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Characterization of prime rings having involution and centralizers

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Abstract

The major goal of this paper is to study the commutativity of prime rings with involution that meet specific identities using left centralizers. The results obtained in this paper are the generalization of many known theorems. Finally, we provide some examples to show that the conditions imposed in the hypothesis of our results are not superfluous.

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1. Introduction

Throughout this paper, χ denotes an associative ring with centre $Z(\chi)$. This study was motivated by [1] and deals with the commutativity of prime rings with involution involving left centralizers. For any $s_1, s_2 \in \chi$, the notation $[s_1, s_2]$ illustrates the commutator $s_1s_2 - s_2s_1$, and $s_1 \circ s_2$ denotes the anti-commutator $s_1s_2 + s_2s_1$. An additive map $s_1 \mapsto s_1^*$ of χ into itself is said to be an involution if it satisfies $(s_1s_2)^* = s_2^*s_1^*$ and $(s_1^*)^* = s_1 \forall s_1, s_2 \in \chi$. Rings with involution, often known as $*$-rings. Let $H(\chi)$ be the collection of Hermitian elements ($s_1^* = s_1$) and $S(\chi)$ be the collection of skew-Hermitian elements ($s_1^* = -s_1$) of χ. If char$(\chi) = 2$, then, obviously, $H(\chi) = S(\chi)$. If $Z(\chi) \subseteq H(\chi)$, the involution is said to be of the first kind; otherwise, it is of the second kind. In the latter case $S(\chi) \cap Z(\chi) \neq (0)$ (e.g. quaternion involution is of the first kind). In [6], there's a mention of these rings as well as additional references.

Following [13], an additive mapping $T : \chi \to \chi$ is called a left (resp. right) centralizer of χ if $T(s_1s_2) = T(s_1)s_2$ (resp. $T(s_1s_2) = s_1T(s_2)) \forall s_1$, $s_2 \in \chi$. The mapping $T : \chi \to \chi$ is called centralizer of χ if it is both left as well as right centralizer of χ. Further, many authors have extended this definition and obtained many results between the ring χ's commutativity and certain types of mappings on χ. One of the result was found by Divinsky [4], who established that if a simple Artinian ring χ has a commuting non-trivial automorphism, then χ is commutative. A number of authors went on to refine and extend this conclusion in diverse directions (viz., [2, 3, 5, 7, 9, 10, 11, 12]). Recently Ali and Dar in [1], investigated the commutativity of $(\chi, *)$ prime ring involving left centralizers. In fact, they have proved that if χ be a prime ring characteristic different from two having involution $*$ of the second kind and having a left centralizer $T(\neq 0)$ satisfying $T([s_1, s_1^*]) = 0 \ \forall s_1 \in \chi$, implies χ is commutative. In this paper we have obtained such results in a more general form.

In our paper, we use only left centralizers and these results are also valid for right centralizers as of its symmetry. Throughout our discussion, $(\chi, *)$ refers as ring χ with involution $*$ of the second kind.

2. Results

We start this section with some basic results.

Lemma 2.1. *[8, Lemma 2.1] Let $(\chi, *)$ be a prime ring with characteristic different from two. Then $[s_1, s_1^*] \in Z(\chi) \forall s_1 \in \chi \iff \chi$ is commutative.*

Lemma 2.2. *[8, Lemma 2.2] Let $(\chi, *)$ be a prime ring with characteristic different from two. Then $s_1 \circ s_1^* \in Z(\chi) \forall s_1 \in \chi \iff \chi$ is commutative.*

Theorem 2.3. *Let $(\chi, *)$ be a prime ring with characteristic different from two. If χ has a left centralizer $T(\neq 0)$ satisfying $T([s_1, s_1^*]) \in Z(\chi) \forall s_1 \in \chi$, then χ is commutative.*

Proof. By the given assumption, we have

$$T([s_1, s_1^*]) \in Z(\chi) \forall s_1 \in \chi. \tag{2.1}$$

Linearizing (2.1), we get

$$T([s_1, s_2^*]) + T([s_2, s_1^*]) \in Z(\chi) \forall s_1, s_2 \in \chi. \tag{2.2}$$

Replacing s_2 by $s_4 s_2$ in (2.2), where $s_4 \in S(\chi) \cap Z(\chi)$, we have

$$\begin{aligned} -T([s_1, s_2^*])s_4 + T([s_2, s_1^*])s_4 \in Z(\chi) \forall s_1, s_2 \in \chi \text{ and} \\ s_4 \in S(\chi) \cap Z(\chi). \end{aligned} \tag{2.3}$$

Multiplying (2.2) by s_4 from right and adding it with (2.3), we arrive at

$$2T([s_2, s_1^*])s_4 \in Z(\chi) \forall s_1, s_2 \in \chi \text{ and } s_4 \in S(\chi) \cap Z(\chi).$$

