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Hyers-Ulam-Rassias stability of some perturbed nonlinear second order ordinary differential equations

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Abstract

In this paper we investigate the Hyers-Ulam-Rassias stability of a perturbed nonlinear second order ordinary differential equation using Gronwall-Bellman-Bihari type integral inequalities. Further, the paper also investigates the Hyers-Ulam-Rassias stability of four different cases of a perturbed nonlinear second order differential equation.

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1. Introduction

There has been continuous interest in the investigation of Hyers-Ulam stability of both linear and nonlinear ordinary differential equations since Ulam [26] started with the stability of functional equation in 1940 during his talk before a mathematical colloquium at the University of Wincosin, Maidison. Hyers continued where Ulam stopped and extended his result to investigate Hyers-Ulam stability [8], which was later extended again to Hyers-Ulam-Rassias stability by Rassias [21] in 1978. In these articles [1, 7, 10, 11, 12, 13, 14, 15, 16, 25, 27] researchers investigated the Hyers-Ulam stability of linear differential equations, while in [2, 4, 5, 6, 17, 18, 19, 20, 23, 24, 22] others considered Hyers-Ulam stability of nonlinear differential equations. In this paper we investigate the Hyers-Ulam-Rassias stability of the following nonlinear second order ordinary differential equation which are perturbed and of the form:

$$[r(t)\phi(u(t))u'(t)]' + g(t,u(t),u'(t))u'(t) + \alpha(t)h(u(t)) = p(t,u(t),u'(t)),\ \forall t > 0 \tag{1.1}$$

with the initial conditions:

$$u(t_0) = u'(t_0) = 0, \tag{1.2}$$

where $r, \alpha, \phi, : \mathbf{R}_+ \to \mathbf{R}_+, \quad g, p : \mathbf{R}_+ \times \mathbf{R^2} \to \mathbf{R}$ are continuous functions. The four variants of equation (1.1) considered are as follows:

i $p(t,u(t),u'(t)) = g(t,u(t),u'(t))$,

ii $p(t,u(t),u'(t)) \neq g(t,u(t),u'(t))$,

iii $g(t,u(t),u'(t)) = 0$,

iv $p(t,u(t),u'(t)) = 0$,

and they are new in the literature. The result obtained extends the previous results by other researchers.

2. Preliminary

The following definitions, lemma and theorems are necessary for subsequent proofs.

Definition 1. Equation (1.1) has Hyers-Ulam-Rassias stability, if there exists a positive constant C_φ and a positive function $\varphi : \mathbf{R}_+ \to \mathbf{R}_+$ with the following property: for every solution $u(t) \in C^2(\mathbf{R}_+)$, of

$$\left|[r(t)\phi(u(t))u'(t)]' + g(t, u(t), u'(t))u'(t) + \alpha(t)h(u(t)) - p(t, u(t), u'(t))\right| \leq \varphi(t) \tag{2.1}$$

satisfying the initial condition (1.2), so that we can find a solution $u_0(t) \in C^2(\mathbf{R}_+)$ of the equation (1.1), such that

$$|u(t) - u_0(t)| \leq C_\varphi \varphi(t) \tag{2.2}$$

where C_φ is independent of $\varphi(t)$ and $u(t)$.

Definition 2. A function $\omega : [0, \infty) \to [0, \infty)$ is said to belong to a class S if

i $\omega(u)$ is nondecreasing and continuous for $u \geq 0$,

ii $(\frac{1}{v})\omega(u) \leq \omega(\frac{u}{v})$ for all u and $v \geq 1$,

iii there exist a function ϕ, continuous on $[0, \infty)$ with $\omega(\alpha u) \leq \phi(\alpha)\omega(u)$ for $\alpha \geq 0$.

Theorem 1. *[3]*Let

i $u(t), r(t), h(t) : \mathbf{R}_+ \to \mathbf{R}_+$ and be continuous,

ii $f, \omega \in S$,

If

$$u(t) \leq K + \int_{t_0}^{t} r(s) f(u(s)) ds + \int_{t_0}^{t} h(s)\omega(u(s)) ds, \tag{2.3}$$

then

$$\begin{aligned} u(t) \leq \Omega^{-1} &\left(\Omega(K) + \int_{t_0}^{t} h(s)\omega\left(F^{-1}\left(F(1)\right.\right.\right. \\ &\left.\left.\left. + \int_{t_0}^{s} r(\delta) d\delta\right)\right) ds\right) F^{-1}\left(F(1) + \int_{t_0}^{t} r(s) ds\right), \quad t \in \mathbf{R}_+ \end{aligned} \tag{2.4}$$

