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Group vertex magic labeling of some special graphs

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Abstract

For any additive abelian group A, a graph $G = (V, E)$ is said to be A-vertex magic graph if there exist an element $\mu \in A$ and a labeling function $f : V \to A \setminus \{0\}$ such that $\omega(v) = \sum_{u \in N(v)} f(u) = \mu$ for any vertex v of G, where $N(v)$ is the set of the open neighborhood of v. In this paper, we prove that graphs such as the wheel, the corona product $C_n \odot mK_1$, the subdivision ladder and the t-fold wheel are A-vertex magic graphs for abelian groups A satisfying certain conditions. Also, we prove that the subdivided wheel, the helm and the closed helm are Z_k-vertex magic graphs for specific values of k. Furthermore, we prove that the triangular book and the t-fold wheel for $t = n, n-2$ are A-vertex magic graphs for every abelian group A.

Mathematics Subject Classification: *05C78.*

Keywords: *A-vertex magic ; Group vertex magic; Weight of the vertex ; Subdivided wheel ; t-fold wheel; Helm ; Triangular book ; Cryptography.*

1. Introduction

By a graph $G = (V, E)$ we consider a finite undirected simple graph with vertex set V and edge set E. The degree of a vertex v in graph G, indicated as $d(v)$, is the number of edges incident with v. We refer to [1] for graph theoretic terminology and to [3] for terminology on group theory. Throughout this paper A denotes an abelian group with identity element 0. The order of element $g \in A$, is the smallest positive integer n such that $ng = 0$, it is denoted by $o(g)$. The group $Z_2 \otimes Z_2 = \{(x, y)|x, y \in Z_2\}$ with binary operation $(x, y) + (x^{'}, y^{'}) = (x + x^{'}, y + y^{'})$ is called Klain's 4-group and is denoted by V_4. The concept of group magic graphs was introduced by Lee et al. [5] as follows: For any abelian group A, a graph G with edge labeling function which assigns to each edge of G an element of A different from the identity such that the sum of the labels of edges incident to any vertex is same for all the vertices is called an A-magic graph. In [4] N.Kamatchi et al. introduced the concept of group vertex magic graphs and obtained the necessary conditions for some graphs to be group vertex magic. Labeled graphs play a vital role in various scientific fields such as coding theory, cryptography, logistics, mathematical modeling, crystallography, radar, astronomy and circuit design [2]. Also, in communication network there are many applications using graph labeling, such as communication network addressing, fault-tolerant system designing, and automated channel allocation [6].

Definition 1. *Let A be any non-trivial abelian group and let μ be any element of A, a graph $G = (V, E)$ is said to be A-vertex magic graph with magic constant μ if there exist a vertex labeling $f : V \to A \setminus \{0\}$ such that $\omega(v) = \sum_{u \in N(v)} f(u) = \mu$ for any vertex v of G.*

Comment: The function f satisfying the condition of the Definition 1 is called an A-vertex magic labeling of G with magic constant μ.

If G has a vertex labeling satisfying the condition in the above definition for every non-trivial abelian group A, then G is called a group vertex magic graph. We use the following definitions in the subsequent section.

Definition 2. *The subdivided wheel $W_n(r, k)$ is a graph derived from the wheel graph W_n, by replacing each external edge v_iv_{i+1} with a v_iv_{i+1}-path of order $r \geq 2$, and every radial edge v_iv, $1 \leq i \leq n$ by a v_iv-path of order $k \geq 2$.*

Fig. 1 shows the subdivided wheel of W_7.

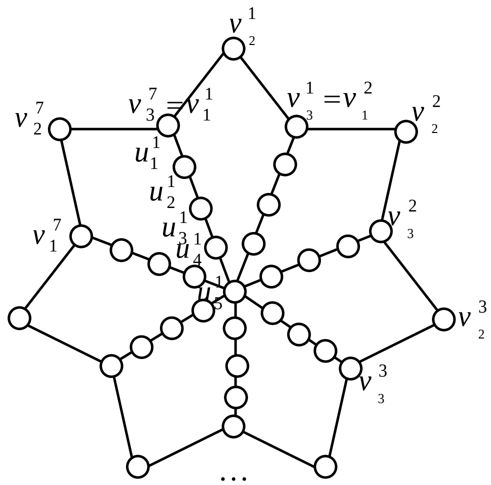

Figure 1.1: The subdivided wheel $W_7(3,5)$

Definition 3. *Let G_1,G_2 be two graphs of order n,m respectively. The corona product of G_1 and G_2, denoted $G_1 \odot G_2$, is the graph obtained by taking one copy of G_1 and n copies of G_2 and then making the ith vertex of G_1 adjacent to each vertex in the ith copy of G_2.*

Fig. 2 shows the corona product of C_8 and $3K_1$.