Since characteristic of χ is different from two, we get

$$T([s_2, s_1^*])s_4 \in Z(\chi) \forall s_1, s_2 \in \chi \text{ and } s_4 \in S(\chi) \cap Z(\chi).$$

Since, χ is prime ring and we know that $S(\chi) \cap Z(\chi) \neq (0)$, we obtain

$$T([s_2, s_1^*]) \in Z(\chi) \forall s_1, s_2 \in \chi.$$

Replace s_1 by s_1^*, we get

$$T([s_2, s_1]) \in Z(\chi) \forall s_1, s_2 \in \chi. \tag{2.4}$$

Replacing s_1 by $s_1 t$ in (2.4), where $t \in \chi$, we get $T([s_2, s_1 t]) \in Z(\chi) \forall s_1, s_2, t \in \chi$ With the help of (2.4), we obtain

$$T(s_1)[[s_2, t], s_3] + [T(s_1), s_3][s_2, t] + T([s_2, s_1])[s_3, t] = 0 \ \forall s_1, s_2, s_3, t \in \chi. \tag{2.5}$$

Substituting s_2 for t in (2.5), we arrive at

$$T([s_2, s_1])[s_2, s_3] = 0 \ \forall s_1, s_2, s_3 \in \chi. \tag{2.6}$$

Replacing s_3 by s_3m in (2.6), where $m \in \chi$ and applying (2.6), we obtain

$$T([s_2, s_1])s_3[s_2, m] = 0 \ \forall s_1, s_2, s_3, m \in \chi. \tag{2.7}$$

Using the primeness of χ, for fixed s_2, we get either $T([s_2, s_1]) = 0 \ \forall s_1 \in \chi$ or $[s_2, m] = 0 \ \forall m \in \chi$. Define $M = \{s_2 \in \chi | \ T([s_2, s_1]) = 0 \ \forall s_1 \in \chi\}$ and $N = \{s_2 \in \chi | [s_2, m] = 0 \ \forall m \in \chi\}$. Now, M and N are additive subgroup of χ such that $\chi = M \cup N$. Then by Brauer's results, either $M = \chi$ or $N = \chi$. If $N = \chi$, then $[s_2, m] = 0 \ \forall s_2, m \in \chi$, we get χ is commutative. Now consider $M = \chi$, in this situation, we have

$$T([s_2, s_1]) = 0 \ \forall s_1, s_2 \in \chi. \tag{2.8}$$

Replace s_2 by s_2u in (2.8), where $u \in \chi$, we get $T(s_2)[u, s_1] + T([s_2, s_1])u = 0 \ \forall s_1, s_2, u \in \chi$. Using (2.8), we obtain $T(s_2)[u, s_1] = 0 \ \forall s_1, s_2, u \in \chi$ Replacing s_2 by s_2s_3 where $s_3 \in \chi$, we get $T(s_2)s_3[u, s_1] = 0 \ \forall s_1, s_2, s_3, u \in \chi$, primeness of χ gives us either T is zero or $[u, s_1] = 0 \ \forall u, s_1 \in \chi$. Since $T \neq 0$, we get $[u, s_1] = 0 \ \forall u, s_1 \in \chi$ and hence χ is commutative. □

Corollary 2.1. *[1, Theorem 3.1] Let $(\chi, *)$ be a prime ring with characteristic different from two. If χ has a left centralizer $T(\neq 0)$ satisfying $T([s_1, s_1^*]) = 0 \ \forall s_1 \in \chi$, then χ is commutative.*

Theorem 2.4. *Let $(\chi, *)$ be a prime ring with characteristic different from two. If χ has a left centralizer $T(\neq 0)$ satisfying $T(s_1 \circ s_1^*) \in Z(\chi) \forall s_1 \in \chi$, then χ is commutative.*

Proof. We have

$$T(s_1 \circ s_1^*) \in Z(\chi) \forall s_1 \in \chi. \tag{2.9}$$

Linearizing (2.9), we get

$$T(s_1 \circ s_2^*) + T(s_2 \circ s_1^*) \in Z(\chi) \forall s_1, s_2 \in \chi. \tag{2.10}$$

Replacing s_2 by s_4s_2 in (2.10) where $s_4 \in S(\chi) \cap Z(\chi)$, we get

$$-T(s_1 \circ s_2^*)s_4 + T(s_2 \circ s_1^*)s_4 \in Z(\chi)\forall s_1, s_2 \in \chi \text{ and } s_4 \in S(\chi) \cap Z(\chi). \tag{2.11}$$

