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On graphs whose chromatic transversal number is two

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Abstract

In this paper we characterize the class of trees, block graphs, cactus graphs and cubic graphs for which the chromatic transversal domination number is equal to two.

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1. Introduction

Let $G = (V, E)$ be a simple graph of order p. A vertex v of G is a *critical vertex* if $\chi(G - v) < \chi(G)$, where $\chi(G)$ is the chromatic number of G. If every vertex of G is a critical vertex, then G is called a *vertex critical graph.* A subset $D \subset V$ is a *dominating set*, if every $v \in V - D$ is adjacent to some $u \in D$. The *domination number* $\gamma = \gamma(G)$ is the minimum cardinality of a dominating set of G. A dominating set D is called a *chromatic transversal dominating set* (ctd-set) if D has non empty intersection with every color class of every chromatic partition of G. The *chromatic transversal domination number* $\gamma_{ct} = \gamma_{ct}(G)$ is the minimum cardinality of a ctd-set of G. The parameter γ_{ct} for a few well known graphs was computed by L.Benedict et al. [1]. For example, if G is a vertex critical graph of order p, then $\gamma_{ct}(G) = p$.

By a *double star* we mean a tree obtained by joining the centers of two stars $K_{1,m}$ and $K_{1,n}$ by an edge. If we subdivide the edge connecting the centers of two stars, then it is called a *double star with one subdivision.* Similarly, a *double star with two subdivisions* is defined. The *diameter* of a graph G is the length of the longest path in G and is denoted by $diam(G)$. A vertex v of a connected graph G is said to be a *cutvertex* if $G - v$ is no longer connected. A connected subgraph B of G is a *block*, if B has no cutvertex and every subgraph $B' \subset G$ with $B \subset B'$ and $B \neq B'$ has at least one cutvertex. A block B of G is called an *end block*, if B contains at most one cutvertex of G; such a cutvertex is called an *end block cutvertex.* A graph G is called a *block graph*, if every block G is a complete graph. A graph G is called a *cactus graph* if every edge of G is in at most one cycle of G. A graph G is said to be a *cubic graph* if it is 3-regular. $\Delta(G) = max\{deg(v) : v \in V(G)\}$. A *support vertex* in G is one which is adjacent to a leaf.

Theorem 1.0.1. [2] *Let G be a connected bipartite graph of order $p \geq 3$ with partition (V_1, V_2) of $V(G)$, where $|V_1| \leq |V_2|$. Then $\gamma_{ct}(G) = \gamma(G) + 1$ if and only if every vertex in V_1 has at least two pendant neighbors.*

Theorem 1.0.2. [2] *For a tree T, $\gamma_{ct}(T) = \gamma(T) + 1$ if and only if either T is K_2 or T satisfies the condition that whenever v is a support vertex, then each vertex w with $d(v, w)$ even is also a support vertex and each support vertex has at least two pendant neighbors. Otherwise $\gamma_{ct}(T) = \gamma(T)$.*

2. Characterization

2.1. Trees

Lemma: 2.1.1. *For a tree T, $\gamma(T) = 2$ if and only if T is one of the following:*

(i) a double star

(ii) a double star with one subdivision

(iii) a double star with two subdivisions.

Proof: Assume that $\gamma(T) = 2$.

Claim: $diam(T) \leq 5$.

If not, let P be the largest path in T with length greater than 5. Then $\gamma(P) \geq 3$ where $\gamma(P)$ refers to the domination number of the path P. Without loss of generality assume that $\gamma(P) = 3$ and let $D = \{x_1, x_2, x_3\}$ be a γ-set of P.

Now, take any γ-set $S = \{x, y\}$ of T. If x or y is not in P, then a cycle will be formed with one of the vertices of D. In fact, if $x = x_1$ and y is not in P, then x_3 must be adjacent to y and at least one of the neighbors of x_3, say u, will be adjacent to y so that the vertices x_3, y, u form a cycle. Thus $x, y \in P$. But if $x, y \in P$ then at least one of the vertices of D will not be dominated by S, contradicting the assumption that $\gamma(T) = 2$.

Case 1. Let $diam(T) = 3$ and $P_4 : u_1u_2u_3u_4$ be the longest path in T. If $S = \{x, y\}$ is a γ-set of T, then as argued earlier, $S \subset V(P_4)$. As P_4 is the longest path in T it follows that u_1 and u_4 are pendant vertices in T. If $x = u_1$ and $y = u_4$, then T is a path P_4. If $x = u_1$ and $y = u_3$, then T is a double star with $K_{1,t}$ at u_3. Similarly, we get a double star if $x = u_2$ and $y = u_4$(or if $x = u_2$ and $y = u_3$).

