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On $*$-reverse derivable maps

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Abstract

Let R be a ring with involution containing a nontrivial symmetric idempotent element e. Let $\delta : R \rightarrow R$ be a mapping such that $\delta(ab) = \delta(b)a^* + b^*\delta(a)$ for all $a, b \in R$, we call δ a $*-$reverse derivable map on R. In this paper, our aim is to show that under some suitable restrictions imposed on R, every $*-$reverse derivable map of R is additive.

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1. Introduction

Let R be a ring, an additive map $\delta : R \to R$ such that $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in R$, is called a *derivation*. A derivation which is not necessarily additive is said to be a *multiplicative derivation* or a *derivable map*. A mapping $\delta : R \to R$ is known as multiplicative Jordan derivation of R if $\delta(ab + ba) = \delta(a)b + a\delta(b) + \delta(b)a + b\delta(a)$ for all $a, b \in R$. In addition, δ is called n-multiplicative derivation of R if $\delta(a_1a_2 \cdots a_n) = \sum_{i=1}^{n} a_1a_2 \cdots \delta(a_i) \cdots a_n$ for all $a_1, a_2, \cdots, a_n \in R$. A mapping $F : R \to R$ (not necessarily additive) associated with a derivation d is called multiplicative generalized derivation if $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$ (see [4]). In [14], Herstein introduced a mapping "$\dagger$" satisfying $(a+b)^\dagger = a^\dagger + b^\dagger$ and $(ab)^\dagger = b^\dagger a + ba^\dagger$ called a *reverse derivation*, which is certainly not a derivation. Moreover, a mapping $\delta : R \to R$ satisfying $\delta(ab) = \delta(b)a + b\delta(a)$ for all $a, b \in R$ is called a *multiplicative reverse derivation* or *reverse derivable map* of R. A mapping $\psi : R \to R$ is said to be a left (resp. right) centralizer if $\psi(ab) = \psi(a)b$ (resp. $\psi(ab) = a\psi(b)$) for all $a, b \in R$. Moreover, if ψ is left and right centralizer, then it is called *centralizer* of R. A left (resp. right) centralizer which is not necessarily additive is called multiplicative left (resp. right) centralizer. By involution, we mean an anti-automorphism $* : R \to R$ such that $(x^*)^* = x$ for all $x, y \in R$. An element $s \in R$ satisfying $s^* = s$ is called a *symmetric element* of R.

Let e be an idempotent element of R such that $e \neq 0, 1$. Then R can be decomposed as follows:

$$R = eRe \bigoplus eR(1-e) \bigoplus (1-e)Re \bigoplus (1-e)R(1-e)$$

This decomposition of R is called *two-sided Peirce decomposition relative to e* ([15], see pg. 48). It is easy to see that the components of this decomposition are the subrings of R and for our convenience, we denote $R_{11} = eRe, R_{12} = eR(1-e), R_{21} = (1-e)Re$ and $R_{22} = (1-e)R(1-e)$. For any $r \in R$, we denote the elements of R_{ij} by r_{ij} for all $i, j \in \{1, 2\}$. We use the notation $e_1 := e$ and define $e_2 \colon R \to R$ and $e_2' \colon R \to R$ by $e_2a = a - e_1a$ and $e_2'a = a - ae_1$. We shall denote $e_2'a$ by ae_2. Note that R need not have an identity element: the operation $x(1 - y)$ for $x, y \in R$ is understood as $x - xy$.

The present study is motivated by various additivity theorems proved by several well-known algebraists (viz. [3, 4, 5, 16, 18, 19]). Studying the interrelationship between the multiplicative and additive structure of rings is a quite interesting subject nowadays. The pursuit of this line of investigation is inspired by a surprising result of Martindale [18], which exhibits that how multiplicative structure of a ring determines its additive structure. Precisely, Martindale [18] proved the following:

Theorem 1.1. *Let R be a ring containing a family $\{e_\lambda : \lambda \in \Lambda\}$ of idempotents which satisfies:*

(1) $xR = 0$ implies $x = 0$.

