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A note on local edge antimagic chromatic number of graphs

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Abstract

Let G be a finite, undirected and simple graph. A bijection $f : V(G) \to [1, |V(G)|]$ is called a local edge antimagic labeling if for any two adjacent edges $uv, vx \in E(G), w(uv) \neq w(vx)$ with $w(uv) = f(u) + f(v)$. By giving every edges $uv \in E(G)$ a coloring with $w(uv)$, then the local edge antimagic labeling of G induces an edge coloring of G. The local edge antimagic chromatic number $\chi'_{lea}(G)$ is the minimum number of colors taken over all edge colorings induced by local edge antimagic labeling of G. In this paper, we investigate characterization of graphs G with small number $\chi'_{lea}(G)$, relationship between local edge antimagic chromatic number $\chi'_{lea}(G)$ and edge independence number $\alpha'(G)$, and bounds of $\chi'_{lea}(G)$ for any graphs.

Keywords: *Edge coloring, edge independence number, local edge antimagic.*

MSC (2020): *05C15, 05C70, 05C78.*

Introduction

Consider graphs in this paper to be simple, undirected and finite. Let nG to be a union of n disjoint copies of graph G. The *edge independence number* $\alpha'(G)$ is the size of maximum independent edge set. If the order of a graph G is n, then the edge independence number is bounded by

$$\alpha'(G) \leq \left\lfloor \frac{1}{2} n \right\rfloor$$

Let G and H be graphs with $v \in V(H)$. We define a *comb product* of G and H, denoted by $G \triangleright_v H$, to be a graph obtained by taking one copy of G and $|V(G)|$ copies of H in which the vertex v in i-th copy of H is identified with the i-th vertex of G [16]. The graph $G \triangleright_v H$ has a vertex set

$$V(G \triangleright_v H) = \{(a, x) \mid a \in V(G), x \in V(H)\}$$

and an edge set

$$E(G \triangleright_v H) = \{(a, x)(b, y) \mid \quad \begin{array}{l} \text{if } a = b \text{ and } xy \in E(H), \\ \text{or } ab \in E(G) \text{ and } v = x = y\} \end{array}$$

Let f be a bijection $f : E(G) \to [1, |E(G)|]$. The map f is called *local antimagic labeling* if for any two adjacent vertices $u, v \in V(G), w(u) \neq w(v)$ with $w(u) = \sum_{e \in E(u)} f(e)$ and $E(u)$ be the set of all edges incident to u. For every vertex v, assign the color $w(v)$ to the vertex v. Consequently, a local antimagic labeling of G will induce a vertex coloring of G. The local antimagic chromatic number $\chi_{la}(G)$ is the minimum number of colors taken over all vertex colorings induced by local antimagic labeling of G. Arumugam et al. [3] have determined the local antimagic chromatic number of several families of graphs, namely paths P_n, cycles C_n, friendship graphs F_n, complete bipartites $K_{m,n}$, and wheels W_n. They also found some bounds of local antimagic chromatic number for trees. There are other studies about local antimagic chromatic number which involves complete full *t-ary* trees [4], wheels and helms [7], corona products related to friendship and fan graph [11], graphs amalgamation [12], generalized friendship graphs [14], and lexicographic product graphs [13]. In addition, Haslegrave [10] has proven that every connected graphs other than K_2 has a local antimagic labeling.

It is natural to consider a variation of such labeling. A bijection $f : V(G) \to [1, |V(G)|]$ is called *local edge antimagic labeling* if for any two adjacent edges $uv, vx \in E(G), w(uv) \neq w(vx)$ with $w(uv) = f(u) + f(v)$.

By assigning the color $w(uv)$ to the edge uv for every edge $uv \in E(G)$, the local edge antimagic labeling of G will induce an edge coloring of G. The local edge antimagic chromatic number $\chi'_{lea}(G)$ (some authors write it as $\gamma_{lea}(G)$) is the minimum number of colors taken over all edge colorings induced by local edge antimagic labeling of G. Agustin et al. [1] have found the local edge antimagic chromatic number of paths P_n, cycles C_n, ladders L_n, stars S_n, complete graphs K_n, and many more. Some of their results are shown below.