where $(0, b) \subset (0, \infty)$,

$$F(u) = \int_{u_0}^{u} \frac{ds}{\beta(s)}, \quad 0 < u_0 \leq u \tag{2.5}$$

and

$$\Omega(u) = \int_{u_0}^{u} \frac{dt}{\omega(t)}, \quad 0 < u_0 < u \tag{2.6}$$

for F^{-1}, Ω^{-1} inverses of F, Ω respectively and t is in the subinterval $(0, b) \in \mathbf{R}_+$, so that

$$F(1) + \int_{t_0}^{t} r(s)ds \in Dom(F^{-1}) \tag{2.7}$$

and

$$\Omega(K) + \int_{t_0}^{t} h(s)\omega\left(\exp\left(\int_{t_0}^{s} r(\delta)d\delta\right)\right) ds \in Dom(\Omega^{-1}). \tag{2.8}$$

Lemma 1. *[9]*Let $r(t)$ be an integrable function, then the n-successive integration of r over the interval $[t_0, t]$ is given by

$$\int_{t_0}^{t} \dots \int_{t_0}^{t} r(s)ds^n = \frac{1}{(n-1)!} \int_{t_0}^{t} (t-s)^{n-1} r(s)ds. \tag{2.9}$$

3. Nonlinear Integral Inequalities

In this section, we will exhibit the development of Gronwall-Bellman-Bihari type inequalities for our results.

Theorem 1. Let u, r, h be defined as in Theorem 1 and $p(t), \omega(u), f(u)$ be nonnegative, monotonic, nondecreasing, continuous functions on $\mathbf{R}_+$ and $\omega(u)$ be submultiplicative for $u > 0$.
If

$$u(t) \le p(t) + A\int_{t_0}^{t} r(s)f(u(s))ds + B\int_{t_0}^{t} h(s)\omega(u(s))ds, \tag{3.1}$$

for positive constants A and B, then

$$u(t) \le p(t)T(t)E(t), \tag{3.2}$$

where

$$T(t) = \Omega^{-1}\left(\Omega(1) + B\int_{t_0}^{t} h(s)\omega\left(E(s)\right)ds\right),$$

$$E(t) = F^{-1}\left(F(1) + A\int_{t_0}^{t} r(s)ds\right) \quad t \in \mathbf{R}_+,$$

with Ω and F as defined in equation (2.5) and (2.6) respectively and F^{-1}, Ω^{-1} the inverses of F, Ω respectively for t in the subinterval $(0,b) \subset \mathbf{R}_+$ so that

$$F(1) + A\int_{t_0}^{t} r(s)ds \in Dom(F^{-1}) \tag{3.3}$$

and

$$\Omega(1) + B\int_{t_0}^{t} h(s)\omega\left(E(s)\right)ds \in Dom(\Omega^{-1}). \tag{3.4}$$

Proof. Since $p(t)$ is nonnegative, monotonic, nondecreasing on $\mathbf{R}_+$, with $\omega \in S$, we then write equation (3.1) as

$$\frac{u(t)}{p(t)} \le 1 + A\int_{t_0}^{t} r(s)f\left(\frac{u(s)}{p(t)}\right)ds + B\int_{t_0}^{t} h(s)\omega\left(\frac{u(s)}{p(t)}\right)ds. \tag{3.5}$$

Setting

$$\frac{u(t)}{p(t)} = z(t) \tag{3.6}$$

and using equation (3.6) in equation (3.5) we obtain

$$z(t) \le 1 + A\int_{t_0}^{t} r(s)z(s)ds + B\int_{t_0}^{t} h(s)\omega(z(s))ds. \tag{3.7}$$

Applying Theorem 1 to equation (3.7), for $K = 1$, we arrive at

$$z(t) \le T(t)E(t). \tag{3.8}$$

Substituting equation (3.6) into equation (3.8), we then arrive at the result (3.2). □

The next result is an extension of the result in Theorem 1.

Theorem 2. Let u, r, h, g and b be as in Theorem 1 and $\omega(u), f(u),\ \gamma(u)$ be nonnegative, monotonic, nondecreasing continuous functions. Let $\gamma(u)$ be submultiplicative for $u > 0$. If

$$u(t) \le K + A\int_{t_0}^{t} r(s)f(u(s))ds + B\int_{t_0}^{t} h(s)\omega(u(s))ds + L\int_{t_0}^{t} g(s)\gamma(u(s))ds, \tag{3.9}$$

for positive constants $K,\ A,\ B$ and L and $t \in \mathbf{R}_+$, then

$$u(t) \leq G^{-1}\left[G(K) + L\int_{t_0}^{t} g(s)\gamma\left[T(s)E(s)\right]ds\right]T(t)E(t), \tag{3.10}$$

where $T(t)$ and $E(t)$ are as defined in Theorem 1, Ω and F are as defined in Theorem 2.1. For

$$G(r) = \int_{t_0}^{t} \frac{ds}{\gamma(s)}, \quad 0 < r_0 \leq r,$$

F^{-1}, Ω^{-1} and G^{-1} inverses of $F,\ \Omega,\ G$ respectively with t in the subinterval $(0, b) \subset (\mathbf{R}_+)$, so that

$$G(K) + L\int_{t_0}^{t} g(s)\gamma\left[T(s)E(s)\right]ds \in Dom(G^{-1}).$$

Proof. Define

$$n(t) = K + L\int_{t_0}^{t} g(s)\gamma(u(s))ds, \quad t \in \mathbf{R}_+. \tag{3.11}$$