Multiplying (2.10) by s_4 from right and on comparing it with (2.11), we get $2T(s_2 \circ s_1^*)s_4 \in Z(\chi)\forall s_1, s_2 \in \chi$ and $s_4 \in S(\chi) \cap Z(\chi)$. Since char $(\chi) \neq 2$, this implies that $T(s_2 \circ s_1^*)s_4 \in Z(\chi)\forall s_1, s_2 \in \chi$ and $s_4 \in S(\chi) \cap Z(\chi)$. Now using the primeness of χ and the fact that $S(\chi) \cap Z(\chi) \neq (0)$, we get $T(s_2 \circ s_1^*) \in Z(\chi)\forall s_1, s_2 \in \chi$. Taking s_1^* for s_1 we get $T(s_2 \circ s_1) \in Z(\chi)\forall s_1, s_2 \in \chi$. Replacing s_2 by h where $h \in H(\chi) \cap Z(\chi)$, we get $2T(s_1)h \in Z(\chi)\forall s_1 \in \chi$ and $h \in H(\chi) \cap Z(\chi)$. Since characteristic of χ is different from two and $S(\chi) \cap Z(\chi) \neq (0)$, we get $T(s_1) \in Z(\chi)\forall s_1 \in \chi$. That is,

$$[T(s_1), s_3] = 0 \ \forall s_1, s_3 \in \chi. \tag{2.12}$$

Replacing s_1 by $s_1 s_2$ where $s_1, s_2 \in \chi$ and using (2.12), we finally arrive at

$$T(s_1)[s_2, s_3] = 0 \ \forall s_1, s_2, s_3 \in \chi.$$

Taking $s_2 = ws_2$ where $w \in \chi$, we get $T(s_1)w[s_2, s_3] = 0 \ \forall s_1, s_2, s_3$ and $w \in \chi$. Since $T \neq 0$ and by the primeness of χ, we get the required result. □

Corollary 2.2. *[1, Theorem 3.2] Let $(\chi, *)$ be a prime ring with characteristic different from two. If χ has a left centralizer $T(\neq 0)$ satisfying $T(s_1 \circ s_1^*) = 0 \ \forall s_1 \in \chi$, then χ is commutative.*

Corollary 2.3. *Let $(\chi, *)$ be a prime ring with characteristic different from two. If χ has a left centralizer $T(\neq 0)$ satisfying $T(s_1 \circ s_1^*) \mp [s_1, s_1^*] \in Z(\chi)\forall s_1 \in \chi$, then χ is commutative.*

Proof. We have

$$T(s_1 \circ s_1^*) \mp [s_1, s_1^*] \in Z(\chi)\forall s_1 \in \chi. \tag{2.13}$$

If $T(s_1 \circ s_1^*)$ is zero, thus, χ is commutative by Lemma 2.1. Now consider $T(s_1 \circ s_1^*)$ is nonzero. Substituting s_1^* for s_1 in (2.13), we obtain

$$T(s_1 \circ s_1^*) \mp [s_1^*, s_1] \in Z(\chi)\forall s_1 \in \chi. \tag{2.14}$$

Combining (2.13) and (2.14), we get $T(s_1 \circ s_1^*) \in Z(\chi)\forall s_1 \in \chi$. Thus in view of Theorem 2.4, we get the required result. □

Theorem 2.5. *Let $(\chi, *)$ be a prime ring with characteristic different from two. If χ has a left centralizer T satisfying $T([s_1, s_1^*]) \mp [s_1, s_1^*] \in Z(\chi) \forall s_1 \in \chi$ and $T(s_1) \neq \mp s_1 \forall s_1 \in \chi$, then χ is commutative.*

Proof. First, we consider the situation

$$T([s_1, s_1^*]) - [s_1, s_1^*] \in Z(\chi) \forall s_1 \in \chi. \tag{2.15}$$

If $T([s_1, s_1^*])$ is zero then we get $[s_1, s_1^*] \in Z(\chi) \forall s_1 \in \chi$. Then by using Lemma 2.1, we get χ is commutative. Later consider $T([s_1, s_1^*])$ to be nonzero. Linearizing (2.15), we get

$$T([s_1, s_2^*]) + T([s_2, s_1^*]) - [s_1, s_2^*] - [s_2, s_1^*] \in Z(\chi) \forall s_1, s_2 \in \chi. \tag{2.16}$$

Replacing s_2 by $s_4 s_2$ in (2.16) where $s_4 \in S(\chi) \cap Z(\chi)$, we get

$$\begin{aligned} -T([s_1, s_2^*])s_4 + T([s_2, s_1^*])s_4 + [s_1, s_2^*]s_4 - [s_2, s_1^*]s_4 \in Z(\chi) \forall s_1, s_2 \in \chi \\ \text{and } s_4 \in S(\chi) \cap Z(\chi). \end{aligned} \tag{2.17}$$

Multiplying (2.16) by s_4 from right and adding it with (2.17), we obtain

$$2(T([s_2, s_1^*]) - [s_2, s_1^*])s_4 \in Z(\chi) \forall s_1, s_2 \in \chi \text{ and } s_4 \in S(\chi) \cap Z(\chi).$$

Since characteristic of χ is different from two and $S(\chi) \cap Z(\chi) \neq (0)$, last relation yields

$$T([s_2, s_1^*]) - [s_2, s_1^*] \in Z(\chi) \forall s_1, s_2 \in \chi. \tag{2.18}$$