Theorem 1. *[1] For $n \geq 3$, the local edge antimagic chromatic number of P_n is $\chi'_{lea}(P_n) = 2$.*

Theorem 2. *[1] For $n \geq 3$, the local edge antimagic chromatic number of C_n is $\chi'_{lea}(C_n) = 3$.*

Theorem 3. *[1] For $n \geq 3$, the local edge antimagic chromatic number of S_n is $\chi'_{lea}(S_n) = n$.*

The study is followed by Rajkumar and Nalliah [15] who investigated the local edge antimagic chromatic number of friendship graphs F_n, wheels W_n, fan graphs f_n, helm graph H_n, and flower graphs Fl_n. Many variations on local antimagic labeling may also be seen in [5, 8, 9]. To see many other kinds of labeling please consult to [6].

For a graph G, let $\Delta(G)$ be the largest degree of a vertex in G and $\chi'(G)$ be the edge chromatic number of G. It is evident that the following inequalities are true

$$\Delta(G) \leq \chi'(G) \leq \chi'_{lea}(G) \leq |E(G)|$$

In this paper, we investigate characterization of graphs G with small number of $\chi'_{lea}(G)$, relationship of $\chi'_{lea}(G)$ and $\alpha'(G)$, and bounds of $\chi'_{lea}(G)$ for any graph G.

Main Results

Unlike the analog labeling [10], proving that every graphs admits local edge antimagic labeling is pretty straightforward. To prove this, consider the following proposition.

Proposition 1. *Let G be a graph, $f : V(G) \to [1, |V(G)|]$ be a bijection, and $w(uv) = f(u)+f(v)$ for an edge $uv \in E(G)$. For any two adjacent edges $uv, vx \in E(G)$, we have $w(uv) \neq w(vx)$ and f is a local edge antimagic labeling.*

Proof. For any distinct vertices $u, v, x \in V(G)$, one may observe that

$$\begin{aligned} w(uv) \neq w(vx) &\iff f(u) + f(v) \neq f(v) + f(x) \\ &\iff f(u) \neq f(x) \\ &\iff f \text{ is injective} \end{aligned}$$

Since f is a bijection, it follows that $w(uv) \neq w(vx)$ and f is a local edge antimagic labeling. □

Fix any graph G and consider any bijection $f : V(G) \to [1, |V(G)|]$. By Proposition 1, f is a local edge antimagic labeling. Therefore, we have shown the following.

Corollary 1. *Every graphs admits local edge antimagic labeling.*

Next, consider graphs G with $\Delta(G) = 1$. If G is connected then $G \cong K_2$. For more general graphs G, it may be seen that if G does not have any isolated vertex, then $G \cong nK_2$ for some positive integer n. It follows that this is the only graph with $\chi'_{lea}(G) = 1$.

Proposition 2. *Let G be a graph without isolated vertices. We have $\chi'_{lea}(G) = 1$ if and only if $G \cong nK_2$ for some positive integer n.*

Proof. Let G be a graph without isolated vertices and $\chi'_{lea}(G) = 1$, then $\Delta(G) \leq 1$ which implies $G \cong nK_2$. For the backward direction, let $G \cong nK_2$. To show $\chi'_{lea}(G) = 1$, consider $X = [1, 2n]$. Create a partition of X into 2-sets namely X_i for $i \in [1, n]$ such that

$$\sum_{t \in X_i} t = 2n + 1$$

Then, let f be a map which labels every two adjacent vertices with X_i for $i \in [1, n]$. It follows that every edges of G has a weight of $2n + 1$. This implies $\chi'_{lea}(G) = 1$. □

Then, consider connected graphs G with $\Delta(G) = 2$. Using results from [1], we may also determine a characterization as follows

Corollary 2. *Let G be a connected graph with the order at least 3. We have $\chi'_{lea}(G) = 2$ if and only if $G \cong P_n$ for some positive integer n.*

Proof. Let G be a connected graph with $\chi'_{lea}(G) = 2$. Consequently, $\Delta(G) \leq 2$. This implies G is isomorphic to either paths P_n or cycles C_n. However, $\chi'_{lea}(C_n) = 3$ due to Theorem 2. This implies that G may only be isomorphic to paths P_n. The backward direction is exactly Theorem 1. $\square$

We may extend this result to conclude that disjoint paths also have the same local edge antimagic chromatic number. For integers $m \geq 1$ and $n \geq 3$, let mP_n be a graph with vertex set

$$V(mP_n) = \{v_{i,j} \mid i \in [1,n], j \in [1,m]\}$$

and with edge set

$$E(mP_n) = \{v_{i,j}v_{i+1,j} \mid i \in [1,n-1], j \in [1,m]\}$$

Theorem 4. *Let $m \geq 1$ and $n \geq 3$ be integers. We have $\chi'_{lea}(mP_n) = 2$.*