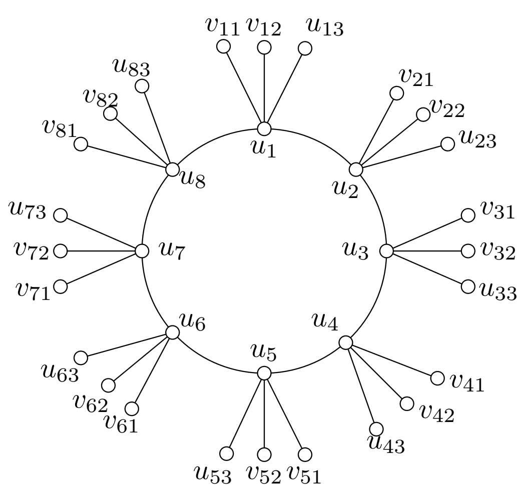

Figure 1.2: Corona product $C_8 \odot 3K_1$

Definition 4. *The t-fold of W_n is the graph W_n^t with t central vertices and n rim vertices, where the n rim vertices form a cycle and each of the central vertices is adjacent to all cycle vertices, but central vertices are not adjacent to each other.*

The graph of 3-fold wheel of W_6 is given in Fig. 3.

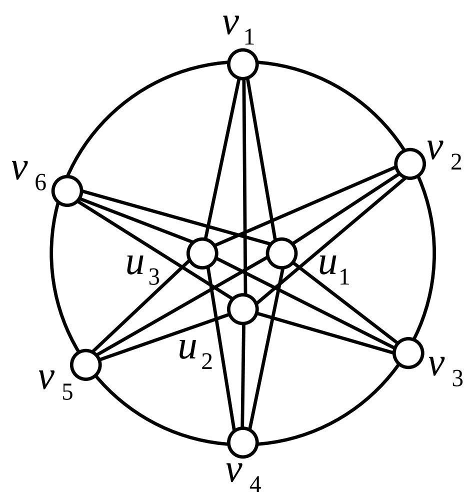

Figure 1.3: 3-fold wheel of W_6

Definition 5. *Let P_n be a path with n vertices, the ladder graph,L_n, is a graph formed from the Cartesian product $P_2 \times P_n$, where $n \geq 2$.*

Definition 6. *The Subdivision of the ladder graph L_n denoted by $S(L_n)$ is created by splitting each edge of L_n by one vertex.*

Definition 7. *The helm H_n is the graph derived from a wheel by attaching a pendant edge at each vertex of the rim.*

Definition 8. *The closed helm CH_n is the graph derived from a helm H_n by attaching each pendant vertex to form cycle.*

The graphs of H_8 and CH_8 are given in Fig. 4.

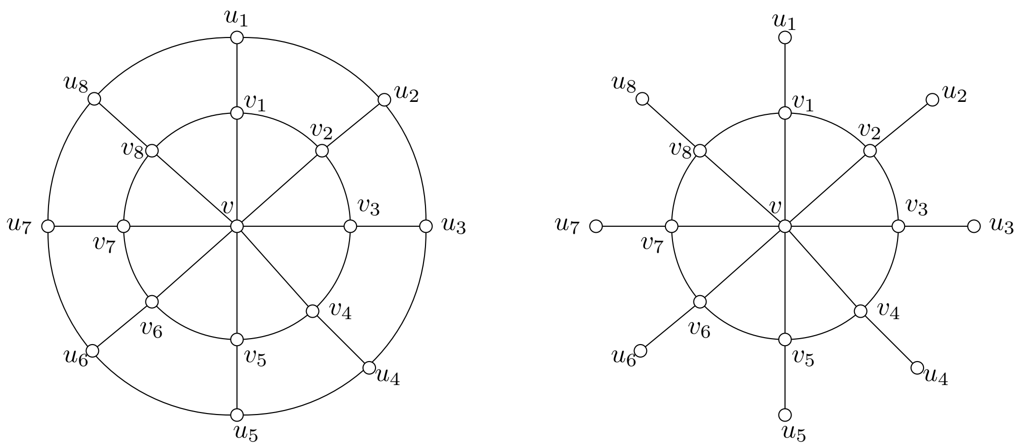

Figure 1.4: H_8 and CH_8 graphs

Definition 9. *The triangular book $B(3,n)$ is a graph made up of n triangles that share a common edge.*

The graph of $B(3,4)$ is given in Fig. 5.

