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How to draw the graphs of the Exponential, Logistic, and Gaussian functions with pencil and ruler in an accurate way

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Abstract

In this work, we will give a novel method to construct a continuous approximation of the Exponential, Logistic, and Gaussian functions that allow us to do a handmade drawing of their graphs for which there is no accuracy of drawing at elementary levels (even at advanced ones!). This method arises from solving the elementary ordinary differential equation $x'(t) = ax(t)$ combined with a suitable piecewise constant argument. The proposed approximation will allow us to generate several numerical schemes in an elementary way, generalizing the classical ones as, Euler's schemes. No sophisticated mathematical tools are needed.

Keywords: *Piecewise Constant Argument, Numerical integration, approximation of solutions, Exponential function.*

MSC (2020): *34K06, 39A06, 65L20, 65L70.*

1. Introduction

It is so strange when someone ask for drawing the graphs of the *Exponential function* $f(t) = e^t$ and the *Logistic function* $g(t) = \dfrac{1}{1+e^{-t}}$

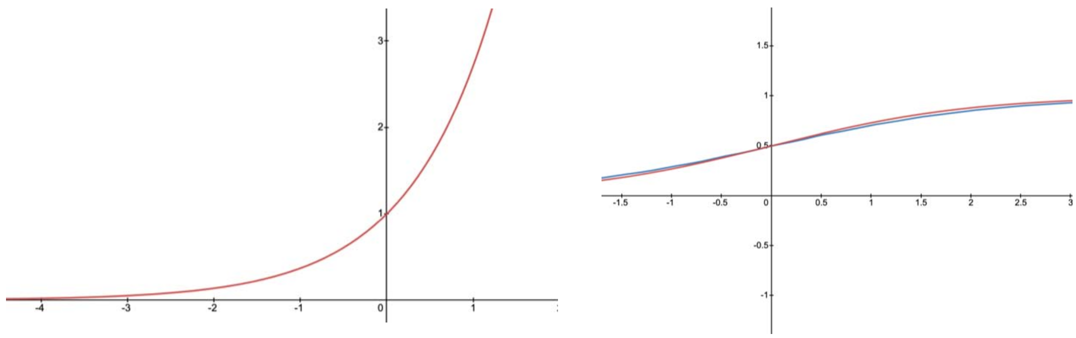

Figure 1: Graphs of $f(t) = e^t$ and $g(t) = 1/(1+e^{-t})$.

Actually, we just know some guide points and some asymptotic behavior for drawing such functions. The mentioned points and behaviors are $(0, 1)$, $(1, e)$ and $f(t) \to 0$ as $t \to -\infty$, $f(t) \to \infty$ as $t \to \infty$ for the case of the exponential function and $(0, 1)$, $g(t) \to 0$ as $t \to -\infty$, $g(t) \to 1$ as $t \to \infty$, for the Logistic function. In simple words they know almost nothing about to sketch an acceptable graph or such functions, because of their lack of basic knowledge. Using some elementary calculus, we can know how to draw the graph of a function locally using some approximations given by, for example, Taylor's series with a certain error. In the case of the exponential function, from elementary calculus, a good approximation near $t = 0$ is given by

$$\lim_{t\to 0} \frac{e^t - 1}{t} = 1. \tag{1.1}$$

The expression given in (1.1) allows us to write

$$e^t \approx 1 + t, \qquad \text{as } t \text{ is small enough.} \tag{1.2}$$

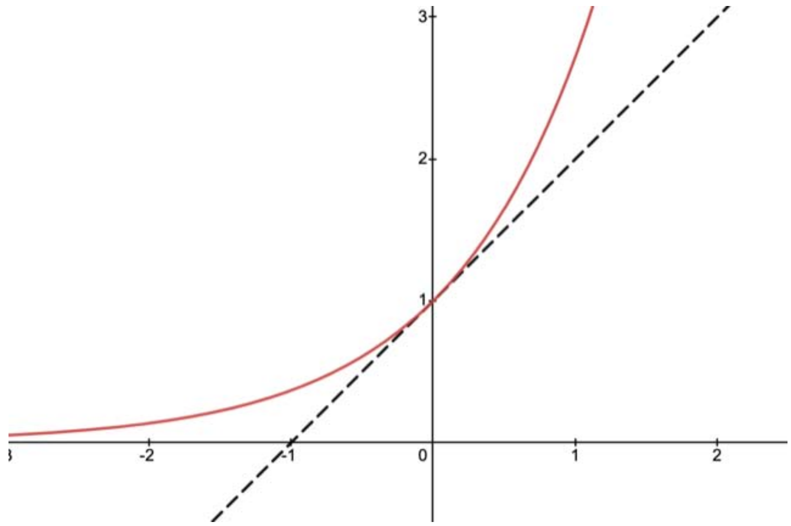

Figure 2: $f(t) = e^t$ vs $r(t) = 1 + t$.

The lack of this situation is that we have only a local approximation of the function $f(t) = e^t$, so the graph is restricted just in a neighborhood of $t = 0$. Due to this reason, using a such local approximation is not a very useful method for drawing the exponential function.

Aiming to fill this gap, we will propose a very simple and elementary numerical scheme for handmade drawings of the Exponential and Logistic functions using straight lines.

2. Preliminaries

In the following, if $x(t)$ is a function, we denote its derivative by the symbols $\frac{dx}{dt}$ or $x'(t)$.

Now, we will show the main tools used in this work: first-order ordinary differential equations with constant coefficients and the Logistic ordinary differential equation.

2.1. First order ordinary differential equations with constant coefficients

The cleric and scholar Thomas R. Malthus claimed that under favorable conditions, a human population experiences exponential growth. This can be explained by the differential equation

$$x'(t) = ax(t), \quad a \in \mathbf{R}, \tag{2.1}$$

where $x(t)$ is the size of the population at time t. It is not hard to see the exponential nature of $x(t)$. By integrating it on $[t_0, t]$, we obtain that

$$x(t) = x(t_0)e^{a(t-t_0)},$$

where t_0 is an initial value for the time $t = t_0$ and $x(t_0)$ is the initial population.

Notice that a is the constant of proportionality or per capita rate of the population ($a = x'/x$). In other words, the rate of growth of a population is directly proportional to its magnitude. It has not only been used for human populations but for many kinds of living organisms. Many authors use a as the difference between the per capita rates of birth b and of death d. So,

$$x'(t) = (b - d)x(t),$$

Notice that we can consider that the rate of death is directly proportional to the total population, i.e $d = mx(t)$. Then we get

$$x'(t) = bx(t) - mx(t)^2.$$

In the following, we study these kinds of equations. If $m \cdot x(t) > 0$ is small enough for t in a suitable small interval $[t_0, t_1]$, we have that (2.2) can be approximated to

$$x'(t) \approx ax(t)$$

and it would be reasonably expected that its solution satisfies that $x(t) \approx x(t_0)e^{a(t-t_0)}$ for t is in $[t_0, t_1]$.