That is,

$$[T([s_2, s_1^*]) - [s_2, s_1^*], s_3] = 0 \ \forall s_1, s_2, s_3 \in \chi.$$

Or

$$[T([s_2, s_1^*]), s_3] - [[s_2, s_1^*], s_3] = 0 \ \forall s_1, s_2, s_3 \in \chi.$$

Taking s_1^* for s_1, we get

$$[T([s_2, s_1]), s_3] - [[s_2, s_1], s_3] = 0 \ \forall s_1, s_2, s_3 \in \chi. \tag{2.19}$$

Replacing s_2 by $s_2 w$ in (2.19), where $w \in \chi$

$$[T([s_2 w, s_1]), s_3] - [[s_2 w, s_1], s_3] = 0 \ \forall s_1, s_2, s_3, w \in \chi. \tag{2.20}$$

This finally yields that

$$
\begin{aligned}
&T(s_2)[[w,s_1],s_3]+[T(s_2),s_3][w,s_1]+T([s_2,s_1])[w,s_3] \\
&+[T([s_2,s_1]),s_3]w-s_2[[w,s_1],s_3]-[s_2,s_3][w,s_1]-[s_2,s_1][w,s_3] \\
&-[[s_2,s_1],s_3]w=0\ \forall s_1,s_2,s_3,w\in\chi. \qquad (2.21)
\end{aligned}
$$

Multiplying (2.19) on the right by w, where $w\in\chi$ and subtracting from (2.21), we obtain

$$
T(s_2)[[w,s_1],s_3]+[T(s_2),s_3][w,s_1]+T([s_2,s_1])[w,s_3]- \qquad (2.22)
$$

$$
s_2[[w,s_1],s_3]-[s_2,s_3][w,s_1]-[s_2,s_1][w,s_3]=0
$$

for all $s_1,s_2,s_3,w\in\chi$. Taking s_1 for w in (2.22), we get

$$
T([s_2,s_1])[s_1,s_3]-[s_2,s_1][s_1,s_3]=0\ \forall s_1,s_2,s_3\in\chi.
$$

This can be further written as

$$
(T([s_2,s_1])-[s_2,s_1])[s_1,s_3]=0\ \forall s_1,s_2,s_3\in\chi. \qquad (2.23)
$$

Replacing s_3 by ts_3 in (2.23), where $t\in\chi$.

$$
(T([s_2,s_1])-[s_2,s_1])t[s_1,s_3]+(T([s_2,s_1])-[s_2,s_1])[s_1,t]s_3=0
$$

$$
\forall s_1,s_2,s_3,t\in\chi.
$$

In view of (2.23), we get

$$
(T([s_2,s_1])-[s_2,s_1])t[s_1,s_3]=0\ \forall s_1,s_2,s_3,t\in\chi.
$$

Now by primeness of χ, for each fixed $s_1\in\chi$, we get either $T([s_2,s_1])-[s_2,s_1]=0\ \forall s_2\in\chi$ or $[s_1,s_3]=0\ \forall s_3\in\chi$. Define $A=\{s_1\in\chi|\ T([s_2,s_1])-[s_2,s_1]=0\ \forall s_2\in\chi\}$ and $B=\{s_1\in\chi|[s_1,s_3]=0\ \forall s_3\in\chi\}$. We observe that A and B are additive subgroups of χ. So, by previous argument we get either $A=\chi$ or $B=\chi$. If $B=\chi$, then $[s_1,s_3]=0\ \forall s_1,s_3\in\chi$, we get χ is commutative. Now consider $A=\chi$, in this situation

$$
T([s_2,s_1])-[s_2,s_1]=0\ \forall s_1,s_2\in\chi. \qquad (2.24)
$$

Replacing s_2 by $s_2T(w)$ in (2.24), where $w\in\chi$, we obtain

$$
T([s_2T(w),s_1])-[s_2T(w),s_1]=0\ \forall s_1,s_2,w\in\chi.
$$

That is,

$$T(s_2)[T(w), s_1] + T([s_2, s_1])T(w) - s_2[T(w), s_1] - [s_2, s_1]T(w) = 0 \quad \forall s_1, s_2, w \in \chi.$$

by using (2.24), we get

$$T(s_2)[T(w), s_1] - s_2[T(w), s_1] = 0 \ \forall s_1, s_2, w \in \chi. \tag{2.25}$$

Substituting $s_2T(m)$ for s_2 in (2.25), where $m \in \chi$, we obtain

$$T(s_2)T(m)[T(w), s_1] - s_2T(m)[T(w), s_1] = 0 \ \forall m, s_1, s_2, w \in \chi. \tag{2.26}$$

Left multiplication by $T(m)$ in (2.25), produces

$$T(m)T(s_2)[T(w), s_1] - T(m)s_2[T(w), s_1] = 0 \ \forall m, s_1, s_2, w \in \chi. \tag{2.27}$$