Proof. Let $f : V(G) \to [1,mn]$ be a labeling of mP_n. We define f according to the parity of n.
If n is even, define f as follows

$$f(v_{i,j}) = \begin{cases} \frac{i+n(j-1)+1}{2}, & \text{for } i \text{ is odd,} \\ mn+1-\frac{i+n(j-1)}{2}, & \text{for } i \text{ is even.} \end{cases}$$

It could be seen that f is a bijection. As a result, we have

$$w(v_{i,j}v_{i+1,j}) = \begin{cases} mn+1, & \text{for } i \text{ is odd,} \\ mn+2, & \text{for } i \text{ is even.} \end{cases}$$

Else, if n is odd, then f is defined by

$$f(v_{i,j}) = \begin{cases} \frac{i+n(j-1)+1}{2}, & \text{for } i+j \text{ is even,} \\ mn+1-\frac{i+n(j-1)}{2}, & \text{for } i+j \text{ is odd.} \end{cases}$$

Notice that f is also a bijection. It follows that

$$w(v_{i,j}v_{i+1,j}) = \begin{cases} mn+1, & \text{for } i+j \text{ is even,} \\ mn+2, & \text{for } i+j \text{ is odd.} \end{cases}$$

It may be concluded that $\chi'_{lea}(mP_n) \leq 2$. Since $\Delta(mP_n) = 2 \leq \chi'_{lea}(mP_n)$, then $\chi'_{lea}(mP_n) = 2$. $\square$

In Figure 1, we present an example of local edge antimagic labeling for $3P_6$ and $4P_5$. The distinct weights implies $\chi'_{lea}(3P_6) = \chi'_{lea}(4P_5) = 2$.

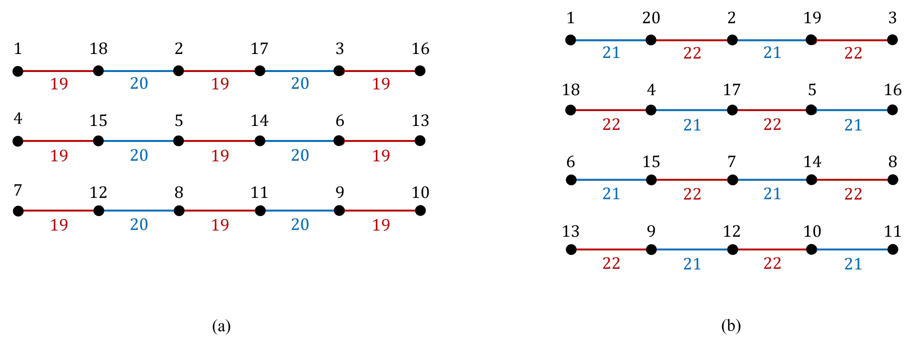


Figure 1: Local edge antimagic labeling of (a) $3P_6$ and (b) $4P_5$.

Results in disjoint graphs may also be found in star forests. For integers $m \geq 1$ and $n \geq 1$, let mS_n be a graph with vertex set

$$V(mS_n) = \{c_j, v_{i,j} \mid i \in [1,n], j \in [1,m]\}$$

and with edge set

$$E(mS_n) = \{c_j v_{i,j} \mid i \in [1,n], j \in [1,m]\}$$

The following result presents the local edge antimagic chromatic number for disjoint stars.

Proposition 3. *Let $m \geq 1$ and $n \geq 3$ be integers. We have $\chi'_{lea}(mS_n) = n$.*

Proof. Let $G = mS_n$. A vertex labeling $f : V(G) \to [1, m(n+1)]$ of mS_n is defined as follows

$$\begin{aligned} f(c_j) &= j \\ f(v_{i,j}) &= m(i+1) - j + 1 \end{aligned}$$

the weights of the edges are

$$w(c_j v_{i,j}) = m(i+1) + 1$$

Therefore, $\chi'_{lea}(mS_n) \geq n$. Since $n = \Delta(mS_n) \leq \chi'_{lea}(mS_n)$, we conclude that $\chi'_{lea}(mS_n) = n$. □

Preceding results may be applied to determine bounds of local edge antimagic chromatic number for any graphs G.