We re-write (3.9) as

$$u(t) \leq n(t) + A\int_{t_0}^{t} r(s)f(u(s))ds + B\int_{t_0}^{t} h(s)\omega(u(s))ds.$$

Since, $n(t)$ is monotonic, nondecreasing on $\mathbf{R}_+$ by applying Theorem 1, we have

$$u(t) \leq n(t)T(t)E(t).$$

Hence,

$$\gamma(u(t)) \leq \gamma\left[n(t)T(t)E(t)\right], \quad t \in \mathbf{I}.$$

Since $\gamma(u)$ is submultiplicative, we have

$$\gamma(u(t)) \leq \gamma(n(t))\gamma\left[T(t)E(t)\right]$$

and it follows that

$$\frac{\gamma(u(t))}{\gamma(n(t))} \leq \gamma\left[T(t)E(t)\right].$$

For $L, g(t) > 0$, we get

$$\frac{\gamma(u(t))}{\gamma(n(t))}Lg(t) \leq Lg(t)\gamma\left[T(t)E(t)\right]. \tag{3.12}$$

Using (3.12) gives

$$\frac{dG(n(t))}{dt} \leq Lg(t)\gamma\left[T(t)E(t)\right]. \tag{3.13}$$

Integrating (3.13), yields

$$G(n(t)) \leq G(n(t_0)) + L\int_{t_0}^{t} g(s)\gamma\left[T(s)E(s)\right]ds,$$

and using (3.11) we have

$$G(n(t)) \leq G(K) + L\int_{t_0}^{t} g(s)\gamma\left[T(s)E(s)\right]ds.$$

Hence,

$$n(t) \leq G^{-1}\left[G(K) + L\int_{t_0}^{t} g(s)\gamma\left[T(s)E(s)\right]ds\right]. \tag{3.14}$$

Substituting for $n(t)$ in (3.14) we arrive at the result in (3.10). □

Theorem 3. Let u, r, h, g, β and b be as in Theorem 1 and $\omega(u)$, $f(u)$, $\gamma(u)$ be nonnegative, monotonic, nondecreasing continuous functions. Let $\gamma(u)$ be submultiplicative for $u > 0$. If

$$u(t) \leq \beta(t) + A\int_{t_0}^{t} r(s)f(u(s))ds + B\int_{t_0}^{t} h(s)\omega(u(s))ds + L\int_{t_0}^{t} g(s)\gamma(u(s))ds, \tag{3.15}$$

for positive constants: K, A, B and L, then

$$u(t) \leq \beta(t)G^{-1}\left[G(K) + L\int_{t_0}^{t} g(s)\gamma\left[T(s)E(s)\right]ds\right]T(t)E(t), \tag{3.16}$$

where $T(t)$ and $E(t)$ are defined in Theorem 1, F and Ω are defined in Theorem 1 and $G(r)$ is given in (3.11), further more, F^{-1}, Ω^{-1} and G^{-1} are the inverses of F, Ω, and G respectively, t is in the subinterval $(0, b) \in (\mathbf{I})$. So that

$$G(K) + L\int_{t_0}^{t} g(s)\gamma\left[T(s)E(s)\right]ds \in Dom(G^{-1}).$$

Proof. Since $\beta(t)$ is a monotonic, nondecreasing and nonnegative continuous function, we rewrite equation (3.15) as:

$$\frac{u(t)}{\beta(t)} \leq 1+A\int_{t_0}^{t} r(s)f(\frac{u(s)}{\beta(s)})ds+B\int_{t_0}^{t} h(s)\omega(\frac{u(s)}{\beta(s)})ds+L\int_{t_0}^{t} g(s)\gamma(\frac{u(s)}{\beta(s)})ds. \tag{3.17}$$

Equation (3.17) is in the form of equation (3.9). Therefore, by carefully following the proof of Theorem 3.2, we arrive at the result in (3.16). □

4. Application of Integral Inequalities to Hyers-Ulam-Rassias Stability

In this section we apply the integral inequalities of section 3 in investigating the Hyers-Ulam-Rassias stability of equation (1.1) with different cases mentioned in section 1.