Combining (2.26) and (2.27), we have

$$([T(s_2), T(m)] + [T(m), s_2])[T(w), s_1] = 0 \ \forall m, s_1, s_2, w \in \chi. \tag{2.28}$$

Replacing s_1 by s_1u, where $u \in \chi$ in (2.28) and using it again, we get

$$([T(s_2), T(m)] + [T(m), s_2])s_1[T(w), u] = 0 \ \forall m, s_1, s_2, w, u \in \chi.$$

Applying the primeness of χ, we get either $[T(w), u] = 0 \ \forall u, w \in \chi$ or $[T(s_2), T(m)] + [T(m), s_2] = 0 \ \forall m, s_2 \in \chi$. If we consider $[T(w), u] = 0 \ \forall u, w \in \chi$, we can easily find that χ is commutative. Consider

$$[T(s_2), T(m)] + [T(m), s_2] = 0 \ \forall m, s_2 \in \chi. \tag{2.29}$$

Substituting s_2u for s_2 in (2.29), where $u \in \chi$, we find that

$$T(s_2)[u, T(m)] + [T(s_2), T(m)]u + s_2[T(m), u] + [T(m), s_2]u = 0 \tag{2.30}$$

for all $m, s_2, u \in \chi$ Combining (2.29) and (2.30), we get that

$$(T(s_2) - s_2)[T(m), u] = 0 \ \forall s_2, m, u \in \chi. \tag{2.31}$$

Taking us_1 for u in (2.31) and using it again, we see that

$$(T(s_2) - s_2)u[T(m), s_1] = 0 \ \forall s_1, s_2, m, u \in \chi. \tag{2.32}$$

Applying the primeness of χ, we obtain either $T(s_2) = s_2 \forall s_2 \in \chi$ or $[T(m), s_1] = 0\ \forall m, s_1 \in \chi$. But $T(s_2) = s_2 \forall s_2 \in \chi$ is not possible by our assumption, therefore $[T(m), s_1] = 0\ \forall m, s_1 \in \chi$. This implies that χ is commutative.
The second portion can be proved in the same way as the first. $\square$

Corollary 2.4. *[1, Theorem 3.3] Let $(\chi, *)$ be a prime ring with characteristic different from two. If χ has a left centralizer $T(\neq 0)$ satisfying $T([s_1, s_1^*]) \mp [s_1, s_1^*] = (0)\ \forall s_1 \in \chi$ and $T(s_1) \neq \mp s_1\ \forall s_1 \in \chi$, then χ is commutative.*

Theorem 2.6. *Let $(\chi, *)$ be a prime ring with characteristic different from two. If χ has a left centralizer T satisfying $T(s_1 \circ s_1^*) \mp (s_1 \circ s_1^*) \in Z(\chi)$ and $T(s_1) \neq \mp s_1 \forall s_1 \in \chi$, then χ is commutative.*

Proof. We have

$$T(s_1 \circ s_1^*) - (s_1 \circ s_1^*) \in Z(\chi) \forall s_1 \in \chi. \tag{2.33}$$

If $T = 0$, using Lemma 2.2, we conclude that χ is commutative. We consider $T \neq 0$. Linearizing (2.33), we get

$$T(s_1 \circ s_2^*) + T(s_2 \circ s_1^*) - (s_1 \circ s_2^*) - (s_2 \circ s_1^*) \in Z(\chi) \forall s_1, s_2 \in \chi. \tag{2.34}$$

Replacing s_2 by $s_4 s_2$ in (2.34), where $s_4 \in S(\chi) \cap Z(\chi)$, we have

$$-T(s_1 \circ s_2^*)s_4 + T(s_2 \circ s_1^*)s_4 + (s_1 \circ s_2^*)s_4 - (s_2 \circ s_1^*)s_4 \in Z(\chi) \tag{2.35}$$

for all $s_1, s_2 \in \chi$ and $s_4 \in S(\chi) \cap Z(\chi)$. Combining (2.34) and (2.35), we find that

$$2(T(s_2 \circ s_1^*) - (s_2 \circ s_1^*))s_4 \in Z(\chi) \forall s_1, s_2 \in \chi \text{ and } s_4 \in S(\chi) \cap Z(\chi).$$

Since characteristic of χ is different from two and $S(\chi) \cap Z(\chi) \neq (0)$, we get

$$T(s_2 \circ s_1^*) - (s_2 \circ s_1^*) \in Z(\chi) \forall s_1, s_2 \in \chi.$$

Substituting h_0 for s_1, we have

$$2(T(s_2) - s_2)h_0 \in Z(\chi) \forall s_2 \in \chi \text{ and } h_0 \in H(\chi) \cap Z(\chi).$$