(2) If $e_\lambda Rx = 0$ for each $\lambda \in \Lambda$, then $x = 0$ (and hence $Rx = 0$ implies $x = 0$).

(3) For each $\lambda \in \Lambda$, $e_\lambda x e_\lambda R(1 - e_\lambda) = 0$, implies $e_\lambda x e_\lambda = 0$.

Then any multiplicative bijective map from a ring R into an arbitrary ring S is additive.

Since then, this set of conditions has been used by a number of authors in order to obtain the additivity of some specific mappings of rings and algebras. In 1991, Daif [3] figured out that Martindale's conditions can also assure the additivity of multiplicative derivations. In this vein, with the same set of conditions, Li and Lu [17] obtained the additivity of maps $M : R \to R'$ and $M^* : R' \to R$ that are surjective and satisfy $M(xM^*(y)z) = M(x)yM(z)$ and $M^*(yM(x)u) = M^*(y)xM^*(u)$ for all $x, z \in R$ and $y, u \in R'$. Moreover, in 2009, Wang [19] extended the results of Martindale and Daif simultaneously, and gave a short proof of [17, Theorem 2.1].

Besides from the Martindale's set of conditions, there are also some studies available in the literature that investigate the additivity of certain mappings of rings. For instance, in a systematic paper [5], Eremita and Iliševic proved the additivity of multiplicative left centralizers that are defined from R into a bimodule M over R and gave a number of applications of the main result. Precisely, they proved the following:

Theorem 1.2. *Let R be a ring and M be a bimodule over R. Further, let $e_1 \in R$ be a nontrivial idempotent (and $1 - e_1 = e_2$) such that for any $m \in M' = \{m \in M : mZ(R) = (0)\}$, where $Z(R)$ denotes the center of R,*

(i) $e_1 m e_1 R e_2 = (0)$ implies $e_1 m e_1 = 0$,

(ii) $e_1 m e_2 R e_1 = (0)$ implies $e_1 m e_2 = 0$,

(iii) $e_1me_2Re_2 = (0)$ *implies* $e_1me_2 = 0$,

(iv) $e_2me_1Re_2 = (0)$ *implies* $e_2me_1 = 0$,

(v) $e_2me_2Re_1 = (0)$ *implies* $e_2me_2 = 0$,

(vi) $e_2me_2Re_2 = (0)$ *implies* $e_2me_2 = 0$.

Then every left centralizer $\phi : R \to M$ *is additive.*

In 2007, Daif and Tammam-El-Sayiad [4] studied the additivity of multiplicative generalized derivations with slight modifications in conditions of Martindale. In a recent paper, Jing and Lu [16] examined the additivity of multiplicative Jordan derivations and obtained the following result:

Theorem 1.3. *Let* R *be a ring containing a nontrivial idempotent and satisfying the following conditions for* $i, j, k \in \{1, 2\}$*:*

(P1) If $a_{ij}x_{jk} = 0$ *for all* $x_{jk} \in R_{jk}$*, then* $a_{ij} = 0$;

(P2) If $x_{ij}a_{jk} = 0$ *for all* $x_{ij} \in R_{ij}$*, then* $a_{jk} = 0$;

(P2) If $a_{ii}x_{ii} + x_{ii}a_{ii} = 0$ *for all* $x_{ii} \in R_{ii}$*, then* $a_{ii} = 0$.

If $\delta : R \to R$ *is a mapping satisfies* $\delta(ab+ba) = \delta(a)b+a\delta(b)+\delta(b)a+b\delta(a)$ *for all* $a, b \in R$*, then* δ *is additive.*

This sort of problems and their solutions are not limited only to the class associative rings. For the case of non-associative rings and algebras having nontrivial idempotents, additivity of various maps defined on them has already been proved in the literature. In alternative rings, we can mention the works in [6, 7, 8, 9, 10, 11, 12, 13].