Theorem 5. *Let H be a subgraph of G. We have*

$$\chi'_{lea}(G) \leq |E(G)| - |E(H)| + \chi'_{lea}(H)$$

In addition, it follows that

$$\chi'_{lea}(G) \leq |E(G)| - \max_{F \subset G}\{|E(F)| - \chi'_{lea}(F)\}$$

Proof. Let g be a local edge antimagic labeling of H which uses $\chi'_{lea}(H)$ colors. Define a bijection $f : |V(G)| \to [1, |V(G)|]$ such that

$$f(v) = g(v)$$

for $v \in V(H)$ and any mapping (such that f is bijection) for the rest of vertices in G. By Proposition 1, f is a local edge antimagic labeling. Therefore,

- the number of edge colors induced in H is exactly $\chi'_{lea}(H)$,
- the number of edge colors induced in $G - E(H)$ is at most $|E(G)| - |E(H)|$.

This implies

$$\chi'_{lea}(G) \leq |E(G)| - |E(H)| + \chi'_{lea}(H)$$

Since H is chosen randomly, then we have

$$\chi'_{lea}(G) \leq |E(G)| - \max_{F \subset G}\{|E(F)| - \chi'_{lea}(F)\}$$

Hence, the theorem holds. □

For a graph G, let $m = \alpha'(G)$. It may be seen that $\chi'_{lea}(mK_2) = 1$ due to Theorem 2. Therefore, by choosing $H = mK_2$ in Theorem 5, we have a relationship between $\chi'_{lea}(G)$ and $\alpha'(G)$.

Corollary 3. *For any graph G, we have $\chi'_{lea}(G) + \alpha'(G) \leq |E(G)| + 1$.*

In some cases, the colors induced may be less than $|E(G)| + 1$. The illustration for this occurence is depicted in Figure 2.

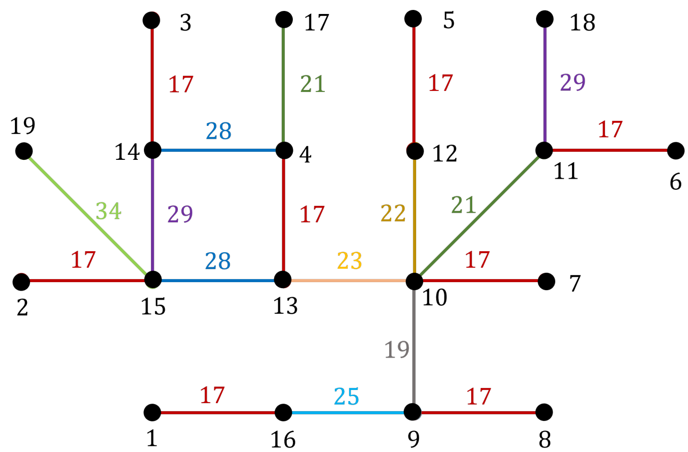


Figure 2: A graph G with $\chi'_{lea}(G) \leq 9$.

Some graphs which satisfy the equality in the preceding corollary are path with 4 vertices P_4 and stars S_n for $n \geq 2$ [1]. Moreover, we may also choose H to be some disjoint stars as in Proposition 3 or disjoint paths as in Theorem 4 to Theorem 5. In this case, we have the following corollaries.

Corollary 4. *Let G be a graph and q_n be the largest integer such that $q_n P_n \subseteq G$ for $n \geq 3$. Then,*

$$\chi'_{lea}(G) \leq |E(G)| + 2 - \max_{n \geq 3}\{q_n(n-1)\}$$

Corollary 5. *Let G be a graph and q_n be the largest integer such that $q_n S_n \subseteq G$ for $n \geq 3$. Then,*

$$\chi'_{lea}(G) \leq |E(G)| - \max_{n \geq 3}\{(q_n - 1)n\}$$

In particular, consider $P_4 \triangleright_v P_n$ with v being a leaf in P_n. Clearly, $\Delta(P_4 \triangleright_v P_n) = 3$. Moreover, it may be seen that $2P_{2n} \subseteq P_4 \triangleright_v P_n$. Then,

$$\chi'_{lea}(P_4 \triangleright_v P_n) \leq (4n-1) + 2 - 2(2n-1) \leq 3$$

due to Corollary 4. Consequently, $\chi'_{lea}(P_4 \triangleright_v P_n) = 3$. This is a counter example of Theorem 2.1 and Theorem 2.2 in [2] proving that those results are incorrect.

In general, for graphs G, and H a subgraph of G, let q_H be the largest integer such that $q_H H \subseteq G$. The largest number of $q_H|E(H)|$ may vary for each G. Indeed, for some integer m and vertex $v \in V(H)$, the graph $K_{1,m} \triangleright_v H$ satisfies the equality in Theorem 5.