Since characteristic of χ is different from two and $S(\chi) \cap Z(\chi) \neq (0)$, we get

$$T(s_2) - s_2 \in Z(\chi) \forall s_2 \in \chi. \tag{2.36}$$

In particular, we get

$$[T(s_2), s_2] = 0 \ \forall s_2 \in \chi.$$

Linearization of above relation gives

$$[T(s_2), s_1] + [T(s_1), s_2] = 0 \ \forall s_1, s_2 \in \chi. \tag{2.37}$$

Substituting $s_2 w$ for s_2 in (2.37), where $w \in \chi$, we get

$$T(s_2)[w, s_1] + [T(s_2), s_1]w + s_2[T(s_1), w] + [T(s_1), s_2]w = 0 \quad \forall s_1, s_2, w \in \chi. \tag{2.38}$$

Combining (2.37) and (2.38), we found

$$T(s_2)[w, s_1] + s_2[T(s_1), w] = 0 \ \forall s_1, s_2, w \in \chi. \tag{2.39}$$

Replacing s_1 by $s_1 m$ in (2.39), where $m \in \chi$, yields that

$$T(s_2)s_1[w, m] + T(s_2)[w, s_1]m + s_2 T(s_1)[m, w] + s_2[T(s_1), w]m = 0 \quad \forall s_1, s_2, w, m \in \chi. \tag{2.40}$$

Using (2.39) in (2.40), it gives

$$(T(s_2)s_1 - s_2 T(s_1))[w, m] = 0 \ \forall s_1, s_2, m, w \in \chi. \tag{2.41}$$

Substituting mu for m in (2.41) and using (2.41), we obtain $(T(s_2)s_1 - s_2 T(s_1))m[w, u] = 0 \ \forall s_1, s_2, m, u, w \in \chi$. Applying the primeness of χ, yields that either $T(s_2)s_1 = s_2 T(s_1) \forall s_1, s_2 \in \chi$ or $[w, u] = 0 \ \forall u, w \in \chi$. If $[w, u] = 0 \ \forall u, w \in \chi$, then χ is commutative. Now consider

$$T(s_2)s_1 = s_2 T(s_1) \forall s_1, s_2 \in \chi. \tag{2.42}$$

The relation (2.36) can be written as

$$[T(s_2), s_3] - [s_2, s_3] = 0 \ \forall s_2, s_3 \in \chi. \tag{2.43}$$

Replacing s_2 by s_1s_2, where $s_1 \in \chi$ in (2.43), we obtain

$$[T(s_1s_2), s_3] - [s_1s_2, s_3] = 0 \ \forall s_1, s_2, s_3 \in \chi. \tag{2.44}$$

Since T is a left centralizer, (2.44), reduces to

$$[T(s_1)s_2, s_3] - [s_1s_2, s_3] = 0 \ \forall s_1, s_2, s_3 \in \chi. \tag{2.45}$$

Using (2.42), we get

$$[s_1T(s_2), s_3] - [s_1s_2, s_3] = 0 \ \forall s_1, s_2, s_3 \in \chi. \tag{2.46}$$

$$s_1[T(s_2), s_3] + [s_1, s_3]T(s_2) - s_1[s_2, s_3] - [s_1, s_3]s_2 = 0 \ \forall s_1, s_2, s_3 \in \chi. \tag{2.47}$$

Application of (2.43), we have

$$[s_1, s_3]T(s_2) - [s_1, s_3]s_2 = 0 \ \forall s_1, s_2, s_3 \in \chi. \tag{2.48}$$

Above relation gives that $[s_1, s_3](T(s_2) - s_2) = 0 \ \forall s_1, s_2, s_3 \in \chi$. Since $T(s_2) \neq s_2 \forall s_2 \in \chi$, by the primeness of χ, we are done.
Similarly we can prove the other part. □

Corollary 2.5. *[1, Theorem 3.4] Let $(\chi, *)$ be a prime ring with characteristic different from two. If χ has a left centralizer $T(\neq 0)$ satisfying $T(s_1 \circ s_1^*) \mp (s_1 \circ s_1^*) = (0) \ \forall s_1 \in \chi$ and $T(s_1) \neq \mp s_1 \forall s_1 \in \chi$, then χ is commutative.*

Theorem 2.7. *Let $(\chi, *)$ be a prime ring with characteristic different from two. If χ has a left centralizer T satisfying $T([s_1, s_1^*]) \mp (s_1 \circ s_1^*) \in Z(\chi) \forall s_1 \in \chi$, then χ is commutative.*

Proof. We first consider the case

$$T([s_1, s_1^*]) - (s_1 \circ s_1^*) \in Z(\chi) \forall s_1 \in \chi. \tag{2.49}$$

If $T = 0$, then by Lemma 2.2, we get χ is commutative. We consider $T \neq 0$. Replacing s_1 by s_1^* in (2.49), we find that

$$T([s_1^*, s_1]) - (s_1^* \circ s_1) \in Z(\chi) \forall s_1 \in \chi. \tag{2.50}$$