In 1838, Pierre Franois Verhulst published the equation

$$y'(t) = ay(t) - cy(t)^2,$$

where $y(t)$ represents number of individuals at time t, a the rate of growth and α the density. He called, in 1845, the solution of the equation a Logistic equation. Subsequently, this equation was popularized by Raymond Pearl and Lowell Reed this equation with $c = \dfrac{a}{K}$, where the maximum number of individuals that the environment can support. So, the *logistic equation* is

$$y'(t) = ay(t)\left(1 - \frac{y(t)}{K}\right). \tag{2.2}$$

The last equation is called *Logistic differential equation*, where $a, K > 0$ for biological reasons. This example will be very important in the rest of the work.
Before solving the equation (2.2), we deal with

$$y'(t) = ay(t) + c, \quad a, c \in \mathbf{R} \tag{2.3}$$

Using the well-known variation of parameters formula, the solution of (2.3) is

$$y(t) = e^{a(t-t_0)}y(t_0) - \frac{c}{a}\left(1 - e^{-a(t-t_0)}\right). \tag{2.4}$$

Now, (2.2) corresponds to a differential equation of the family of Bernoulli's type. Hence, we make the change of variable of the form $u(t) = \dfrac{1}{y(t)}$. By differentiation we have $y'(t) = -\dfrac{1}{u^2(t)}u'(t)$. So, by applying the last change of variables to (2.2), we have

$$-\frac{1}{u^2(t)}u'(t) - \frac{a}{u(t)} + \frac{a}{K}\frac{1}{u^2(t)} = 0,$$

i.e.

$$u'(t) = -au(t) + \frac{a}{K}. \tag{2.5}$$

Notice that (2.5), is a particular case of 2.3. Then we can apply to (2.5) the constant variation formula (2.4). So,

$$u(t) = e^{-a(t-t_0)}u(t_0) + \frac{1}{K}\left(1 - e^{-a(t-t_0)}\right).$$

For the solution of (2.2), we have

$$y(t) = \frac{y(t_0)Ke^{a(t-t_0)}}{K + y(t_0)(e^{a(t-t_0)} - 1)}, \tag{2.6}$$

since $y(t) = \dfrac{1}{u(t)}$.

When $0 < y(t_0) < K$, $y(t)$ is an increasing function and $y(t) \to K$ as $t \to +\infty$. In other words, the population grows and it can be so close to K as we want if t is as large as we need.

Particularly, if $t_0 = 0$, $a = K = 1$, and $y(0) = 1/2$ we have the Logistic function $g(t) = \dfrac{e^t}{e^t + 1}$ or $g(t) = \dfrac{1}{1 + e^{-t}}$. Note that $g(t)$ is increasing and $g(t) \to 1$ as $t \to +\infty$. If we indicate that $g(t)$ is the number of millions of individuals, $y(0) = 500$ individuals and the maximum number of individuals that the environment can support is one million individuals.

For the reader interested in ordinary differential equations, we very recommend the excellent elementary introductions given by [1] and [2].

2.2. Piecewise constant argument of generalized type

The reader may be familiarized with the function $[\cdot]$, which denotes the floor or greatest integer function. Most exactly, $[t]$ gives as output the greatest integer less than or equal to t. This is equivalent to say

$$[t] = n \Leftrightarrow n \leq t < n + 1, \quad n \in \mathbf{Z}.$$

It is not difficult to see that the greatest integer function has jump discontinuities at $\mathbf{Z}$, the set of integer numbers.
Now, we can see that

$$[t/h]\, h = nh \Rightarrow nh \leq t < (n+1)\, h, \quad \forall n \in \mathbf{Z}, h > 0, \tag{2.7}$$

and (2.7) has also jump discontinuities at points $\{nh : n \in \mathbf{Z}\}$. The greatest integer function is an example of a family of locally constant functions called **piecewise constant functions.** More precisely, $\gamma(t)$ is a piecewise constant function, if there are sequences of real numbers $(t_n)_{n\in\mathbf{Z}}$ and $(\zeta_n)_{n\in\mathbf{Z}}$ such that $t_n < t_{n+1}\,, \forall n \in \mathbf{Z},\ \lim\limits_{n\to\pm\infty} t_n = \pm\infty$ and

$$\gamma(t) = \zeta_{n(t)} \in [t_{n(t)}, t_{n(t)+1}], \quad \text{if } t \in I_{n(t)} = \Big[t_{n(t)}, t_{n(t)+1}\Big)\,.$$

In this work, we use the notation $n(t)$ for the unique integer number such that $t \in I_{n(t)} = [t_{n(t)}, t_{n(t)+1})$. Below, we will denote $n(t)$ as n, except in the situations where we need to clarify the dependence of t.

In simple words, $\gamma(t)$ is a function that is constant in every set of some family of disjoint intervals $(I_n)_{n\in\mathbf{Z}}$ which cover the field of real numbers; this is the reason why the use of the name "piecewise constant"; every interval I_n is a "piece" of $\mathbf{R}$.

The use of such γ divides every interval I_n into two parts: an advanced part and an retarded one with respect to the function $f(t) = t$, i.e $I_n = I_n^+ \bigcup I_n^-$, where

$$I_n^+ = [t_n, \zeta_n] \text{ and } I_n^- = [\zeta_n, t_{n+1}).$$

As we have seen before, $\gamma(t) = [t]$ has a constant value in every interval of the form $I_n = [n, n+1)$. In this case $t_n = \zeta_n = n$ and it has discontinuities at $t = n$, where $n \in \mathbf{Z}$.
Consider now, the step function

$$\gamma(t) = \left[\frac{t}{h}\right] h + \beta h, \tag{2.8}$$

where $\beta \in [0, 1]$ and $h > 0$.

This step function is constant, with value $\gamma(t) = (n + \beta)h$ in every interval of the form $I_{n(t)} = [nh, (n + 1)h)$.
If $t \in I_n$, then

$$\begin{aligned} \gamma(t) - t &\geq 0 \Leftrightarrow t \leq (n + \beta)h \\ \gamma(t) - t &\leq 0 \Leftrightarrow t \geq (n + \beta)h. \end{aligned}$$

So, $I_n^+ = [nh, (n + \beta)h)$ and $I_n^- = [(n + \beta)h, (n + 1)h)$.

Remark 1.

1. *If $\beta = 0$, then $I_n^+ = \phi$ and if $\beta = 1$, then $I_n^- = \phi$.*

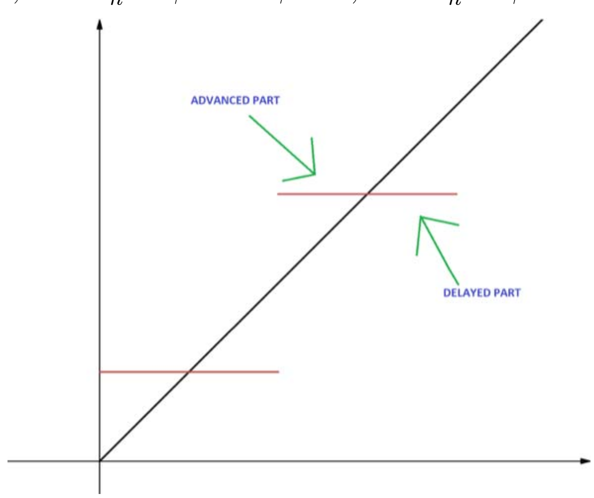

Figure 3: $\gamma(t) = [t/h]h + \beta h$ with $h = 0.5 = \beta$ vs $f(t) = t$.