First, we consider equation (1.1) when $g(t, u(t), u'(t)) = p(t, u(t), u'(t))$.

Theorem 1. Let $r(t) > 0$ be a polynomial function of degree n, $n \in \mathbf{N}$, that is continuous on $\mathbf{R}_+$, if $u(t) \in C^2(R_+)$ is a solution of (2.1) and $p(t, u(t), u'(t)) = g(t, u(t), u'(t))$, then the problem (1.1) is stable in the sense of Hyers-Ulam-Rassias, provided:

i $R(u(t)) = \int_{u(t_0)}^{u(t)} \phi(s)ds$,

ii $|g(t, u(t), u'(t))| = |p(t, u(t), u'(t))| \leq \kappa(t)\omega(|u(t)|)\,|u'(t)|$,

iii $\int_{t_0}^{t} \varphi(s)ds \leq \varphi(t)$,

with Hyers-Ulam-Rassias constant:

$$C_\varphi = TE, \tag{4.1}$$

where

$$T = \Omega^{-1}\left(\Omega(1) + m(\eta + \eta^2)\omega\left(E\right)\right)$$

and

$$E = F^{-1}\left(F(1) + l\right).$$

Proof. From inequality (2.1), we get

$$-\varphi(t) \leq \left[r(t)\phi(u(t))u'(t)\right]'$$
$$+g(t,u(t),u'(t))u'(t) + \alpha(t)h(u(t)) - p(t,u(t),u'(t)) \leq \varphi(t). \tag{4.2}$$

Integrating (4.2) twice and using Lemma 1, we obtain
$-t\int_{t_0}^{t}\varphi(ds) \leq r(t)\int_{t_0}^{t}\phi(u(s))u'(s)ds + t\int_{t_0}^{t}g(s,u(s),u'(s))u'(s)ds$
$+t\int_{t_0}^{t}\alpha(s)h(u(s))ds - t\int_{t_0}^{t}P(s,u(s),u'(s))ds \leq t\int_{t_0}^{t}\varphi(s)ds.$

Since $t > 0$, considering the upper inequality, we have
$\frac{r(t)}{t}\int_{t_0}^{t}\phi(u(s))u'(s) + \int_{t_0}^{t}g(s,u(s),u'(s))u'(s)ds$
$+\int_{t_0}^{t}\alpha(s)h(u(s)) - \int_{t_0}^{t}P(s,u(s),u'(s))ds \leq \int_{t_0}^{t}\varphi(s)ds.$

By applying condition (i) of Theorem 1, we obtain
$\frac{r(t)}{t}\int_{t_0}^{t}\frac{d}{ds}R(u(t))ds + \int_{t_0}^{t}g(s,u(s),u'(s))u'(s)ds$
$+\int_{t_0}^{t}\alpha(s)h(u(s)) - \int_{t_0}^{t}p(s,u(s),u'(s))ds \leq \int_{t_0}^{t}\varphi(s)ds.$

Evaluating first term using the condition (1.2) we obtain
$\frac{r(t)}{t}R(u(t)) + \int_{t_0}^{t}g(s,u(s),u'(s))u'(s)ds + \int_{t_0}^{t}\alpha(s)h(u(s))$
$-\int_{t_0}^{t}p(s,u(s),u'(s))ds \leq \int_{t_0}^{t}\varphi(s)ds,$

rearranging and taking absolute value, we have

$\frac{r(t)}{t}\left|R(u(t))\right| \leq \int_{t_0}^{t}\varphi(s)ds + \int_{t_0}^{t}\left|g(s,u(s),u'(s))\right|\left|u'(s)\right|ds$
$+\int_{t_0}^{t}\alpha(s)\left|h(u(s))\right| + \int_{t_0}^{t}\left|p(s,u(s),u'(s))\right|ds.$

Using the condition (ii), we obtain
$\frac{r(t)}{t}|R(u(t))| \leq \int_{t_0}^{t}\varphi(s)ds + \int_{t_0}^{t}\kappa(s)\omega(|u(s)|)|u'(s)|^2ds + \int_{t_0}^{t}\alpha(s)h(|u(s)|)+$
$\int_{t_0}^{t}\kappa(s)\omega(|u(s)|)|u'(s)|ds$

and factorising we get

$\frac{r(t)}{t}\left|R(u(t))\right| \leq \int_{t_0}^{t}\varphi(s)ds$
$+\int_{t_0}^{t}\alpha(s)h(|u(s)|) + \int_{t_0}^{t}(|u'(s)| + |u'(s)|^2)\kappa(s)\omega(|u(s)|)ds.$