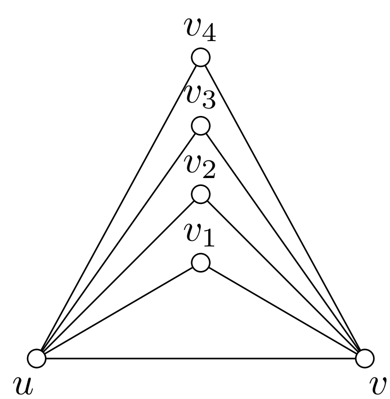

Figure 1.5: $B(3,4)$ graph

Observation 1. *[4] Any r-regular graph G is group vertex magic with magic constant rg, where g is a nonzero element of abelian group A.*

2. Main Results

Theorem 1. *The wheel W_n for $n \geq 3$ is Z_n-vertex magic.*

Proof. Let $\{v_1, v_2, ..., v_n\}$ be the vertices of n-cycle of the wheel W_n and let v be its central vertex. Let $g \neq 0$ be any element of the group Z_n such that $g \neq \frac{n}{2}$ when n is even. Assign labels $f(v_i) = g$ where $1 \leq i \leq n$ and label the central vertex v by $(n-2)g$ i.e. $f(v) = (n-2)g$. This defines a Z_n-vertex magic labeling of W_n with magic constant 0. □

Theorem 2. *Let $n > 2$, $r \geq 2$ and let $k \geq 2$. The subdivided wheel $W_n(r,k)$ is Z_n-vertex magic if r is even or r is odd and n is even.*

Proof. Let v be the central vertex of the wheel W_n and let v_i, $1 \leq i \leq n$ be the vertices of the cycle C_n and let each edge v_iv, $1 \leq i \leq n$ be replaced by the path P_k^i of order k and each edge of v_iv_{i+1}, $1 \leq i \leq n-1$ and v_nv_1 are replaced by the paths P^{*i}_r, P^{*n}_r of order r respectively. Let $v_i = v_1^i, v_2^i, ..., v_r^i = v_{i+1}$ be the vertices of ith copy of the path of order r and let $v_i = u_1^i, u_2^i, ..., u_k^i = v$ be the vertices of ith copy of the path of order k. As $n > 2$, there exists $g \in Z_n \setminus \{0\}$ such that $g \neq \frac{n}{2}$ when n is even. For instance $g = 1$.

Case(i). Suppose r is even then $r = 2s$, where $s \geq 1$. If s even, then define f by:

For $1 \leq i \leq n$

$$f(v_j^i) = \begin{cases} g, & j \equiv 1, 4 \bmod 4 \\ n-g, & j \equiv 2, 3 \bmod 4. \end{cases}$$

$$f(u_\ell^i) = \begin{cases} g, & \ell \equiv 1 \bmod 4 \\ 2g, & \ell \equiv 2 \bmod 4 \\ n-g, & \ell \equiv 3 \bmod 4 \\ n-2g, & \ell \equiv 4 \bmod 4. \end{cases}$$

Thus, f is a Z_n-vertex magic of $W_n(r,k)$ with magic constant 0.
If s odd, hence define f by:

For $1 \leq i \leq n$.

$$f(v_j^i) = \begin{cases} g, & j \equiv 1,2 \bmod 4 \\ n-g, & j \equiv 3,4 \bmod 4. \end{cases}$$

$$f(u_\ell^i) = \begin{cases} g, & \ell \equiv 1 \bmod 4 \\ n-2g, & \ell \equiv 2 \bmod 4 \\ n-g, & \ell \equiv 3 \bmod 4 \\ 2g, & \ell \equiv 4 \bmod 4. \end{cases}$$

Clearly, f is a Z_n-vertex magic labeling of $W_n(r,k)$ with magic constant 0.