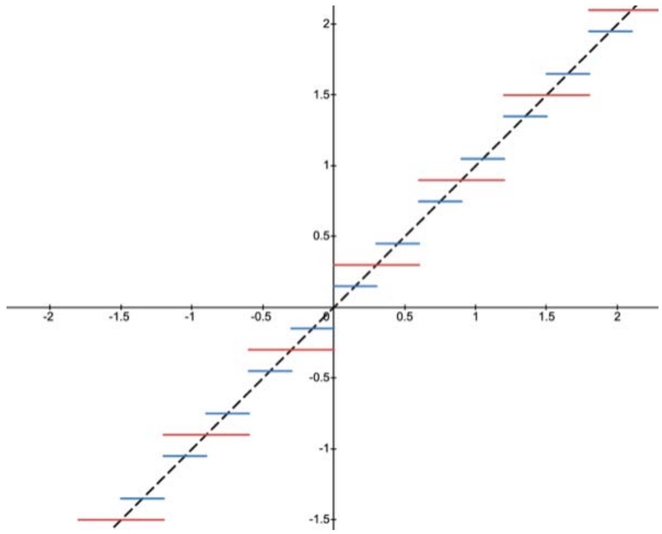

Figure 4: Approximation of the identity function by the piecewise constant argument $\gamma(t) = [t/h]h + \beta h$ with $\beta = 0.5$, $h = 0.3$ and $h = 0.6$
h=0.6 h=0.3 f(t)=t.

2.3. Differential equations with a piecewise constant argument (DEPCA)

In the '70s, the Ukrainian mathematician A.D.Myshkis proposed a new type of differential equations

$$x'(t) = f(t, x(t), x(\rho(t)));$$

where $\rho(t)$ corresponds to particular cases of piecewise constant functions or deviated arguments as, for example, $\rho(t) = [t]$ (see [3]). These equations are called **Differential Equations with Piecewise Constant Arguments** (in short DEPCA). The exhaustive study of this type of equation began in the '80s with the works of *S. Busenberg* and *K.L. Cooke.* Those authors worked DEPCA in models of vertically transmitted diseases (see [3]). Subsequently, new authors applied DEPCA to several branches of knowledge, for example, ecology, medicine, and engineering (see [5], [6]).
In the 2000s, the also Ukrainian mathematician M.U. Akhmet generalized the Myshkis' works by defining the systems

$$x'(t) = f(t, x(t), x(\gamma(t))), \tag{2.9}$$

where $\gamma(t)$ is a **piecewise constant argument of generalized type**. These equations were called Differential Equations with Piecewise Constant

Argument of Generalized Type (in short DEPCAG). Now, we will give a definition of a solution of a DEPCAG:

Definition 1. *A function $x(t)$ is understood to be a solution of the DEPCAG (2.9) if:*

(a) *$x(t)$ is continuous on $\mathbf{R}_0^+$. In particular, x is continuous on every interval I_n, $\forall n \in \mathbf{N}_0$.*

(b) *The derivative $\dfrac{dx}{dt}$ of x exists with the possible exception in $t = t_n$ for $n \in \mathbf{N}_0$, where the unilateral derivative exists;*

(c) *On each interval I_n, the ordinary differential equation*

$$\frac{dx}{dt} = f(t, x(t), x(\zeta_n))$$

is satisfied, where $\gamma(t) = \xi_n$, $\forall t \in I_n$.

DEPCAGs are quite special, since they have continuous solutions, even though $\gamma(t)$ is discontinuous. At both ends of each constancy interval of γ, a recursive law is produced and it defines a finite difference equation. Due to this attribute, DEPCAGs are also called Hybrid Equations. This means that they combine discrete and continuous dynamics.

2.3.1. An elementary and illustrative example of DEPCA

Consider the following DEPCA

$$x'(t) = ax([t]), \tag{2.10}$$

with $a \in \mathbf{R}$. If $t \in [n, n+1)$ for some $n \in \mathbf{Z}$, we can rewrite last equation as

$$x'(t) = ax(n). \tag{2.11}$$

For simplicity, we will assume $t_0 = 0$. Then, integrating on $[n, n+1)$ from n to t is easy to see that

$$x(t) = x(n)(1 + a(t - n)). \tag{2.12}$$

Assuming continuity at $t = n + 1$ we can see that

$$x(n+1) = (1 + a)x(n).$$

This is a so-called finite-difference equation. It also can be easily solved by simple induction getting

$$x(n) = (1+a)^n x(0). \tag{2.13}$$

Now, we can replace (2.13) in (2.12) in order to solve it. Hence we have

$$x(t) = (1+a)^{[t]}(1+a(t-[t]))x(0). \tag{2.14}$$

From (2.14) taking into account, for example, $|1+a| < 1$, we can deduce the behavior of the solutions. The case $a = -1$ is left as an exercise. Interested readers in Difference equations can see the excellent book [7] and for DEPCA we recommend [8, 9].

Behavior of solutions	**Condition**
$x(t) \to 0$ exponentially as $t \to +\infty$	$-2 < a < 0$
$x(t) \to 0$ exponentially and oscillatory as $t \to +\infty$	$-2 < a < -1$
$x(t)$ is oscillatory	$a < -1$
$x(t)$ is periodic	$a = -2$
$x(t) = x(0)$ is constant	$a = 0$
$x(t) \to +\infty$ exponentially as $t \to +\infty$	$a > 0$
$x(t) \to +\infty$ exponentially as $t \to +\infty$	$-2 \neq a < -1$, $t = 2n$, $n \in \mathbf{Z}$.
$x(t) \to -\infty$ exponentially as $t \to +\infty$	$-2 \neq a < -1$, $t = 2n+1$, $n \in \mathbf{Z}$.

Table 2.1: Behavior of solutions of (2.14)

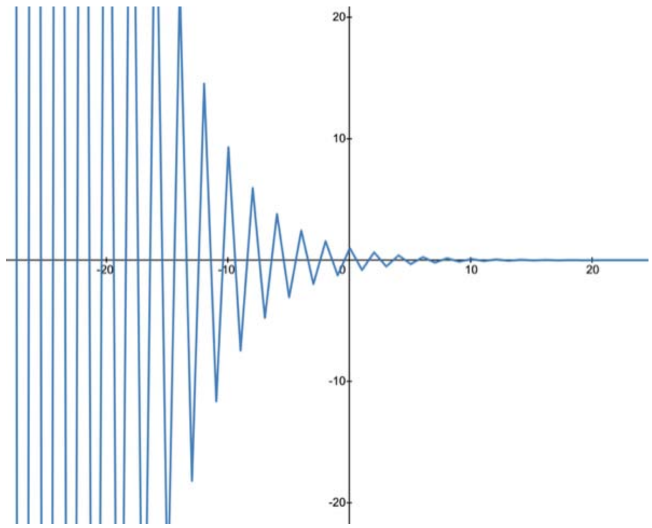

Figure 5: Solution of (2.14) with $a = -1.8$.

2.4. Approximation of solutions of differential equations

A simple and natural question arises at this moment: what happens if we can not solve a differential equation? If there is no known method for solving it, we can construct approximations of the solutions. This is the starting point of the motivation for this work.

The next step is to replace the original equation with a simpler equation that involves only arithmetic operations. In this way, we can obtain an approximate solution by means of paper-and-pencil calculations, with a slide rule.

Consider now the following ordinary differential equation

$$x'(t) = g(t, x(t)),\ x(\tau) = x_0 \tag{2.15}$$

Finding an explicit solution $x(t)$ for (2.15) is, in general, an impossible task. This is the starting point for approximating solutions of differential equations like (2.15).

An amazing idea was conceived by L. Euler known as Euler's Polygonal Paths, published in his work called *Institutiones calculi integralis* (1768-1770).