Theorem 6. *Let m be a positive integer. Let H be a graph with $\Delta(H) = \chi'_{lea}(H)$ and $v \in V(H)$ with $\deg(v) = \Delta(H)$. If $G \cong K_{1,m} \rhd_v H$, then $\chi'_{lea}(G) = m + \Delta(H)$.*

Proof. Let $V(K_{1,m}) = \{c, v_i \mid i \in [1, m]\}$ with c being the center of $K_{1,m}$. By the construction of $G \cong K_{1,m} \rhd_v H$, it may be observed that $|E(G)| = m + (m + 1)|E(H)|$ and $\Delta(G) = m + \Delta(H)$. It follows that $\chi'_{lea}(G) \geq m + \Delta(H)$.

For $i \in [1, m + 1]$, let $H^{(i)}$ be a subgraph of G with chosen vertices as follows

$$
\begin{aligned}
H^{(1)} &= \{(v_1, u) \mid u \in V(H)\},\\
H^{(2)} &= \{(v_2, u) \mid u \in V(H)\},\\
&\vdots\\
H^{(m)} &= \{(v_m, u) \mid u \in V(H)\},\\
H^{(m+1)} &= \{(c, u) \mid u \in V(H)\}.
\end{aligned}
$$

Hence, there are $m+1$ subgraph of H in G. $q_k H \subseteq K_{1,m} \rhd_o H$. Clearly,

$$(m + 1)|E(H)| - \Delta(H) \leq \max_{F \subset G}\{|E(F)| - \chi'_{lea}(F)\}$$

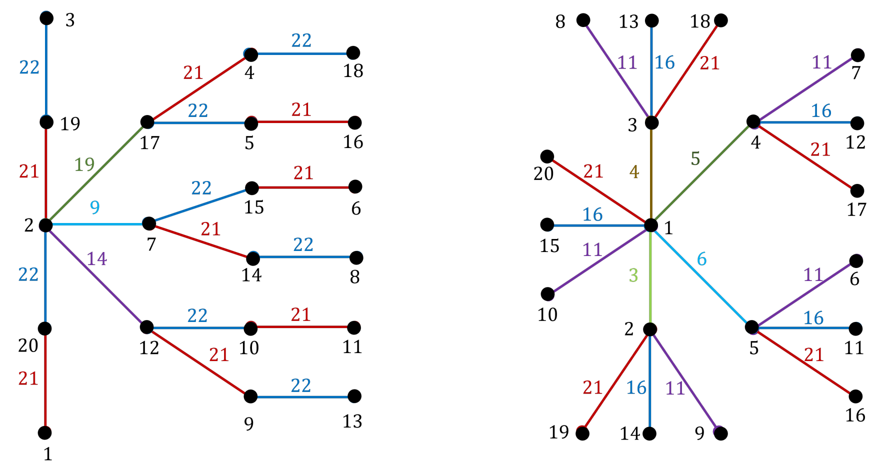
By Theorem 5, we have

$$
\begin{aligned}
\chi'_{lea}(G) &\leq |E(G)| - \max\nolimits_{F\subset G}\{|E(F)| - \chi'_{lea}(F)\}\\
&\leq m + (m + 1)|E(H)| - ((m + 1)|E(H)| - \Delta(H))\\
&\leq m + \Delta(H)
\end{aligned}
$$

It may be concluded that $\chi'_{lea}(G) = m + \Delta(H)$. $\square$

This implies that the bound in Theorem 5 is sharp. Some graph G which satisfies $\Delta(G) = \chi'_{lea}(G)$ are $G \cong mP_n$ and $G \cong mS_n$ for some integers m, n. For instance, we present $K_{1,3} \rhd_v P_5$ for some $v \in V(P_5)$ which has $\chi'_{lea}(K_{1,m} \rhd_v P_5) = 5$ and $K_{1,4} \rhd_v K_{1,3}$ for some $v \in V(K_{1,3})$ which has $\chi'_{lea}(K_{1,4} \rhd_v K_{1,3}) = 7$ in Figure 3. Motivated by these results, we proposed a problem shown below.

Problem 1. *Characterize graphs G with $\chi'_{lea}(G) = \Delta(G)$.*



(a) (b)

Figure 3: Graphs with (a) $\chi'_{lea}(K_{1,m} \triangleright_v P_5) = 5$ and (b) $\chi'_{lea}(K_{1,4} \triangleright_v K_{1,3}) = 7$.

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