Combining (2.49) and (2.50), we obtain $-2(s_1 \circ s_1^*) \in Z(\chi) \forall s_1 \in \chi$. Since characteristic of χ is different from two, we conclude that $s_1 \circ s_1^* \in Z(\chi) \forall s_1 \in \chi$. By an application of Lemma 2.2, we are done. The second part can be proved on similar lines. □

Theorem 2.8. *Let $(\chi, *)$ be a prime ring with characteristic different from two. If χ has a left centralizer $T(\neq 0)$ satisfying $T([s_1, s_1^*]) \mp [T(s_1), s_1^*] \in Z(\chi) \forall s_1 \in \chi$, then χ is commutative or T is a centralizer.*

Proof. We first consider the case

$$T([s_1, s_1^*]) - [T(s_1), s_1^*] \in Z(\chi) \forall s_1 \in \chi. \tag{2.51}$$

Linearizing (2.51), we get

$$T([s_1, s_2^*]) + T([s_2, s_1^*]) - [T(s_1), s_2^*] - [T(s_2), s_1^*] \in Z(\chi) \forall s_1, s_2 \in \chi. \tag{2.52}$$

Replacing s_2 by $s_4 s_2$ in (2.52) and using (2.52), we get

$$2(T([s_2, s_1^*]) - [T(s_2), s_1^*])s_4 \in Z(\chi) \forall s_1, s_2 \in \chi \text{ and } s_4 \in S(\chi) \cap Z(\chi).$$

Since characteristic of χ is different from two and $S(\chi) \cap Z(\chi) \neq (0)$, we have

$$T([s_2, s_1^*]) - [T(s_2), s_1^*] \in Z(\chi) \forall s_1, s_2 \in \chi. \tag{2.53}$$

Taking s_1^* for s_1 in (2.53), we get

$$T([s_2, s_1]) - [T(s_2), s_1] \in Z(\chi) \forall s_1, s_2 \in \chi. \tag{2.54}$$

Replacing s_2 by s_1 in (2.54) we get $[T(s_1), s_1] \in Z(\chi) \forall s_1 \in \chi$. On linearizing the last equation, we have

$$[T(s_1), s_2] + [T(s_2), s_1] = 0 \ \forall s_1, s_2 \in \chi. \tag{2.55}$$

Replacing s_1 by $s_1 m$ in (2.55) and using it, we find that

$$T(s_1)[m, s_2] + s_1[T(s_2), m] = 0 \ \forall s_1, s_2, m \in \chi. \tag{2.56}$$

Replacing s_2 by $s_2 u$ in (2.56) and using it, we get

$$(T(s_1)s_2 - s_1 T(s_2))[m, u] = 0 \ \forall s_1, s_2, m, u \in \chi. \tag{2.57}$$

Taking m by mw in (2.57), we get

$$(T(s_1)s_2 - s_1T(s_2))m[w,u] = 0 \ \forall s_1, s_2, m, u, w \in \chi.$$

Since χ is prime, we get the required result. The second part can be proved on similar lines. □

Theorem 2.9. *Let $(\chi, *)$ be a prime ring with characteristic different from two. If χ has a left centralizer $T(\neq 0)$ satisfying $T([s_1, s_1^*]) \mp T(s_1) \circ s_1^* \in Z(\chi) \forall s_1 \in \chi$, then χ is commutative.*

Proof. First we assume that

$$T([s_1, s_1^*]) - T(s_1) \circ s_1^* \in Z(\chi) \forall s_1 \in \chi. \tag{2.58}$$

Linearizing (2.58), we get

$$T([s_1, s_2^*]) + T([s_2, s_1^*]) - T(s_1) \circ s_2^* - T(s_2) \circ s_1^* \in Z(\chi) \forall s_1, s_2 \in \chi. \tag{2.59}$$

Replacing s_2 by s_4s_2 in (2.59) and using it, we find that

$$2(T([s_2, s_1^*]) - T(s_2) \circ s_1^*)s_4 \in Z(\chi) \forall s_1, s_2 \in \chi \text{ and } \ s_4 \in S(\chi) \cap Z(\chi).$$

Since characteristic of χ is different from two and $S(\chi) \cap Z(\chi) \neq (0)$, we obtain

$$T([s_2, s_1^*]) - T(s_2) \circ s_1^* \in Z(\chi) \forall s_1, s_2 \in \chi.$$

Taking s_1^* for s_1, we have

$$T([s_2, s_1]) - T(s_2) \circ s_1 \in Z(\chi) \forall s_1, s_2 \in \chi. \tag{2.60}$$

Replacing s_1 by $z \in Z(\chi)$ in (2.60), we get $2T(s_2)z \in Z(\chi) \forall s_2 \in \chi$ and $z \in Z(\chi)$. Since characteristic of χ is different from two and $S(\chi) \cap Z(\chi) \neq (0)$, we get $T(s_2) \in Z(\chi) \forall s_2 \in \chi$. That is, $[T(s_2), s_3] = 0 \ \forall s_2, s_3 \in \chi$. Then reasoning as for equation (2.12), we get our result. The second part can be proved on similar lines. □