In 1957, Herstein [14] introduced the notion of reverse derivation, and proved that if R is a prime ring and d is a reverse derivation of R, then R is a commutative integral domain, and hence d is an ordinary derivation of R. Later, this result has been extended by Brešar and Vukman [1, 2]. The notion of reverse derivation is related to some generalization of derivation, for instance, every reverse derivation is a Jordan derivation. Therefore, under the hypothesis taken by Jing and Lu [16, Theorem 1.2], every reverse derivation is additive.

In view of the above discussion, in this study we object to investigate the additivity of a mapping $\delta : R \to R$ satisfies $\delta(xy) = \delta(y)x^* + y^*\delta(x)$ for all $x, y \in R$, where $*$ is the involution of R. If δ is additive, then it is called $*$-reverse derivation, which is clearly neither a derivation nor a reverse derivation. The basic example of $*$-reverse derivation is a mapping $x \mapsto [a, x^*]$, where $a \in R$ a fixed element, called the inner $*-$reverse derivation. In addition, one can easily observe from the following example that the theorem of Herstein [14, Theorem 2.1] does not hold for $*-$reverse derivations:

Example 1.1. *Let* $R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbf{Z} \right\}$, *where* $\mathbf{Z}$ *is the ring of integers. Define a mapping* $\delta : R \to R$ *such that*

$$\delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & c \\ -b & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

the standard involution of R*. Clearly,* δ *is a* $*-$*reverse derivation and* R *is a noncommutative prime ring.*

2. Main Results

Definition 2.1. *Let* R *be a ring and* $'*'$ *be an involution on* R*. Then a mapping* $\delta : R \to R$ *(not necessarily additive) is called* $*-$*reverse derivable if* $\delta(ab) = \delta(b)a^* + b^*\delta(a)$ *for all* $a, b \in R$*.*

The main result of this paper reads as follows:

Theorem 2.1. *Let* R *be a ring with involution containing a nontrivial symmetric idempotent element* e *and any element* $a \in R$ *such that the following conditions are satisfied*

(i) If $x_{ii}a_{ij} = 0$ *for all* $x_{ii} \in R_{ii}$*, then* $a_{ij} = 0$*;*

(ii) If $a_{ii}x_{ij} = 0$ *for all* $x_{ij} \in R_{ij}$ *with* $i \neq j$*, then* $a_{ii} = 0$*.*

Then every $*-$*reverse derivable map* $\delta : R \to R$ *is additive.*

It is easy to see that an unital prime ring with a nontrivial symmetric idempotent e satisfies the conditions (i) and (ii) of the Theorem 2.1, so we get the following

Corollary 2.1. *Let R be an unital prime ring with a nontrivial symmetric idempotent e. Then every $*-$reverse derivable map of R is additive.*

Corollary 2.2. *Let R be the ring same as in Theorem 2.1 and*

$$\mathcal{R} = \left\{ \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} : r_{ij} \in R_{ij} \right\} \cong R_{11} \oplus R_{12} \oplus R_{21} \oplus R_{22} = R.$$

Moreover, $R_{11} \equiv \left\{ \begin{pmatrix} r_{11} & 0 \\ 0 & 0 \end{pmatrix} : r_{11} \in R_{11} \right\}$. *Similarly to other spaces R_{12}, R_{21} and R_{22}. Let* $\mathcal{E} = \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix}$ *be the non-trivial idempotent in $\mathcal{R}$. Define $\delta : \mathcal{R} \to \mathcal{R}$ such that $\delta(XY) = \delta(Y)\tau(X)+\tau(Y)\delta(X)$ for all $X, Y \in \mathcal{R}$, where τ is the transpose map, which is named transpose reverse derivable map. Under the same conditions of Theorem 2.1, every transpose reverse derivable map is additive.*

It is easy to note that $\delta(e) = a_{11} + a_{12} + a_{21} + a_{22}$. Since $\delta(e) = \delta(e^2) = \delta(e)e^* + e^*\delta(e)$, it follows that $\delta(e) = a_{12}+a_{21}$. Define a mapping $\wp : R \to R$ such that $\wp(x) = [a_{21} - a_{12}, x^*]$. It is not difficult to check that $\wp$ is an additive $*-$reverse derivable map. Thus, we set $\Delta = \delta - \wp$, which is also a $*-$reverse derivable map and Δ is additive if and only if δ is so. Moreover it is easy to observe that $\Delta(e) = 0$.