Euler proposed the following approximation scheme for the solution of (2.15):

$$
\begin{aligned}
\varphi(\tau) &= x_0, \\
\varphi(t) &= \varphi(t_k) + g(t_k, \varphi(t_k))(t - t_k), \qquad (2.16)\\
t_k &\leq t \leq t_{k+1}, \qquad k = 1, 2, \ldots n-1, \quad \forall t \in [t_0, t_n],\ \tau = t_0,
\end{aligned}
$$

or its discrete version

$$
\begin{aligned}
\varphi(\tau) &= x_0, \\
\varphi(t_{k+1}) &= \varphi(t_k) + g(t_k, \varphi(t_k))(t_{k+1} - t_k), \\
t_k &\leq t \leq t_{k+1}, \qquad k = 1, 2, \ldots n-1, \quad \forall t_k \in [t_0, t_n],\ \tau = t_0.
\end{aligned} \tag{2.17}
$$

For more details, the reader can see the first pages of [10]).

2.4.1. A useful property about the approximating piecewise constant argument

Next, we will illustrate the importance of the piecewise constant argument defined in (2.8):

$$\gamma(t) = \left[\frac{t}{h}\right] h + \beta h, \quad h > 0,\ \beta \in [0, 1].$$

It has the very interesting property

$$\left[\frac{t}{h}\right] h + \beta h \to t, \qquad \text{uniformly as } h \to 0.$$

In fact,

$$\left| t - \left(\left[\frac{t}{h}\right] h + \beta h \right) \right| = |t - (k + \beta)h| < (1 - \beta)h$$

if $t \in I_k = [kh, (k+1)h]$. Hence

$$\lim_{h \to 0} |t - \gamma(t)| = 0.$$

Let the case when $\beta = 0$. Following the idea of Euler, we consider the following DEPCA:

$$
\begin{aligned}
x' &= g\left([t/h]\, h, x\left([t/h]\, h\right)\right), \qquad t \in [nh, (n+1)\, h), \\
x_0 &= x(nh),
\end{aligned} \tag{2.18}
$$

where $n \in \mathbf{Z}$ and $h > 0$ is fixed. h is known as discretization.
If $t \in I_n = [nh, (n+1)h]$, we have that (2.18) can be rewritten as

$$x'(t) = g\left(nh, x(nh)\right).$$

Integrating the last expression on I_n we have

$$x(t) = x(nh) + g\left(nh, x(nh)\right)(t - nh). \tag{2.19}$$

Finally, assuming continuity at $t = (n+1)h$ we get

$$x((n+1)h) = x(nh) + hg\left(nh, x(nh)\right). \tag{2.20}$$

Remark 2. *DEPCA equation (2.18) can be considered as an approximation for solutions of (2.15), recovering the scheme proposed by Euler in (2.16) and (2.17). Notice the similarity between (2.16)-(2.17) and (2.19)-(2.20). This is a very remarkable fact. Hence, the piecewise constant argument can be used widely in approximation theory.*

2.4.2. Some new approximation schemes using $\gamma(t) = [t/h]h + \beta h$

In general, by using $\gamma(t)$ as defined on (2.8), we can construct the following approximating system

$$\varphi' = f\left([t/h]\, h + \beta h, \varphi\left([t/h]\, h + \beta h\right)\right), \qquad t \in [nh, (n+1)\, h)$$

I.e, if $t \in [nh, (n+1)h[$, we have

$$\varphi'(t) = f\left((n+\beta)h, \varphi\left((n+\beta)h\right)\right), \qquad t \in [nh, (n+1)\, h). \tag{2.21}$$

Integrating over the advanced and delayed intervals, we have that for $t \in I_n^+ = [nh, (n+\beta)\, h)$

$$\varphi(t) = \varphi(nh) + (t - nh) f\left((n+\beta)h, \varphi\left((n+\beta)h\right)\right),$$

Applying continuity at $t = (n+\beta)h$ we have

$$\varphi((n+\beta)h) = \varphi(nh) + (\beta h) f\left((n+\beta)h, \varphi\left((n+\beta)h\right)\right), \tag{2.22}$$

In the same way for the delayed interval, I.e $t \in I_n^- = t \in [(n+\beta)h, (n+1)\, h)$,

$$\varphi(t) = \varphi((n+\beta)h) + (t - (n+\beta)h) f\left((n+\beta)h, \varphi\left((n+\beta)h\right)\right)$$

Applying, again, continuity at $t=(n+1)h$

$$\varphi((n+1)h)=\varphi((n+\beta)h)+((1-\beta)h)f\left((n+\beta)h,\varphi\left((n+\beta)h\right)\right). \tag{2.23}$$

Using (2.22) in (2.23) we have

$$\varphi((n+1)h)=\varphi(nh)+hf\left((n+\beta)h,\varphi\left((n+\beta)h\right)\right).$$

The last expression allows us to construct several numerical approximation schemes for (2.15).

The interested reader in the approximation of solutions using piecewise constant arguments can see [11].

3. Main results

3.1. A model for drawing the Exponential function using straight lines

Let the following homogeneous linear differential equation

$$x'(t)=ax(t) \tag{3.1}$$

The function $f(t)=e^{at}$ is the only solution of the differential equation

$$x'(t)=ax(t),\quad a\in\mathbf{R},\, x(0)=1. \tag{3.2}$$

Hence, any numerical scheme for solving (3.2) will approximate with certain accuracy e^{at}.
Consider now the DEPCA

$$x_h'(t)=ax_h\left(\left[\tfrac{t}{h}\right]h+h\beta\right) \tag{3.3}$$

where $h>0$, $a\neq 0$ and $\beta\in[0,1]$.

Here, $I_n=[nh,(n+\beta)h]$, $I_n^+=[nh,(n+\beta)h]$ and $I_n^-=[(n+\beta)h,(n+1)h]$.

It seems very reasonable to think about a close relationship between solutions of (3.2) and (3.3), because $t-\left(\left[\dfrac{t}{h}\right]h+\beta h\right)\to 0$, as $h\to 0$. I,e we can wonder if the solutions of (3.2) are approximated by the solutions of (3.3) :

$$x_h'(t) = ax_h\left(\left[\frac{t}{h}\right]h + h\beta\right) \to x'(t) = ax, \qquad \text{as } h \to 0.$$

In the following, actually, we will show that the solutions of (3.1) can be approximated by the solutions of the DEPCA (3.3).

I.e, we will show that

$$\left(\frac{1+(1-\beta)ha}{1-\beta ha}\right)^{\left[\frac{t}{h}\right]} \approx e^{at}(1+ha) \tag{3.4}$$

when $h > 0$ is small.

The proposed scheme for drawing the function $f(t) = e^{at}$ with pencil-and-paper is given by the following theorem:

Theorem 1. *Let $h > 0$, $\beta \in [0,1]$, $a \in \mathbf{R}$ and*

$$1 - \beta ha \neq 0, \quad 1 + (1-\beta)ha \neq 0. \tag{3.5}$$

Then, the function $f(t) = e^{at}$ can be discretely approximated by

$$x_h(nh) = \left(\frac{1+(1-\beta)ha}{1-\beta ha}\right)^{n-n_0} x_h(n_0h), \quad n_0 = \left[\tfrac{t_0}{h}\right]. \tag{3.6}$$

for all $t \in \{nh : n \in \mathbf{Z}\}$.
Moreover, the continuous approximation for all $t \in [t_0, \infty)$ is given by

$$x_h(t) = \left(1 + \frac{a\left(t - \left[\frac{t}{h}\right]h\right)}{1-\beta ha}\right)\left(\frac{1+(1-\beta)ha}{1-\beta ha}\right)^{\left[\frac{t}{h}\right]-\left[\frac{t_0}{h}\right]} x_h(n_0h), \tag{3.7}$$

Proof. In the following, we will solve (3.3) in order to prove the theorem. First, integrating (3.3) on $t \in I_n^+ = [nh, (n+\beta)h]$ we get

$$x_h(t) = x_h(nh) + a(t-nh)x_h((n+\beta)h). \tag{3.8}$$

Assuming continuity at $t = (n+\beta)h$ we have
$x_h((n+\beta)h) = x_h(nh) + a\beta h \cdot x_h((n+\beta)h)$.