Case(ii). Suppose r is odd and n is even. Then $r \equiv 1 \bmod 4$ or $r \equiv 3 \bmod 4$. When $r \equiv 1 \bmod 4$, define f as follows:

For i odd.

$$f(v_j^i) = \begin{cases} \frac{n}{2}, & j \equiv 1 \bmod 2 \\ g, & j \equiv 2 \bmod 4 \\ n-g, & j \equiv 4 \bmod 4. \end{cases}$$

For i even.

$$f(v_j^i) = \begin{cases} \frac{n}{2}, & j \equiv 1 \bmod 2 \\ \frac{n}{2}+g, & j \equiv 2 \bmod 4 \\ \frac{n}{2}-g, & j \equiv 4 \bmod 4. \end{cases}$$

$$f(u_\ell^i) = \frac{n}{2}, \qquad 1 \leq \ell \leq k, \quad 1 \leq \ i \leq n.$$

It is clear that f is a Z_n-vertex magic labeling of $W_n(r,k)$ with magic constant 0.

When $r \equiv 3 \bmod 4$, now define f as follows:

For $1 \leq i \leq n$

$$f(v_j^i) = \begin{cases} \frac{n}{2}, & j \equiv 1 \bmod 2 \\ g, & j \equiv 2 \bmod 4 \\ n-g, & j \equiv 4 \bmod 4. \end{cases}$$

$$f(u_\ell^i) = \begin{cases} \frac{n}{2}, & \ell \equiv 1 \bmod 2 \\ n-2g, & \ell \equiv 2 \bmod 4 \\ 2g, & \ell \equiv 4 \bmod 4. \end{cases}$$

Hence, f is a Z_n-vertex magic labeling of $W_n(r,k)$ with magic constant 0.

□

Corollary 2.1. *The Jahangir graph $J_{m,n}$ is Z_n-vertex magic if $nm+n$ is even.*

Proof. Note that the Jahangir graph $J_{m,n}$ can be defined as the especial case from the subdivided wheel $W_n(r,k)$, by taking $r=m+1$ and $k=2$. From the above theorem it follows that $J_{m,n}=W_n(m+1,2)$ is a Z_n-vertex magic when $nm+n$ is even. □

Proposition 2.1. *If r,k are odd and $n\equiv 0 \bmod 3$, then $W_n(r,k)$ is V_4-vertex magic.*

Proof. Consider $A=V_4=\{0,a,b,c\}$, where $2a=2b=2c=0$ and the sum of any two elements other than zero gives the third, then we can define f as follows:

$$f(v_j^i)=\begin{cases} a, & j\equiv 1 \bmod 2\ , i\equiv 1 \bmod 3\\ c, & j\equiv 2 \bmod 2\ , i\equiv 1 \bmod 3\\ a, & 1\leq j\leq r, i\equiv 2 \bmod 3\\ a, & j\equiv 1 \bmod 2\ , i\equiv 3 \bmod 3\\ b, & j\equiv 2 \bmod 2\ , i\equiv 3 \bmod 3. \end{cases}$$

$$f(u_\ell^i)=\begin{cases} a, & 1\leq \ell\leq k, i\equiv 1 \bmod 3\\ a, & \ell\equiv 1 \bmod 2\ , i\equiv 2 \bmod 3\\ b, & \ell\equiv 2 \bmod 2\ , i\equiv 2 \bmod 3\\ a, & \ell\equiv 1 \bmod 2\ , i\equiv 3 \bmod 3\\ c, & \ell\equiv 2 \bmod 2\ , i\equiv 3 \bmod 3. \end{cases}$$

This defines a V_4-vertex magic labeling of $W_n(r,k)$ with magic constant 0. □

Theorem 3. *The t-fold of the wheel W_n is A-vertex magic for some abelian group A. Furthermore, W_n^t is group vertex magic for $t=n,n-2$.*

Proof. The 1-fold wheel is a wheel and is A-vertex magic graph by the Theorem 2.1. Let v_i, $1\leq i\leq n$ be the vertices of rim of wheel W_n and let u_j, $1\leq j\leq t$ be the t vertices hub.

Case(i). When $t=n$. Let A be an abelian group, $|A|\geq 2$ and $g\in A\setminus\{0\}$, now define the labeling f by:

$f(v_i)=g, \quad 1\leq i\leq n$
$f(u_j)=g, \quad 1\leq j\leq n-1$
$f(u_n)=-g.$

This gives a group vertex magic labeling of G with magic constant ng.