We shall use the following fact very frequently in the sequel.

Proposition 2.1. *Let $s \in R$ ($s_{ij} \in R_{ij}$, where $i, j \in \{1, 2\}$). Then $s^*_{ij} = r_{ji}$, where $r = s^* \in R$. Moreover, $s_{ij} = r^*_{ji}$.*

Proof. Let $s \in R$ be any element. Then for $es(1 - e) = s_{12} \in R_{12}$, we have $(es(1 - e))^* = (1 - e)^*s^*e^* = (1 - e)s^*e$. It gives that $s^*_{12} = r_{21}$, where $r = s^*$. Similarly, one can easily observe that $s^*_{21} = r_{12}$, $s^*_{11} = r_{11}$ and $s^*_{22} = r_{22}$. Moreover, for each $s_{ij} \in R$ there exists unique $r \in R$ such that $r^*_{ji} = s_{ij}$ as $*$ is bijective. □

Lemma 2.1. $\Delta(0) = 0$.

Proof. The proof is trivial. □

Lemma 2.2. $\Delta(R_{ij}) \subset R_{ji}$, *where* $i, j = \{1, 2\}$.

Proof. For any $x_{11} \in R_{11}$, we have $\Delta(x_{11}) = \Delta(ex_{11}e) = \Delta(x_{11}e)e^* = e^*\Delta(x_{11})e^* = e\Delta(x_{11})e \in R_{11}$. Hence $\Delta(R_{11}) \subset R_{11}$.
For any $x_{22} \in R_{22}$, $\Delta(x_{22}) \in R$, we put $\Delta(x_{22}) = r_{11} + r_{12} + r_{21} + r_{22}$. Now $0 = \Delta(ex_{22}) = \Delta(x_{22})e^* = (r_{11} + r_{12} + r_{21} + r_{22})e = r_{11} + r_{21}$. Likewise $0 = \Delta(x_{22}e) = e^*\Delta(x_{22}) = r_{11} + r_{12}$. It implies $r_{11} = r_{21} = r_{12} = 0$. Therefore $\Delta(x_{22}) = r_{22}$ and hence $\Delta(R_{22}) \subset R_{22}$.
For any $x_{12} \in R_{12}$, $\Delta(x_{12}) = b_{11}+b_{12}+b_{21}+b_{22}$. Now $\Delta(x_{12}) = \Delta(ex_{12}) = \Delta(x_{12})e^* = b_{11}+b_{21}$ and $0 = \Delta(x_{12}e) = e^*\Delta(x_{12}) = e(b_{11}+b_{12}+b_{21}+b_{22}) = b_{11} + b_{12}$. Thus $\Delta(x_{12}) = b_{21}$ and hence $\Delta(R_{12}) \subset R_{21}$.
Let be $x_{21} \in R_{21}$ then $\Delta(x_{21}) = c_{11} + c_{12} + c_{21} + c_{22}$. Now $\Delta(x_{21}) = \Delta(x_{21}e) = e^*\Delta(x_{21}) = c_{11}+c_{12}$ and $0 = \Delta(ex_{21}) = \Delta(x_{21})e^* = (c_{11}+c_{12}+c_{21}+c_{22})e = c_{11}+c_{21}$. That yields $\Delta(x_{21}) = c_{12}$ and hence $\Delta(R_{21}) \subset R_{12}$. □

The following Steps have the same hypotheses of Theorem 2.1 and we need these Steps for the proof of the main result.