It follows that
$\frac{r(t)}{t}\left|R(u(t))\right| \leq \int_{t_0}^{t}\varphi(s)ds$
$+\int_{t_0}^{t}\alpha(s)h(|u(s)|) + (|u'(s)| + |u'(t)|^2)\int_{t_0}^{t}\kappa(s)\omega(|u(s)|)ds,$

setting $\frac{r(t)}{t}|R(u(t)| \geq |u(t)|$ and $|u'(t)| \leq \eta$, for $\eta > 0$, we have

$$|u(t)| \leq \int_{t_0}^{t} \varphi(s)ds + \int_{t_0}^{t} \alpha(s)h(|u(s)|) + (\eta + \eta^2) \int_{t_0}^{t} \kappa(s)\omega(|u(s)|)ds.$$

Applying Theorem 1 we get

$$|u(t)| \leq T(t)E(t) \int_{t_0}^{t} \varphi(s)ds,$$

where

$$T(t) = \Omega^{-1}\left(\Omega(1) + (\eta + \eta^2) \int_{t_0}^{t} \kappa(s)\omega\left(E(s)\right) ds\right),$$

$$E(t) = F^{-1}\left(F(1) + \int_{t_0}^{t} \alpha(s)ds\right),$$

and $B = \eta + \eta^2$, $A = 1$.

Setting

$$\int_{t_0}^{t} \varphi(s)ds \leq \varphi(t), \quad \lim_{t\to\infty} \int_{t_0}^{t} \kappa(s)ds = m < \infty$$

and $\lim_{t\to\infty} \int_{t_0}^{t} \alpha(s)ds = l < \infty$, for $l,\ m > 0$, it follows that

$$|u(t)| \leq \varphi(t)TE.$$

Therefore,

$$|u(t) - u_0(t)| \leq |u(t)|$$

and hence,

$$|u(t) - u_0(t)| \leq \varphi(t)TE.$$

□

Example 1. Consider Hyers-Ulam-Rassias stability of the second order nonlinear differential equation

$$\left[(t^3 + t^2 + 1)u^2(t)u'(t)\right]' + \frac{1}{t^2}u^2(t)u'^3(t) + \frac{1}{t^4}u^2(t) = \frac{1}{t^2}u^2(t)u'^2(t)$$

where $r(t) = t^3 + t^2 + 1$, that is a polynomial of degree 3, $\phi(u(t)) = u^2(t),\ ; g(t, u(t), u'(t)) = \frac{1}{t^2}u^2(t)u'^3(t),\ P(t, u(t), u'(t)) = \frac{1}{t^2}u^2(t)u'^2(t)$ by conditions in the Theorem 1 and the conditions analysed in the proof of Theorem 1, the nonlinear differential equation is Hyers-Ulam-Rassias stable for Ω^{-1} and F^{-1} are finite.

In the next theorem we consider the case $g(t,u(t),u'(t)) \neq p(t,u(t),u'(t))$ in equation (1.1).

Theorem 2. *Let $r(t)$ possess the same features as in Theorem 1, if $u(t) \in C^2(R_+)$ is a solution of (2.1) and $p(t,u(t),u'(t)) \neq g(t,u(t),u'(t))$ are continuous functions on $(\mathbf{I} \times \mathbf{R^2})$. Then the problem (1.1) satisfies the Hyers-Ulam-Rassias stability, provided:*

(ii)' $|g(t,u(t),u'(t))| \leq \kappa(t)\omega(|u(t)|)\,|u'(t)|$,

(iii)' $|p(t,u(t),u'(t))| \leq g(t)\gamma(|u(t)|)\,|u'(t)|^n$ *for* $n \in \mathbf{N}$,

and

$$|u(t) - u(t_0)| \leq C_\varphi \varphi(t),$$

where C_φ is Hyers-Ulam-Rassias constant.

Proof. From equation (2.1), it follows that

$$-\varphi(t) \leq [r(t)\phi(u(t))u'(t)]'$$
$$+g(t,u(t),u'(t))u'(t) + \alpha(t)h(u(t)) - p(t,u(t),u'(t)) \leq \varphi(t).$$

Integrating twice and using the Lemma 1, we have

$$-t\int_{t_0}^t \varphi(ds) \leq r(t)\int_{t_0}^t \phi(u(s))u'(s)ds + t\int_{t_0}^t g(s,u(s),u'(s))u'(s)ds$$
$$+t\int_{t_0}^t \alpha(s)h(u(s))ds - t\int_{t_0}^t p(s,u(s),u'(s))ds \leq t\int_{t_0}^t \varphi(s)ds.$$