Then, if $1-\beta ha \neq 0$, we have

$$x_h((n+\beta)h) = \left(\frac{1}{1-\beta ha}\right) x_h(nh). \tag{3.9}$$

Now, analogously for $t \in I_n^- = [(n+\beta)h, (n+1)h]$ we get
$x_h(t) = x_h((n+\beta)h) + (t-(n+\beta)h)a \cdot x_h((n+\beta)h)$.
Assuming continuity at $t = (n+1)h$ we have
$x_h((n+1)h) = x_h((n+\beta)h) + (1-\beta)ha \cdot x_h((n+\beta)h)$.
I.e
$x_h((n+1)h) = (1+(1-\beta)ha)x_h((n+\beta)h)$.
Again, if $1+(1-\beta)ha \neq 0$ we have

$$x_h((n+\beta)h) = \left(\frac{1}{1+(1-\beta)ha}\right) x_h((n+1)h). \tag{3.10}$$

Now, by (3.9) and (3.10) we conclude that

$$x_h((n+1)h) = \left(\frac{1+(1-\beta)ha}{1-\beta ha}\right) x_h(nh). \tag{3.11}$$

Hence, we obtained a finite-differences equation, where

$$x_h(nh) = \left(\frac{1+(1-\beta)ha}{1-\beta ha}\right)^{n-n_0} x_h(n_0h), \quad n_0 = \left[\tfrac{t_0}{h}\right]. \tag{3.12}$$

The last expression corresponds to the discrete solution of (3.3). Finally, applying (3.9),(3.12) in (3.8) we have

$$x_h(t) = \left(1 + \frac{a\left(t - \left[\frac{t}{h}\right]h\right)}{1-\beta ha}\right)\left(\frac{1+(1-\beta)ha}{1-\beta ha}\right)^{\left[\frac{t}{h}\right]-\left[\frac{t_0}{h}\right]} x_h(n_0h), \tag{3.13}$$

with $h > 0$, $\beta \in [0,1]$ and

$$1-\beta ha \neq 0, \quad 1+(1-\beta)ha \neq 0. \tag{3.14}$$

The expressions given in (1.1) and (1.2) allow us to write
$e^{-a\beta h} \approx 1 - a\beta h \Rightarrow e^{a\beta h} \approx \dfrac{1}{1-\beta ah}, \qquad e^{(1-\beta)ah} \approx 1+(1-\beta)ah.$
Keeping in mind the solution of (3.3), it easy to see that
$\left(\dfrac{1+(1-\beta)ha}{1-\beta ha}\right)^{\left[\frac{t}{h}\right]} \approx \left(e^{a\beta h}e^{(1-\beta)ah}\right)^{\left[\frac{t}{h}\right]} = e^{a\beta\left[\frac{t}{h}\right]h}e^{(1-\beta)a\left[\frac{t}{h}\right]h}.$

for every fixed value of t.
Hence, the expression

$$\left(\frac{1+(1-\beta)ha}{1-\beta ha}\right)^{\left[\frac{t}{h}\right]}$$

which corresponds to the solution of (3.3) naturally approximates the solution of (3.1) uniformly on $[0,\infty)$. The last fact and the error of convergence can be seen as follows:

$$\begin{aligned}
\left|e^{at}-\left(\frac{1+(1-\beta)ha}{1-\beta ha}\right)^{\left[\frac{t}{h}\right]}\right| &\approx \left|e^{at}-e^{a\beta\left[\frac{t}{h}\right]h}e^{(1-\beta)a\left[\frac{t}{h}\right]h}\right| = \left|e^{at}-e^{a\left[\frac{t}{h}\right]h}\right| \\
e^{at}\left|1-e^{-a\left(t-\left[\frac{t}{h}\right]h\right)}\right| &\approx e^{at}\left|1-\left(1-a\left(t-\left[\frac{t}{h}\right]h\right)\right)\right| \qquad \approx \\
e^{at}|a|h &
\end{aligned}$$

(3.15)

for a fixed value of t, where $\left|t-\left[\frac{t}{h}\right]h\right| = |t-nh| < (n+1)h-nh = h, \quad t \in [nh,(n+1)h)$. $\square$

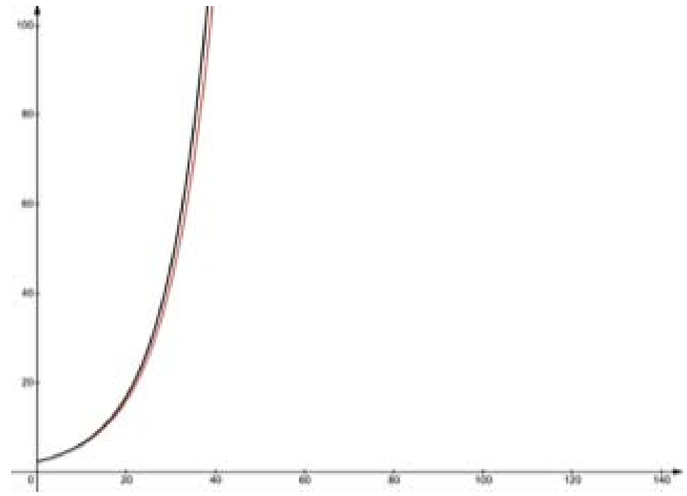

Figure 6: Approximation of the solutions of (3.3) by solutions of system (3.1) with $\beta = 0$, $a = 0.1$ and $h = 0.6$
Approximating system (3.3) Approximated system (3.1).

Remark 3. *As we have seen, the stability of the linear approximated system (represented by the coefficient $a \in \mathbf{R}$ in (3.2)) is very important in order to approximate the solutions uniformly in terms of the parameter $h > 0$ for $t \in [t_0, \infty[$, because the error of approximation grows exponentially if $a > 0$. Nevertheless, if $a \leq 0$ or $a = a(t) \in L^1[t_0, \infty[$*[1]*, we have uniform approximation over $[t_0, \infty[$.*

Remark 4. *Conditions given by (3.5) correspond to a particular case of more general existence conditions for linear DEPCA (see [9]). In our case, they are always satisfied with $h > 0$ small enough.*

As our main results, next, we will give the numerical scheme to draw the graph of the Exponential function $f(t) = e^t$ with a pencil and a ruler using straight lines:

Corollary 1. *Let $h > 0$, $\beta \in [0, 1]$, $a \in \mathbf{R}$ and $1 - \beta h \neq 0$.*
Then, the function $f(t) = e^t$ can be discretely approximated, for $t \geq 0$, by

$$x_h(nh) = \left(\frac{1 + (1 - \beta)h}{1 - \beta h} \right)^n,$$

[1] The space of all absolutely integrable real valued functions f defined on $[t_0, \infty)$. I.e such that satisfy $\int_{t_0}^{\infty} |f(s)| ds < \infty$.

for all $\{nh : n \in \mathbf{N}\}$, *where* $x_h(0) = 1$.
Moreover, the continuous approximation for all $t \in [0, \infty)$ *is given by*

$$x_h(t) = \left(1 + \frac{(t - \left[\frac{t}{h}\right] h)}{1 - \beta h}\right) \left(\frac{1 + (1 - \beta)h}{1 - \beta h}\right)^{\left[\frac{t}{h}\right]}.$$

In simple words, we can draw in accuracy and an elementary way the graph of $f(t) = e^t$ *on* $[t_0, \infty)$ *joining every point of the following expression end-to-end by a straight line:*

$$x_h(nh) = \left(\frac{1 + (1 - \beta)h}{1 - \beta h}\right)^{n - n_0} x_h(n_0 h), \quad \text{where } n_0 = \left[\tfrac{t_0}{h}\right], n = \left[\tfrac{t}{h}\right] \text{ and}$$

$x(n_0 h) = e^{n_0 h}$.