Step 1: For $i \neq j$, $\Delta(a_{ii} + a_{ij}) = \Delta(a_{ii}) + \Delta(a_{ij})$ and $\Delta(a_{ii} + a_{ji}) = \Delta(a_{ii}) + \Delta(a_{ji})$.
Let us work just with $\Delta(a_{ii} + a_{ij}) = \Delta(a_{ii}) + \Delta(a_{ij})$ because the other case have a similar proof. Let t_{ii} be an element of R_{ii}. First, observe that, if $i \neq j$ then $0 = \Delta(a_{ij}t_{ii}) = t_{ii}^*\Delta(a_{ij}) + \Delta(t_{ii})a_{ij}^*$ which implies $\Delta(t_{ii})a_{ij}^* = -t_{ii}^*\Delta(a_{ij})$.

Now,

$$\begin{aligned}\Delta((a_{ii} + a_{ij})t_{ii}) &= t_{ii}^*\Delta(a_{ii} + a_{ij}) + \Delta(t_{ii})(a_{ii} + a_{ij})^* \\ &= t_{ii}^*\Delta(a_{ii} + a_{ij}) + \Delta(t_{ii})a_{ii}^* - t_{ii}^*\Delta(a_{ij}).\end{aligned}$$

On the other hand,

$$\Delta((a_{ii} + a_{ij})t_{ii}) = \Delta(a_{ii}t_{ii}) = t_{ii}^*\Delta(a_{ii}) + \Delta(t_{ii})a_{ii}^* \tag{2.1}$$

Then $t_{ii}^*[\Delta(a_{ii} + a_{ij}) - \Delta(a_{ii}) - \Delta(a_{ij})] = 0$ which implies that, by condition (i), $[\Delta(a_{ii}+a_{ij})-\Delta(a_{ii})-\Delta(a_{ij})]_{ik} = 0$ for $i, k = 1, 2$ and hence,

$$\Delta(a_{ii} + a_{ij}) = \Delta(a_{ii}) + \Delta(a_{ij}).$$

Step 2: For $i \neq j$, $\Delta(a_{ij} + a_{ji}) = \Delta(a_{ij}) + \Delta(a_{ji})$.
Let t_{ii} be an element of R_{ii} and recall that, if $i \neq j$, we have $\Delta(t_{ii})a_{ij}^* = -t_{ii}^*\Delta(a_{ij})$. First,

$$\Delta((a_{ij} + a_{ji})t_{ii}) = \Delta(a_{ji}t_{ii}) = t_{ii}^*\Delta(a_{ji}) + \Delta(t_{ii})a_{ji}^*.$$

On the other hand,

$$\Delta((a_{ij} + a_{ji})t_{ii}) = t_{ii}^*\Delta(a_{ij} + a_{ji}) + \Delta(t_{ii})(a_{ji} + a_{ji})^*.$$

Then, $t_{ii}^*[\Delta(a_{ij}+a_{ji})-\Delta(a_{ij})-\Delta(a_{ji})] = 0$ which implies that, by condition (i), $[\Delta(a_{ij} + a_{ji}) - \Delta(a_{ij}) - \Delta(a_{ji})]_{ik} = 0$ for $i, k = 1, 2$ and hence,

$$\Delta(a_{ij} + a_{ji}) = \Delta(a_{ij}) + \Delta(a_{ji}).$$

Step 3: For $i \neq j$, $\Delta(a_{ij} + b_{ij}c_{jj}) = \Delta(a_{ij}) + \Delta(b_{ij}c_{jj})$.
Notice that $(e_i + b_{ij})(a_{ij} + c_{jj}) = a_{ij} + b_{ij}c_{jj}$. By Step 1, we have

$$\begin{aligned}\Delta((e_i + b_{ij})(a_{ij} + c_{jj})) &= (a_{ij} + c_{jj})^*\Delta(e_i + b_{ij}) + \Delta(a_{ij} + c_{jj})(e_i + b_{ij})^* \\ &= (a_{ij} + c_{jj})^*\Delta(b_{ij}) + (\Delta(a_{ij}) + \Delta(c_{jj}))(e_i + b_{ij})^*.\end{aligned}$$