Case 2. Let $diam(T) = 4$ and $P_5 : u_1u_2u_3u_4u_5$ be the longest path in T with u_1 and u_5 as pendant vertices in T. Then for any γ-set $S = \{x, y\}$ of T,we have $S \subset V(P_5)$. We claim that $x = u_2$ and $y = u_4$. Suppose $x = u_3$ and $y = u_4$. Then u_1 will not be dominated by S. Similarly, the other possibilities for x and y except $x = u_2$ and $y = u_4$. Thus T is a double star with one subdivision.

Case 3. Let $diam(T) = 5$ and $P_6 : u_1u_2u_3u_4u_5u_6$ be the longest path in T. Then any γ-set $S = \{x, y\}$ of T is a subset of $V(P_6)$ and as argued earlier $x = u_2$ and $y = u_4$ making T a double star with two subdivisions.

The converse is obvious.

Theorem 2.1.1. *For a tree T, $\gamma_{ct}(T) = 2$ if and only if T is one of the following:*

(i) a double star

(ii) a double star with two subdivisions

(iii) a star graph.

Proof: Assume that $\gamma_{ct}(T) = 2$. According to Theorem 1.0.2, $\gamma_{ct}(T)$ is either $\gamma(T)$ or $\gamma(T) + 1$. If $\gamma_{ct}(T) = \gamma(T)$, then $\gamma(T) = 2$ and so by Lemma 2.1.1, T is a double star or a double star with one subdivision or a double star with two subdivisions. But T cannot be a double star with one subdivision in view of Theorem 1.0.2.

If $\gamma_{ct}(T) = \gamma(T) + 1$, then $\gamma(T) = 1$ and so T is a star graph.

The converse is obvious.

2.2. Block graphs

Proposition: 2.2.1. *For a block graph G, $\gamma_{ct}(G) = 2$ if and only if G is a star graph.*

Proof: Assume that $\gamma_{ct}(G) = 2$. Let K_n be a maximal clique with maximum number of cutvertices. Then one can show that $\gamma_{ct}(G) = n + \gamma(G')$ where $G' = G - V(K_n) - L$ and L is the set of all leaves with supports at some vertices of K_n. Then $\gamma_{ct}(G) = 2$ if and only if $n = 2$ and $\gamma(G') = 0$ or $n = 1$ and $\gamma(G') = 1$. In either case G is a star graph.

The converse is obvious.

Note: For a block graph G, one can easily verify that $\gamma(G) = 2$ if and only if G has exactly two end vertices and at most two internal cut vertices.

2.3. Cubic graphs

Proposition: 2.3.1. *For a connected cubic graph G of order p, $\gamma_{ct}(G) = 2$ if and only if $p \leq 8$.*

Proof: Let G be a cubic graph with $\gamma_{ct}(G) = 2$. Then since $\gamma_{ct}(G) \geq \chi(G)$, $\chi(G) = 2$. That is G is a bipartite graph. Therefore by Theorem 1.0.1, $\gamma_{ct}(G) = \gamma(G)$ and so $\gamma(G) = 2$. But then we have $\frac{p}{1+\Delta(G)} \leq 2$ which implies $\frac{p}{4} \leq 2$. That is $p \leq 8$. Conversely, if G is a cubic graph with $p \leq 8$, one can easily verify that $\gamma_{ct}(G) = 2$.

This proves the result.

2.4. Cactus graphs

Proposition: 2.4.1. *If G is a cactus graph with at least one cycle, then $\gamma_{ct}(G) = 2$ if and only if G is either C_4 with at most two support vertices that are adjacent or C_6 with a pair of support vertices u_i and u_j where $j = i + 3$ $(mod\ 6)$ if $V(C_6) = \{u_0, u_1, ..., u_5\}$.*

Proof: Let us assume that $\gamma_{ct}(G) = 2$. If G has an odd cycle, $\chi(G) = 3$ and so $\gamma_{ct}(G) \geq 3$, a contradiction. Therefore G cannot have an odd cycle.

Suppose G has an even cycle of length greater than or equal to 8. Then $\gamma_{ct}(G) = \lceil \frac{8}{3} \rceil = 3$, which is a contradiction. Therefore G has an even cycle of length 4 or 6. Furthermore G is unicyclic. If not, $\gamma_{ct}(G) \geq \gamma(G) \geq 3$.

Case 1. Let G be a unicyclic graph with C_4, a cycle of length 4. Let X be the set of all vertices of degree 2 in C_4. Now $\gamma_{ct}(G) = 2$ implies $|X| \geq 2$. If $|X| = 4$, G is just C_4. If $|X| = 3$, G is C_4 with one support vertex. Similarly, if $|X| = 2$, G is C_4 with two support vertices and as $\gamma_{ct}(G) = 2$, these support vertices are adjacent.

Case 2. Let G be a unicyclic graph with C_6, cycle of length 6.

The proof of this case is just similar to Case 1 except that two support vertices require to be of distance 3 to form a γ_{ct}-set of G.

The converse is obvious.

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