It follows that

$$\text{r(t)t}\int_{t_0}^t \phi(u(s))u'(s) \quad + \quad \int_{t_0}^t g(s,u(s),u'(s))u'(s)ds$$
$$+\int_{t_0}^t \alpha(s)h(u(s)) - \int_{t_0}^t P(s,u(s),u'(s))ds \leq \int_{t_0}^t \varphi(s)ds$$

and by condition(i) of Theorem 1, we have

$$\text{r(t)t R(u(t))} \leq \int_{t_0}^t \varphi(s)ds - \int_{t_0}^t g(s,u(s),u'(s))u'(s)ds$$
$$-\int_{t_0}^t \alpha(s)h(u(s)) \quad + \quad \int_{t_0}^t P(s,u(s),u'(s))ds.$$

Taking the absolute value of both sides, using conditions $(ii)'$ and $(iii)'$ we have

$$\frac{r(t)}{t}|R(u(t))| \leq \int_{t_0}^t |\varphi(s)|\,ds + \int_{t_0}^t \alpha(s)h(|u(s)|) + \int_{t_0}^t \kappa(s)\omega(|u(s)|)\,|u'(s)|^2\,ds$$
$$+\int_{t_0}^t g(s)\gamma(|u(s)|)\,|u'(s)|^n\,ds. \tag{4.3}$$

Since $|u(t)| \leq \frac{r(t)}{t}|R(u(t))$ and $|u'(t)| \leq \eta$, equation (4.3) becomes

$|u(t)| \leq \int_{t_0}^{t} |\varphi(s)|\,ds + \int_{t_0}^{t} \alpha(s)h(|u(s)|) + \eta^2 \int_{t_0}^{t} \kappa(s)\omega(|u(s)|)ds$
$+\eta^n \int_{t_0}^{t} g(s)\gamma(|u(s)|)ds$
and the application of Theorem 3, gives

$$|u(t)| \leq \int_{t_0}^{t} |\varphi(s)|\,dsG^{-1}\left[G(K) + \eta^n \int_{t_0}^{t} g(s)\gamma\left[T(s)E(s)\right]ds\right]T(t)E(t),$$

where

$$T(t) = \Omega^{-1}\left(\Omega(1) + \eta^2 \int_{t_0}^{t} \kappa(s)\omega\left(E(s)\right)ds\right)$$

and $B = \eta^2$.
Let the limit of integrals be as in the proof of Theorem 1 and let $\lim\limits_{t\to\infty} \int_{t_0}^{t} g(s)ds \leq K$, we obtain

$$|u(t)| \leq \varphi(t)G^{-1}\left[G(K) + k\eta^n\gamma\left[TE\right]\right]TE,$$

where we define

$$T = \Omega^{-1}\left(\Omega(1) + m\eta^2\omega\left(E\right)\right),$$

and

$$E = F^{-1}\left(F(1) + l\right).$$

Thus,

$$|u(t) - u_0(t_0)| \leq |u(t)| \leq C_\varphi\varphi(t),$$

and therefore,

$$|u(t) - u_0(t)| \leq \varphi(t)G^{-1}\left[G(K) + k\eta^n\gamma\left[TE\right]\right]TE,$$

where

$$C_\varphi = G^{-1}\left[G(K) + k\eta^n\gamma\left[TE\right]\right]TE.$$

□

Example 2. Investigate the Hyers-Ulam-Rassias stability of the second order nonlinear differential equation

$$\left[(t^4 + t^3 + t^2 + 1)tu^2(t)u'(t)\right]' + \frac{1}{t^2}u^2(t)u'^3(t) + \frac{1}{t^4}u^4(t) = \frac{1}{t^2}u^3(t)u'^4(t)$$

where

$$r(t) = (t^3 + t^2 + 1)t, \;\; \phi(u(t)) = u^4(t),$$

$$g(t,u(t),u'(t)) = \frac{1}{t^2}u^2(t)u'^2(t) \text{ and } P(t,u(t),u'(t)) = \frac{1}{t^2}u^3(t)u'^4(t).$$

By the conditions in Theorem 2 and the conditions analysed in the proof of the Theorem, the nonlinear differential equation is Hyers-Ulam-Rassias stable for finite Ω^{-1} and F^{-1}.

In the third result we consider the case $g(t,u(t),u'(t)) = 0$.