Finally, comparing $\Delta((e_i+b_{ij})(a_{ij}+c_{jj}))$ and $\Delta(a_{ij}+b_{ij}c_{jj})$, a straightforward calculation shows us that

$$\Delta(a_{ij} + b_{ij}c_{jj}) = \Delta(a_{ij}) + \Delta(b_{ij}c_{jj}).$$

Step 4: For $i \neq j$, $\Delta(a_{ij} + b_{ii}c_{ij}) = \Delta(a_{ij}) + \Delta(b_{ii}c_{ij})$.
Step 4 can be proved as Step 3, using the relation

$$(a_{ij} + b_{ii})(e_j + c_{ij}) = a_{ii} + b_{ii}c_{ij}.$$

Step 5: For $i \neq j$, $\Delta(a_{ij} + b_{ij}) = \Delta(a_{ij}) + \Delta(b_{ij})$.
For $t_{ij} \in R_{jj}$, using Step 3,

$$\begin{aligned}\Delta((a_{ij}+b_{ij})t_{jj}) &= \Delta(a_{ij}t_{jj}+b_{ij}t_{jj})\\ &= \Delta(a_{ij}t_{jj})+\Delta(b_{ij}t_{jj})\\ &= t_{jj}^*\Delta(a_{ij})+\Delta(t_{jj})a_{ij}^*+t_{jj}^*\Delta(b_{ij})+\Delta(t_{jj})b_{ij}^*.\end{aligned}$$

On the other hand, we have

$$\Delta((a_{ij}+b_{ij})t_{jj}) = t_{jj}^*\Delta(a_{ij}+b_{ij})+\Delta(t_{jj})(a_{ij}+b_{ij})^*.$$

Then, $t_{jj}^*[\Delta(a_{ij}+b_{ij})-\Delta(a_{ij})-\Delta(b_{ij})]=0$, which implies, by condition (i), $[\Delta(a_{ij}+b_{ij})-\Delta(a_{ij})-\Delta(b_{ij})]_{jk}=0$. Since, by Lemma 2.2, $\Delta(R_{ij})\subset R_{ji}$, we get

$$\Delta(a_{ij}+b_{ij})=\Delta(a_{ij})+\Delta(b_{ij}).$$

Step 6: $\Delta(a_{ii}+b_{ii})=\Delta(a_{ii})+\Delta(b_{ii})$.
For $t_{21}\in R_{21}$, by Step 5, we have

$$\begin{aligned}\Delta(t_{21}(a_{11}+b_{11})) &= \Delta(t_{21}a_{11}+t_{21}b_{11})\\ &= \Delta(t_{21}a_{11})+\Delta(t_{21}b_{11})\\ &= a_{11}^*\Delta(t_{21})+\Delta(a_{11})t_{21}^*+b_{11}^*\Delta(t_{21})+\Delta(b_{11})t_{21}^*.\end{aligned}$$

We also have

$$\Delta(t_{21}(a_{11}+b_{11}))=(a_{11}+b_{11})^*\Delta(t_{21})+\Delta(a_{11}+b_{11})t_{21}^*.$$

Then, we obtain

$$[\Delta(a_{11}+b_{11})-\Delta(a_{11})-\Delta(b_{11})]t_{21}^*=0,$$

which implies, by Proposition 2.1, Lemma 2.2 and condition (ii), $\Delta(a_{11}+b_{11})-\Delta(a_{11})-\Delta(b_{11})=0$. Now for $t_{12}\in R_{12}$, by Step 5, we have

$$\begin{aligned}\Delta(t_{12}(a_{22}+b_{22})) &= \Delta(t_{12}a_{22}+t_{12}b_{22})\\ &= \Delta(t_{12}a_{22})+\Delta(t_{12}b_{22})\\ &= a_{22}^*\Delta(t_{12})+\Delta(a_{22})t_{12}^*+b_{22}^*\Delta(t_{12})+\Delta(b_{22})t_{12}^*.\end{aligned}$$