Case(ii). When $t = n - 1$. Let A be an abelian group and let g be a nonzero element of A, $o(g) \neq 2$. The vertices of the rim are labeled as in the above case, now we label $n-3$ vertices of the hub by label g i.e. $f(u_j) = g$, $1 \leq j \leq n-3$. Now define the label of the remaining two vertices by $f(u_{n-2}) = 2g$, $f(u_{n-1}) = -g$. This gives an A-vertex magic labeling of the t-fold wheel with magic constant ng.

Case(iii). When $t = n - 2$. Here all vertices of G of degree n i.e. $d(v_i) = d(u_j) = 2$ for all i, j. Then it follows from Observation 1 that t-fold wheel is a group vertex magic with magic constant ng.

Case(iv). When $n - t > 2$. Let A be an abelian group and $g \in A \setminus \{0\}$ such that $(n - t - 1)g \neq 0$. Obviously we can define the labeling f by:

$f(v_i) = g, \quad 1 \leq i \leq n$
$f(u_j) = g, \quad 1 \leq j \leq t - 1$
$f(u_t) = (n - t - 1)g.$

This defines an A-vertex magic labeling of the t-fold wheel with magic constant ng.

Case(v). When $n - t < 2$. Clearly, $d(v_i) > d(u_j)$. Then we can define the labeling f as follows:

$f(v_i) = g, \quad 1 \leq i \leq n$
$f(u_j) = g, \quad 1 \leq j \leq n - 2.$

For $n - 1 \leq j \leq t - 1$, define the labels $f(u_j)$ to be any elements of A such that $\sum_{j=n-1}^{t-1} f(u_i) = a$ is nonzero. Now define $f(u_t) = -a$. This gives an A-vertex magic labeling of the t-fold wheel with magic constant ng. □

Theorem 4. *For $n > 2$ and $m \geq 2$, the corona product $C_n \odot mK_1$ is A-vertex magic for some abelian group A.*

Proof. Let $G = C_n \odot mK_1$ be the graph obtained from the corona product of a cycle C_n with m copies of isolated vertices.

Case(i). When $m = 1$ The $C_n \odot mK_1$ graph is obtained from a cycle C_n by adding a pending neighbor to each vertex. We consider the partition $\{X, Y\}$ of V, with $X = V(C_n)$ and $Y = V - V(C_n)$. Let x (resp. y) be the label of each vertex of X (resp. Y). Each vertex u of X has 2 neighbors in X and 1 neighbor in Y, so $w(u) = 2x + y$ Each vertex v of Y has 1 neighbor in X and 0 neighbor in Y, so $w(v) = x$. We obtain the equation $2x + y = x$, i.e. $y = -x$, and therefore the solution $x = g$ and $y = -g$, where g is an element of $A \setminus \{0\}$.

Case(ii). When $m \geq 2$. Let A be any abelian group where $|A| \geq 3$. We can define f by:
(1) Let $g \in A \setminus \{0\}$, for $1 \leq i \leq n$ label each vertex u_i by g i.e. $f(u_i) = g$.
(2) Assign label $f(v_j)$, $1 \leq j \leq m-1$ by arbitrary element of A such that $\sum_{j=1}^{m-1} f(v_j) + g = a$, $a \in A \setminus \{0\}$ and define $f(v_m) = -a$. Hence we obtain an A-vertex magic labeling of G with magic constant g. □

Theorem 5. *The subdivision of Ladder graph $S(L_n)$, $n \geq 3$ is A-vertex magic for some abelian group A.*

Proof. Let u_i, v_i, $1 \leq i \leq n$ be the vertices of the ladder graph L_n and let $\acute{u}_i, \acute{v}_i, \acute{w}_i$ be the newly added vertices to the edges u_iu_{i+1}, v_iv_{i+1} and u_iv_i respectively. Then we obtain the graph $S(L_n)$. Let A be any abelian group where $|A| \geq 3$ and let g be an arbitrary nonzero element of A such that $o(g) \neq 2$. Thus define the labeling $f : V(S(L_n)) \to A$ by :
For $1 \leq i \leq n$.