Theorem 3. If $u(t) \in C^2(R_+)$ is a solution of (2.1) when $g(t,u(t),u'(t)) = 0$, then there exists a solution $u_0(t) \in C^2(\mathbf{R}_+)$ problem (1.1) such that

$$|u(t) - u_0(t)| \le |u(t)| \le C_\varphi \varphi(t),$$

therefore equation (1.1) is stable in the sense of Hyers-Ulam-Rassias with the Hyers-Ulam-Rassias constant $C_\varphi = HE$, where

$$H = G^{-1}\left(G(1) + m\eta^n \gamma\left(E\right)\right)$$

and

$$E = F^{-1}\left(F(1) + l\right).$$

Proof. From equation (2.1), it follows that

$$(4.4) - \varphi(t) \le \left[r(t)\phi(u(t))u'(t)\right]' + \alpha(t)h(u(t)) - p(t,u(t),u'(t)) \le \varphi(t).$$

Integrating (4.4) twice and using Lemma 2.2, we get
$-\int_{t_0}^t (t-s)\varphi(ds) \le \int_{t_0}^t r(s)\phi(u(s))u'(s)ds$
$+\int_{t_0}^t (t-s)\alpha(s)h(u(s))ds - \int_{t_0}^t (t-s)p(s,u(s),u'(s))ds \le \int_{t_0}^t (t-s)\varphi(s)ds.$

It follows that
r(t)t$\int_{t_0}^t \phi(u(s))u'(s)ds$
$+\int_{t_0}^t \alpha(s)h(u(s))ds - \int_{t_0}^t p(s,u(s),u'(s))ds \le \int_{t_0}^t \varphi(s)ds.$

Using the definition of $R(u(t))$ in Theorem 4.1 we get

$$\frac{r(t)}{t}R(u(t)) + \int_{t_0}^t \alpha(s)h(u(s))ds - \int_{t_0}^t p(s,u(s),u'(s))ds \le \int_{t_0}^t \varphi(s)ds.$$

Recall that $|u(t)| \le \frac{r(t)}{t}|R(u(t))$, using this we obtain

$$|u(t)| \le \int_{t_0}^t |\varphi(s)|\,ds + \int_{t_0}^t \alpha(s)\,|h(u(s))|\,ds + \int_{t_0}^t |p(s,u(s),u'(s))|\,ds.$$

Let $|p(t,u(t),u'(t))| \leq g(t)\gamma(|u(t)|)\,|u'(t)|^n$ where $n \in \mathbf{N}$, then we have

$$u(t)| \leq \int_{t_0}^{t} |\varphi(s)|\,ds + \int_{t_0}^{t} \alpha(s)h(|u(s)|)ds + \int_{t_0}^{t} g(s)\gamma(|u(s)|)\,|u'(s)|^n\,ds.$$

It follows that

$$|u(t)| \leq \int_{t_0}^{t} |\varphi(s)|\,ds + \int_{t_0}^{t} \alpha(s)h(|u(s)|)ds + (|u'(t)|^n \int_{t_0}^{t} g(s)\gamma(|u(s)|)ds.$$

Applying Theorem 2, we get

$$|u(t)| \leq \int_{t_0}^{t} |\varphi(s)|\,dsH(t)E(t),$$

where

$$H(t) = G^{-1}\left(G(1) + \eta^n \int_{t_0}^{t} g(s)\gamma\,(E(s))\,ds\right),$$

and

$$E(t) = F^{-1}\left(f(1) + \int_{t_0}^{t} \alpha(s)ds\right).$$

Let the limit of the integrals be as in the proof of Theorems 3 and 1, then we obtain

$$|u(t)| \leq \varphi(t)HE.$$

Therefore,

$$|u(t) - u_0(t)| \leq |u(t)| \leq \varphi(t)HE.$$

Hence,

$$|u(t) - u_0(t)| \leq \varphi(t)HE.$$

□

Example 3. Investigate the Hyers-Ulam-rassias stability of the following differential equation using conditions of the Theorem 3

$$\left[(t^4 + t^3 + t^2 + 1)u^2(t)u'(t)\right]' + \frac{1}{t^2}u^3(t) = \frac{1}{t^3}u^3(t)u'^4(t)$$

where $r(t) = (t^4 + t^2 + 1)$, that is, a polynomial of degree 4, $\phi(u(t)) = u^2(t)$, $\alpha(t)h(t) = \frac{1}{t^2}u^3(t)$ $P(t,u(t),u'(t)) = \frac{1}{t^3}u^3(t)u'^4(t)$ by the conditions of Theorem 3 and the conditions analysed in the proof of the Theorem, the nonlinear differential equation is Hyers-Ulam-Rassias stable.

In this last result we investigate the Hyers-Ulam-Rassias stability of equation (1.1) for $p(t, u(t), u'(t)) = 0$

Theorem 4. If $u(t) \in C^2(R_+)$ is a solution of (2.1) when $p(t, u(t), u'(t)) = 0$. Then the initial valued problem (1.1) is stable in the sense of Hyers-Ulam-Rassias, provided there exists a solution $u_0(t) \in C^2(\mathbf{R}_+)$ of equation(1.1) such that

$$|u(t) - u_0(t)| \leq |u(t)| \leq C_\varphi \varphi(t)$$

where C_φ is the Hyers-Ulam-Rassias constant.