We also have

$$\Delta(t_{12}(a_{22}+b_{22}))=(a_{22}+b_{22})^*\Delta(t_{12})+\Delta(a_{22}+b_{22})t_{12}^*.$$

Then, we obtain

$$[\Delta(a_{22}+b_{22})-\Delta(a_{22})-\Delta(b_{22})]t_{12}^{*}=0,$$

which implies, by Proposition 2.1, Lemma 2.2 and condition (ii), $\Delta(a_{22}+b_{22})-\Delta(a_{22})-\Delta(b_{22})=0$.

Step 7: $\Delta(a_{11}+b_{12}+c_{21}+d_{22})=\Delta(a_{11})+\Delta(b_{12})+\Delta(c_{21})+\Delta(d_{22})$ and, hence, Δ is additive.
For $t_{11}\in R_{11}$, we conclude that

$$\begin{aligned}
& \Delta(t_{11})(a_{11}+b_{12}+c_{21}+d_{22})^{*}+t_{11}^{*}\Delta(a_{11}+b_{12}+c_{21}+d_{22}) \\
= \; & \Delta((a_{11}+b_{12}+c_{21}+d_{22})t_{11}) \\
= \; & \Delta(a_{11}t_{11}+c_{21}t_{11}) \\
= \; & \Delta(a_{11}t_{11})+\Delta(c_{21}t_{11}) \\
= \; & \Delta(a_{11}t_{11})+\Delta(b_{12}t_{11})+\Delta(c_{21}t_{11})+\Delta(d_{22}t_{11}) \\
= \; & \Delta(t_{11})a_{11}^{*}+t_{11}^{*}\Delta(a_{11})+\Delta(t_{11})b_{12}^{*}+t_{11}^{*}\Delta(b_{12}) \\
+ \; & \Delta(t_{11})c_{21}^{*}+t_{11}^{*}\Delta(c_{21})+\Delta(t_{11})d_{22}^{*}+t_{11}^{*}\Delta(d_{22}).
\end{aligned}$$

By condition (i), $t_{11}^{*}[\Delta(a_{11}+b_{12}+c_{21}+d_{22})-\Delta(a_{11})-\Delta(b_{12})-\Delta(c_{21})-\Delta(d_{22})]=0$ implies

$$[\Delta(a_{11}+b_{12}+c_{21}+d_{22})-\Delta(a_{11})-\Delta(b_{12})-\Delta(c_{21})-\Delta(d_{22})]_{1k}=0.$$

By a similar calculation, using $t_{22}\in R_{22}$, one can conclude that

$$[\Delta(a_{11}+b_{12}+c_{21}+d_{22})-\Delta(a_{11})-\Delta(b_{12})-\Delta(c_{21})-\Delta(d_{22})]_{2k}=0.$$

Therefore, $\Delta(a_{11}+b_{12}+c_{21}+d_{22})=\Delta(a_{11})+\Delta(b_{12})+\Delta(c_{21})+\Delta(d_{22})$. The conclusion that Δ is additive is straightforward now.

We would like to end the article by noting a result on additivity for the case of multiplicative derivable maps:

Theorem 2.2. *Let R be a ring with involution containing a nontrivial symmetric idempotent element e and the following conditions are satisfied*

(i) If $x_{ii}a_{ij}=0$ for all $x_{ii}\in R_{ii}$, then $a_{ij}=0$;

(ii) If $a_{ii}x_{ij}=0$ for all $x_{ij}\in R_{ij}$ with $i\neq j$, then $a_{ii}=0$.

Then every multiplicative derivable map $\delta : R\to R$ is additive.

Proof. It is sufficient to note that for a multiplicative derivable map $\delta : R \to R$, we have that $* \circ \delta : R \to R$ is a $*$-reverse derivable map. So, by Theorem 2.1 we have the desired result. $\square$

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