,m),(r',m')]$ becomes as follows

$$d_1 \circ i_1(r).s_2(r') + s_2(r).d_1 \circ i_1(r') - d_1 \circ i_1(r').s_2(r) - s_2(r').d_1 \circ i_1(r)$$

$$= r.m' + m.r' - r'.m - m'.r.$$

If $r' = 0$, then we get that $r.m' - m'.r = 0$, for every $r \in R$ and $m' \in M$; namely, $Z_M(R) = M$. Thus, Proposition 2.4 shows that $R \propto M$ is commutative. □

4. Conclusion

In conclusion, our study of derivations on trivial extensions has provided a more complete understanding of the relationships between these mappings, trivial extensions, and prime rings. We characterized the structure of derivations on trivial extensions, and we provided the necessary and sufficient conditions for derivations to be centralizing or commuting. Additionally, we have investigated the primeness of trivial extensions, and we have explored the structure of derivations in the context of prime rings satisfying the conditions of Herstein's theorem, Posner's theorem and Bell's theorem. This work advances our knowledge of these mathematical concepts and lays the foundation for further study in this area.

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Conflicts of Interest

The authors declare no conflicts of interest.

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Brahim Boudine
Department of Mathematics
Faculty of Sciences Dhar El Mahraz
Sidi Mohamed Ben Abdellah University
Fez 30000,
Morocco
e-mail: brahimboudine.bb@gmail.com
Corresponding author

and

Mohammed Zerra
Department of Mathematics
Faculty of Sciences and Technology
Sidi Mohamed Ben Abdellah University
Fez 30000,
Morocco
e-mail: Mohamed.zerra@gmail.com