Remark 5. *Also, from the last expression we can recover some classical discrete numerical schemes of approximation of* $f(t) = e^t$ *for all* $t \in [t_0, \infty)$ *using some values of* β*:*

1. $\beta = 0$ **(Euler's classical delayed scheme)***:*

$$x_h(nh) = (1 + h)^{n - n_0}\, x_h(n_0 h), \tag{3.16}$$

2. $\beta = 1$ **(Euler's classical advanced scheme)***:*

$$x_h(nh) = \left(\frac{1}{1 - h}\right)^{n - n_0} x_h(n_0 h),$$

3. $\beta = \frac{1}{2}$ **(Trapezoidal classical scheme)***:*

$$x_h(nh) = \left(\frac{1 + \frac{h}{2}}{1 - \frac{h}{2}}\right)^{n - n_0} x_h(n_0 h),$$

where $n_0 = \left[\frac{t_0}{h}\right]$, $n = \left[\frac{t}{h}\right]$ *and* $x(n_0 h) = e^{n_0 h}$.

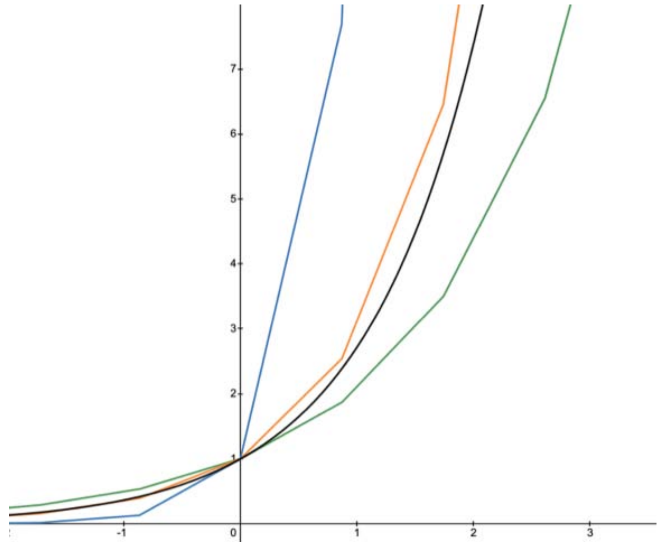

Figure 7: Approximations of $f(t) = e^t$ with $h = 0.87$ and $a = 1$. Euler delayed scheme $f(t) = e^t$, Euler advanced scheme trapezoidal scheme.

Remark 6. *By (3.15), we can see that if $a < 0$, we have a uniform approximation over the entire semiaxes $[t_0, \infty)$. The parameters* **h** *and* **a** *are very useful in order to get a good approximation because in this case, the approximation will depend on the time and the smallness of the parameter h.*
On the other hand, if $a > 0$ the approximation obtained is very robust and, for practical uses, is very useful, despite the fact that it will not converge uniformly over the entire semiaxes $[t_0, \infty)$, due to the exponential growth of the error of approximation.

3.2. A model for drawing the Logistic function using straight lines

In the following, we will conclude a way for drawing by pencil-and-ruler the Logistic function $g(t) = \frac{1}{1+e^{-t}}$ which is a solution of the logistic differential equation (2.2).

3.2.1. The construction of the handmade graph of the Logistic function

Now, we are going to present one of our main results concerning the approximation of solutions of (2.2) (i.e an approximation of (2.6)) using DEPCA:

$$u_h'(t) = -au_h\left(\left[\frac{t}{h}\right]h + \beta h\right) + \frac{a}{K}. \tag{3.17}$$

Theorem 2. *Let $h > 0$, $\beta \in [0,1]$, $a \in \mathbf{R}^+$ and*

$$1 + a\beta h \neq 0. \tag{3.18}$$

Then, the Logistic function (2.6) can be discretely approximated by $y_h(nh) = \dfrac{1}{u_h(nh)}$, where

$$u_h(nh) = \left(\frac{1 - a(1-\beta)h}{1 + a\beta h}\right)^{n-n_0} u_h(n_0h) + \frac{1}{K}\left(1 - \left(\frac{1 - a(1-\beta)h}{1 + a\beta h}\right)^{n-n_0}\right). \tag{3.19}$$

with $n \in \mathbf{Z}^+$. Moreover, the continuous approximation for all $t \in [0,\infty)$ is given by $y_h(t) = \dfrac{1}{u_h(t)}$, where

$$\begin{aligned} u_h(t) &= \left(\frac{1 + a\beta h - a(t - [t/h]h)}{1 + a\beta h}\right)\left(\frac{1 - a(1-\beta)h}{1 + a\beta h}\right)^{[t/h]-[t_0/h]} u_h([t_0/h]h) \\ &+ \frac{1}{K}\left(\frac{1 + a\beta h - a(t - [t/h]h)}{1 + a\beta h}\right)\left(1 - \left(\frac{1 - a(1-\beta)h}{1 + a\beta h}\right)^{[t/h]-[t_0/h]}\right) \\ &+ \frac{1}{K}\left(\frac{a(t - [t/h]h)}{1 + a\beta h}\right). \end{aligned} \tag{3.20}$$

Proof. If $t \in I_n = [nh, (n+1)h[$, (3.17) can be rewritten as

$$u_h'(t) = -au_h\left((n+\beta)h\right) + \frac{a}{K}. \tag{3.21}$$

Integrating last expression on $t \in I_n^+ = [nh, (n+\beta)h]$, we get

$$u_h(t) = -a(t - nh)u_h((n+\beta)h) + u(nh) + \frac{a}{K}(t - nh). \tag{3.22}$$

Now, assuming continuity at $t = (n+\beta)h$ we have

$$u_h((n+\beta)h) = -a\beta h u_h((n+\beta)h) + u(nh) + \frac{a\beta h}{K}.$$

Next, as (3.18) holds, we have

$$u_h((n+\beta)h) = \left(\frac{1}{1+a\beta h}\right)\left(u_h(nh) + \frac{a\beta h}{K}\right) \tag{3.23}$$

Similarly, integrating (3.21) over $I_n^- = [(n+\beta)h, (n+1)h[$ we have

$$u_h(t) = (1 - a(t-(n+\beta)h))u_h((n+\beta)h) + \frac{a}{K}(t-(n+\beta)h).$$

Assuming again continuity at $t = (n+1)h$, we have

$$u_h((n+1)h) = (1 - a(1-\beta)h)u_h((n+\beta)h) + \frac{a}{K}(1-\beta)h.$$

Applying (3.23) in last equation we get

$$u_h((n+1)h) = \left(\frac{1-a(1-\beta)h}{1+a\beta h}\right)u_h(nh) + \frac{ah}{K(1+a\beta h)}. \tag{3.24}$$