Theorem 2.10. *Let $(\chi, *)$ be a prime ring with characteristic different from two. If χ has a left centralizer $T(\neq 0)$ satisfying $T(s_1 \circ s_1^*) + T(s_1) \circ s_1^* \in Z(\chi) \forall s_1 \in \chi$, then χ is commutative.*

Proof. First, we consider the case

$$T(s_1 \circ s_1^*) + T(s_1) \circ s_1^* \in Z(\chi) \forall s_1 \in \chi. \tag{2.61}$$

Linearizing (2.61), we get

$$T(s_1 \circ s_2^*) + T(s_2 \circ s_1^*) + T(s_1) \circ s_2^* + T(s_2) \circ s_1^* \in Z(\chi) \forall s_1, s_2 \in \chi. \tag{2.62}$$

Replacing s_2 by $s_4 s_2$ in (2.62), we have

$$-T(s_1 \circ s_2^*)s_4 + T(s_2 \circ s_1^*)s_4 - (T(s_1) \circ s_2^*)s_4 + (T(s_2) \circ s_1^*)s_4 \in Z(\chi) \tag{2.63}$$

for all $s_1, s_2 \in \chi$ and $s_4 \in S(\chi) \cap Z(\chi)$. Using (2.62) in (2.63), we arrive at

$$2(T(s_2 \circ s_1^*) + T(s_2) \circ s_1^*)s_4 \in Z(\chi) \forall s_1, s_2 \in \chi \text{ and } s_4 \in S(\chi) \cap Z(\chi).$$

Since characteristic of χ is different from two and $S(\chi) \cap Z(\chi) \neq (0)$, we get

$$T(s_2 \circ s_1^*) + T(s_2) \circ s_1^* \in Z(\chi) \forall s_1, s_2 \in \chi. \tag{2.64}$$

Replacing s_1 by h in (2.64), where $h \in H(\chi) \cap Z(\chi)$, we have

$$2(T(s_2))h \in Z(\chi) \forall s_2 \in \chi \text{ and } h \in H(\chi) \cap Z(\chi).$$

As characteristic of χ is different from two and $H(\chi) \cap Z(\chi) \neq (0)$, we get $T(s_2) \in Z(\chi) \forall s_2 \in \chi$. This yields that $[T(s_2), s_3] = 0 \ \forall s_2, s_3 \in \chi$. Then reasoning as for equation (2.12), we arrive at our conclusion. $\square$

3. Examples

The example of this concluding section shows that our results does not hold in case the involution is of the first kind.

Example 3.1. *Let* $\chi = \left\{ \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{pmatrix} \middle| \beta_1, \beta_2, \beta_3, \beta_4 \in \mathbf{Z} \right\}$. *Of course,* χ *with matrix addition and matrix multiplication is a non commutative prime ring. Define mappings* $*, T : \chi \longrightarrow \chi$ *such that*
$\begin{pmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{pmatrix}^* = \begin{pmatrix} \beta_4 & -\beta_2 \\ -\beta_3 & \beta_1 \end{pmatrix}$ *and* $T \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{pmatrix} = \begin{pmatrix} \beta_1 & \beta_2 \\ 0 & 0 \end{pmatrix}$.

Obviously, $Z(\chi) = \left\{ \left(\begin{array}{cc} \beta_1 & 0 \\ 0 & \beta_1 \end{array} \right) \middle| \beta_1 \in \mathbf{Z} \right\}$. *Then* $s_1^* = s_1 \forall s_1 \in Z(\chi)$, *and hence* $Z(\chi) \subseteq H(\chi)$, *which shows that the involution* $*$ *is of the first kind. Moreover,* T *is a nonzero left centralizer of* χ *such that it satisfy the identities of Theorems 2.3, 2.5 and 2.7. However,* χ *is not commutative. Thus, the second kind is necessary in our theorems.*

At last we provide an example to show that the Theorems 2.3, 2.5 and 2.7 can not hold for semi-prime rings.

Example 3.2. *Let* $S = \chi \times \mathbf{C}$, *where* χ *is same as in example 3.1 having involution* $*$ *and left centralizer* T *same as in above example,* $\mathbf{C}$ *is the ring of complex numbers with conjugate involution* τ. *Hence,* S *is a non commutative semi-prime ring. Now define an involution* α *on* S *as* $(s_1, s_2)^\alpha = (s_1^*, s_2^\tau)$. *It can be easily proved that* α *is second kind. Further, we define the mappings* β *from* S *to* S *as follows* $\beta(s_1, s_2) = (T(s_1), 0)\ \forall (s_1, s_2) \in S$ *where* T *is same as in the example above. It can be easily checked that* β *is a nonzero left centralizer on* S *and satisfy the identities of Theorems 2.3, 2.5 and 2.7, but* S *is not commutative. Hence, the hypothesis of primeness is a necessary condition.*

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