Proof. From equation (2.1), it follows that

$$(4.5) - \varphi(t) \leq \left[r(t)\phi(u(t))u'(t)\right]' + \alpha(t)h(u(t)) + g(t, u(t), u'(t))u'(t) \leq \varphi(t).$$

Integrating (4.5) twice and using Lemma 1 we get
$-\int_{t_0}^t (t-s)\varphi(ds) \leq \int_{t_0}^t r(s)\phi(u(s))u'(s)ds$
$+\int_{t_0}^t (t-s)\alpha(s)h(u(s))ds + \int_{t_0}^t (t-s)g(s, u(s), u'(s))u'(s)ds$
$\leq \int_{t_0}^t (t-s)\varphi(s)ds.$

It follows that
r(t)t$\int_{t_0}^t \phi(u(s))u'(s)ds$
$+\int_{t_0}^t \alpha(s)h(u(s))ds + \int_{t_0}^t g(s, u(s), u'(s))u'(s)ds \leq \int_{t_0}^t \varphi(s)ds.$

By the application of condition(i) of Theorem 1 we get

$$\frac{r(t)}{t}R(u(t)) + \int_{t_0}^t \alpha(s)h(u(s))ds + \int_{t_0}^t g(s, u(s), u'(s))u'(s)ds \leq \int_{t_0}^t \varphi(s)ds.$$

It follows that

$$\frac{r(t)}{t}R(u(t)) \leq \int_{t_0}^t \varphi(s)ds - \int_{t_0}^t \alpha(s)h(u(s))ds - \int_{t_0}^t g(s, u(s), u'(s))u'(s)ds,$$

and taking the absolute value of both sides, we get

$$\frac{r(t)}{t}|R(u(t))| \leq \int_{t_0}^t |\varphi(s)|ds + \int_{t_0}^t \alpha(s)|h(u(s))|ds + \int_{t_0}^t |g(s, u(s), u'(s))||u'(s)|ds.$$

With $|g(t, u(t), u'(t))| \leq \kappa(t)\omega(|u(t)|)\,|u'(t)|^2$ we have

$$\frac{r(t)}{t}|R(u(t))| \leq \int_{t_0}^t |\varphi(s)|ds + \int_{t_0}^t \alpha(s)h(|u(s)|)ds + \int_{t_0}^t \kappa(s)\omega(|u(s)|)|u'(s)|^2 ds.$$

Thus, it follows that

$$\frac{r(t)}{t}|R(u(t))| \leq \int_{t_0}^{t} |\varphi(s)|ds + \int_{t_0}^{t} \alpha(s)h(|u(s)|)ds + (|u'(t)|^2 \int_{t_0}^{t} \kappa(s)\omega(|u(s)|)ds.$$

If $\frac{r(t)}{t}|R(u(t))| \geq |u(t)|$, applying Theorem 1 gives

$$|u(t)| \leq T(t)E(t) \int_{t_0}^{t} |\varphi(s)|ds,$$

where

$$T(t) = \Omega^{-1}\left(\Omega(1) + \eta^2 \int_{t_0}^{t} \kappa(s)\left(E(s)\right) ds\right)$$

and

$$E(t) = F^{-1}\left(F(1) + \int_{t_0}^{t} \alpha(s)ds\right).$$

Following the definition of the limit of integrals in the previous results we have

$$|u(t)| \leq \varphi(t)TE,$$

$$T = (\Omega^{-1}\left(\Omega(1) + m\eta^2\omega\left(E\right)\right),$$

and

$$E = F^{-1}\left(F(1) + l\right).$$

Therefore,

$$|u(t) - u_0(t)| \leq |u(t)| \leq \varphi(t)TE$$

and hence,

$$|u(t) - u_0(t)| \leq \varphi(t)\varphi(t)TE.$$

□

Example 4. Investigate the Hyers-Ulam-Rassias stability of the second order nonlinear differential equation:

$$\left[(t^4 + t^3 + t^2 + 1)u^2(t)u'(t)\right]' + \frac{1}{t^2}u^2(t)u'^3(t) + \frac{1}{t^2}u^3(t) = 0$$

where $r(t) = (t^4 + t^3 + t^2 + 1)$, that is, a polynomial of degree 4, $\phi(u(t)) = u^2(t)$, $g(t, u(t), u'(t)) = \frac{1}{t^2}u^2(t)u'^3(t)$ and $\alpha(t)h(u(t)) = \frac{1}{t^2}u^3(t)$. By conditions of Theorem 4 and the conditions analysed in the proof of the last Theorem, the nonlinear differential equation is Hyers-Ulam-Rassias stable for finite Ω^{-1} and F^{-1}.

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