By induction, it is easy to see that the solution of the last finite difference equation is

$$u_h(nh) = \left(\frac{1-a(1-\beta)h}{1+a\beta h}\right)^{n-n_0} u_h(n_0h) + \frac{1}{K}\left(1 - \left(\frac{1-a(1-\beta)h}{1+a\beta h}\right)^{n-n_0}\right).$$

Now, applying (3.23) in (3.22) we get

$$u_h(t) = \left(1 - \frac{a(t-nh)}{1+a\beta h}\right)u_h(nh) + \frac{1}{K}\left(\frac{a(t-nh)}{1+a\beta h}\right)$$

Hence, applying (3.25) to last expression and considering $nh = [t/h]h$, we get

$$\begin{aligned} u_h(t) &= \left(\frac{1+a\beta h - a(t-[t/h]h)}{1+a\beta h}\right)\left(\frac{1-a(1-\beta)h}{1+a\beta h}\right)^{[t/h]-[t_0/h]} u_h(n_0h) \\ &+\frac{1}{K}\left(\frac{1+a\beta h - a(t-[t/h]h)}{1+a\beta h}\right)\left(1 - \left(\frac{1-a(1-\beta)h}{1+a\beta h}\right)^{[t/h]-[t_0/h]}\right) \\ &+\frac{1}{K}\left(\frac{a(t-[t/h]h)}{1+a\beta h}\right), \end{aligned} \tag{3.25}$$

Finally, in view of the following estimations

$$\begin{aligned}
&\left(\frac{1-a(1-\beta)h}{1+a\beta h}\right)^{[t/h]} && \approx e^{-a(1-\beta)[t/h]h}e^{-a\beta[t/h]h} \\
&= e^{-a[t/h]h} \to e^{-at} \text{ as } h \to 0, && \\
&\left(\frac{1+a\beta h - a\left(t-[t/h]\,h\right)}{1+a\beta h}\right) && \approx e^{a\beta h - a(t-[t/h]h)}e^{-a\beta h} \\
&= e^{-a(t-[t/h]h)} \to 1 \text{ as } h \to 0, && \\
(3.26)\quad &\frac{1}{K(1+a\beta h)}\left(t-\left[\frac{t}{h}\right]h\right) && \approx \frac{1}{K}e^{-a\beta h}h \to 0 \text{ as } h \to 0, \\
(3.27)\quad &1-\left(\frac{1-a(1-\beta)h}{1+a\beta h}\right)^{[t/h]} && \approx 1-e^{-a[\frac{t}{h}]h} \to 1-e^{-at} \text{ as } h \to 0,
\end{aligned}$$

by (3.28), comparing (3.26) with (2.6),
we have

$$|u(t)-u_h(t)| \to 0 \text{ as } h \to 0 \text{ uniformly in } t.$$

□

Finally, we can give the numerical scheme to draw the graph of the Logistic function $g(t) = 1/(1+e^{-t})$ with a pencil and a ruler using straight lines:

Corollary 2. *Let (2.6) with $a = 1$, $K = 1$, $t_0 = 0$, $y(0) = 1/2$ and $\beta = [0,1]$. From (3.20) we define the following numerical scheme of approximation of $g(t) = \dfrac{1}{1+e^{-t}}$ for $t \in [0,\infty)$ considering $y_h(t) = 1/u_h(t)$ and $u_h(0) = 2$, where*

$$\begin{aligned}
u_h(t) &= \left(\frac{1+\beta h-(t-[t/h]h)}{1+\beta h}\right)\left(\frac{1-(1-\beta)h}{1+\beta h}\right)^{[t/h]} u_h(0) \\
&+ \left(\frac{1+\beta h-(t-[t/h]h)}{1+\beta h}\right)\left(1-\left(\frac{1-(1-\beta)h}{1+\beta h}\right)^{[t/h]}\right) \\
&+ \left(\frac{(t-[t/h]h)}{1+\beta h}\right), \qquad (3.28)
\end{aligned}$$

or

$$u_h(nh) = \left(\frac{1-(1-\beta)h}{1+\beta h}\right)^{n-n_0} u_h(n_0h) + \left(1-\left(\frac{1-(1-\beta)h}{1+\beta h}\right)^{n-n_0}\right).$$

In simple words, we can draw in accuracy and an elementary way the graph of $g(t) = 1/(1+e^{-t})$ on $[t_0, \infty)$ joining every point of $u_h(nh)$ end-to-end by a straight line, where $n_0 = [\frac{t_0}{h}]$, $n = [\frac{t}{h}]$ and $u_h(n_0h) = \dfrac{1}{(1/(1+e^{-n_0h}))} = 1 + e^{-n_0h}$.

Remark 7. *Also, from the last expression we can recover some classical discrete numerical schemes of approximation of $g(t) = 1/(1+e^{-t})$ for all $t \in [t_0, \infty)$ using some values of β:*

1. $\beta = 0$ **(Euler's classical delayed scheme)**:
$$u_h(nh) = (1-h)^{n-n_0}\, u_h(n_0h) + \left(1 - (1-h)^{n-n_0}\right).$$
2. $\beta = 1$ **(Euler's classical advanced scheme)**:
$$u_h(nh) = \left(\frac{1}{1+h}\right)^{n-n_0} u_h(n_0h) + \left(1 - \left(\frac{1}{1+h}\right)^{n-n_0}\right).$$
3. $\beta = 1/2$ **(Trapezoidal classical scheme)**:
$$u_h(nh) = \left(\frac{1-\frac{h}{2}}{1+\frac{h}{2}}\right)^{n-n_0} u_h(n_0h) + \left(1 - \left(\frac{1-\frac{h}{2}}{1+\frac{h}{2}}\right)^{n-n_0}\right),$$

where $n_0 = [\frac{t_0}{h}]$, $n = [\frac{t}{h}]$ and $u_h(n_0h) = \dfrac{1}{1/(1+e^{-n_0h})} = 1 + e^{-n_0h}$.

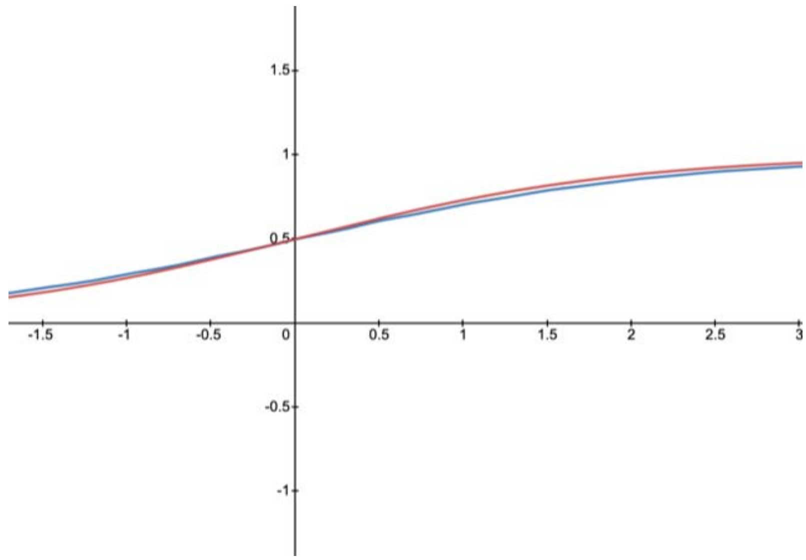

Figure 8: Solution of (2.2) approximated by solutions of (3.20) with $\beta = 0.8$, $a = 1$, $y(0) = 0.5$, $K = 1$ and $h = 0.55$

Approximating system (3.20) Approximated system (2.2).

4. A simple approximation scheme for $r(t) = e^{-t^2}$.

In this last and additional section, we will show a drawing scheme for $r(t) = e^{-t^2}$.
Consider the non-autonomous first-order ordinary differential equation

$$x'(t) = -2tx(t). \tag{4.1}$$

The last equation is very simple to solve. In fact, it belongs to the variable separable family. Assuming that $x(t) \neq 0, \forall t \in \mathbf{R}$, we have $\dfrac{x'(t)}{x(t)} = -2t$. Integrating last expression in $s \in [t_0, t]$, we get

$$\int_{t_0}^{t} \frac{x'(s)}{x(s)} ds = -2 \int_{t_0}^{t} s ds.$$

Then, easily we have $ln\left(\dfrac{x(t)}{x(t_0)}\right) = -(t^2 - t_0^2)$. Hence, the solution of (4.1) is

$$x(t) = Ke^{-t^2}, \quad K = x(t_0)e^{t_0^2}. \tag{4.2}$$

If we consider $x(t_0) = 1$ and $t_0 = 0$, we get the famous Gaussian function $r(t) = e^{-t^2}$. As done before, we will approximate the solution of (4.1) with $\gamma(t)$ considered in (2.8) in order to get a scheme for drawing its graph by straight lines.

Let the following DEPCA

$$z_h'(t) = -2t\, z_h\left(\left[\tfrac{t}{h}\right] h + \beta h\right). \tag{4.3}$$

If $t \in [nh, (n+1)h)$, the last expression can be rewritten as

$$z_h'(t) = -2t\, z_h\left((n+\beta)h\right).$$

By integrating for $t \in I_n^+ = [nh, (n+\beta)h)$ we get

$$z_h(t) = z_h(nh)(1 - (t^2 - (nh)^2)). \tag{4.4}$$

Assuming continuity at $t = (n+\beta)h$ we get

$$z_h((n+\beta)h) = z_h(nh)(1 - \beta h^2(2n+\beta)). \tag{4.5}$$

Now, integrating (4.3) for $t \in I_n^- = [(n+\beta)h, (n+1)h)$ we get

$$z_h(t) = z_h((n+\beta)h)(1 - (t^2 - ((n+\beta)h)^2)).$$

Again, assuming continuity at $t = (n+1)h$ we get

$$z_h((n+1)h) = z_h((n+\beta)h)(1 - (1-\beta)h^2(2n+1+\beta)). \tag{4.6}$$

Replacing (4.5) in (4.6) we get

$$z_h((n+1)h) = z_h(nh)(1-\beta h^2(2n+\beta))(1-(1-\beta)h^2(2n+1+\beta)). \tag{4.7}$$

The solution of the last finite-differences equation is

$$z_h(nh) = \left(\prod_{i=n_0}^{n-1} (1 - \beta h^2(2i+\beta))(1 - (1-\beta)h^2(2i+1+\beta))\right) z_h(n_0 h). \tag{4.8}$$

This expression corresponds to a non-autonomous finite-difference equation (see [7]). Applying (4.8) in (4.4) we have

$$z_h(t) = \left(\prod_{n=n_0}^{[t/h]-1} (1-\beta h^2(2i+\beta))(1-(1-\beta)h^2(2i+1+\beta))\right) \cdot \left(1-\left(t^2-\left(\left[\tfrac{t}{h}\right]h\right)^2\right)\right) z_h(n_0 h). \tag{4.9}$$

where $n = [\frac{t}{h}]$, $n_0 = [\frac{t_0}{h}]$ and $z_h(n_0h) = e^{-(n_0h)^2}$. Hence, if we consider $t_0 = 0$ and $z_h(0) = 1$, we get a drawing scheme for $r(t) = e^{-t^2}$ for $t \in [0, \infty)$:

$$z_h(t) = \left(\prod_{n=0}^{[t/h]-1} (1-\beta h^2(2i+\beta))(1-(1-\beta)h^2(2i+1+\beta))\right) \cdot \left(1-\left(t^2-\left(\left[\tfrac{t}{h}\right]h\right)^2\right)\right). \tag{4.10}$$

Finally, using (4.9) with $t_0 = -3$ and $\beta = 0$ we present the scheme for $r(t) = e^{-t^2}$:

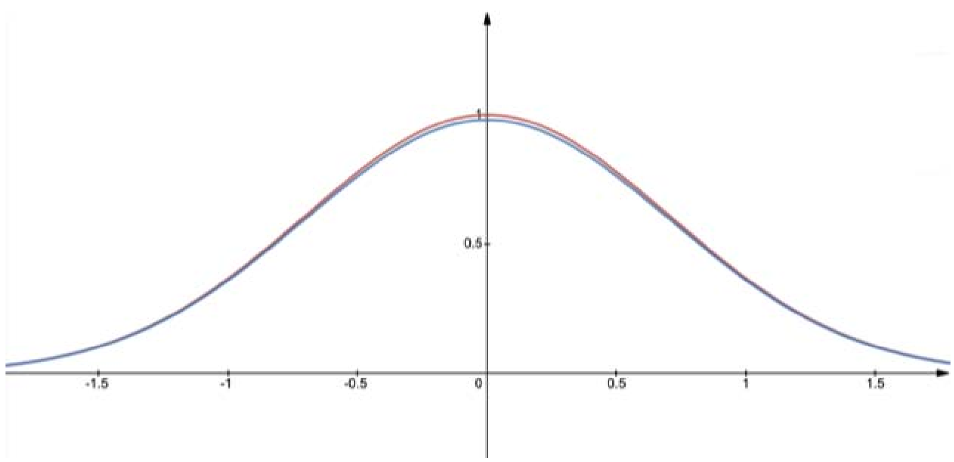

Figure 9: Drawing scheme using $h = 0.0011$.

Drawing scheme (4.10) Solution of (4.2).

Remark 8. *1. The case $\beta = 0$ shows a very simple discrete scheme of approximation (Euler's classical delayed scheme):*

$$z_h(nh) = \left(\prod_{i=0}^{n-1} (1 - h^2(2i+1)) \right).$$

2. *The drawing scheme presented corresponds to a uniform approximation of the solution of (4.2) for all $t \in [t_0, \infty)$ due to the asymptotic behavior of this function.*

3. *The rate of approximation is slower than the others examples presented in this work. This is because a term t^2 has to be defeated by $([t/h]h)^2$. Hence, very small values of h must be considered to sketch a good graph.*

4. *If we put our attention now on (4.8) with $\beta = 0$, $h = 1/1000$ and recalling that*

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} dx = 1; \quad \int_{-3}^{3} e^{-t^2} dt = 2 \int_{0}^{3} e^{-t^2} dt,$$

we compute the following estimation [2]

$$\int_{-3}^{3} e^{-t^2} dt \; - \; 2\int_0^3 \left(\prod_{n=0}^{[t/0.001]-1} (1 - 0.001^2(2n+1)) \right)$$
$$\left(1 - \left(t^2 - ([\tfrac{t}{0.001}]\, 0.001)^2\right)\right) dt$$
$$\approx 0.000665727.$$

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