$$f(u_i) = \begin{cases} g, & i \text{ is odd} \\ 2g, & i \text{ is even.} \end{cases}$$

$$f(v_i) = \begin{cases} 2g, & i \text{ is odd} \\ g, & i \text{ is even.} \end{cases}$$

$$f(\acute{u}_i) = f(\acute{v}_i) = g$$

$$f(\acute{w}_i) = \begin{cases} 2g, & \text{for } i = 1,\, i = n \\ g, & \text{for } 2 \leq i \leq n-1. \end{cases}$$

Obviously, the graph $S(L_n)$ is A-vertex magic with magic constant $3g$. □

Theorem 6. *The helm graph H_n is Z_{n-1}-vertex magic for $n \geq 4$. Furthermore, H_3 is Z_m-vertex magic, where m is any positive even integer.*

Proof. Let W_n be the wheel graph and let v_i, $1 \leq i \leq n$ be the vertices of the rim of W_n. Hence the helm graph H_n obtained by adding a pendent edge for each v_i. Let u_i, $1 \le i \le n$ be the pendant vertices.

Case(i). When $n = 3$. Let A be Z_m the abelian group of integers modulo m, where m is a positive even integer and let $g \in \{1, 2, 3, ..., m-1\}$. So we can define $f : V(H_3) \to Z_m$ by:

$f(v) = \frac{m}{2} - g,$ where v is the central vertex

$f(v_i) = \frac{m}{2}, \quad f(u_i) = g,$ where $1 \le i \le 3$.

Hence the H_3 is Z_m-vertex magic with magic constant $\frac{m}{2}$.

Case(ii). When $n \ge 3$. In this case, let A be Z_{n-1} and let $g \in \{1, 2, 3, ..., n-1\}$. Now define $f : V \to A$ as follows:

$f(v) = (n-3)g,$ where v is the central vertex

$f(v_i) = g, \quad f(u_i) = g,$ where $1 \le i \le n$.

Clearly the H_n is Z_{n-1}-vertex magic with magic constant g. □

Theorem 7. *Let CH_n be a closed helm graph of order n.*
(i) The closed helm graph CH_3 is Z_m-vertex magic, where m is any positive even integer.
(ii) For $n \ge 4$, if n is even CH_n is Z_n-vertex magic, and if n is odd CH_n is Z_{n-1}-vertex magic.

Proof. Let H_n be helm graph, by attaching any two consecutive a pendant vertices by edge we obtained the closed helm graph CH_n.

Case(i). When $n = 3$. In this case take $A = Z_m$, where m is a positive even integer and let $g \in Z_m$. Hence we can define $f : V \to A$ by:

$f(v) = \frac{m}{2},$ where v is the central vertex

$f(v_i) = g, \quad f(u_i) = \frac{m}{2} + g,$ where $1 \le i \le n$.

Thus CH_3 is Z_m-vertex magic with magic constant $3g$.

Case(ii). When $n \ge 4$ and n is even. Let A be Z_n the abelian group of integers modulo n. For any integer $g \in \{1, 2, 3, ..., n-1\}$, $g \ne \frac{n}{2}$ and $o(g) \ne 3$, define f by:

$f(v) = n - 3g,$ where v is the central vertex

$f(v_i) = 2g, \quad f(u_i) = n - g,$ where $1 \le i \le n$.

Clearly f is a Z_n-vertex magic labeling of CH_n with magic constant 0.

Case(iii). When $n \geq 4$ and n is odd. We take $A = Z_{n-1}$. For any odd integer $g \in \{1, 2, 3, ..., n-1\}$, $g \neq \frac{n-1}{2}$, define f by:

$f(v) = \frac{n-3}{2}g$, where v is the central vertex
$f(v_i) = g$, $\quad f(u_i) = \frac{n-1}{2}g$, where $1 \leq i \leq n$.

Then f is a Z_{n-1}-vertex magic labeling of CH_n with magic constant g. $\square$

Theorem 8. *The triangular book $B(3, n)$ with n pages is group vertex magic for $n \geq 3$.*

Proof. Consider $B(3, n)$ is the triangular book graph with n pages. Let u, v be the vertices of the base of the book and v_i, $1 \leq i \leq n$ the vertices of n pages, let A be an arbitrary abelian group. Now define $f : V \to A$ as follows:
(1) Assign labels $f(u), f(v)$ by the same arbitrary nonzero element of A. i.e.$f(u) = f(v) = g$, where $g \in A \setminus \{0\}$.
(2) For $1 \leq i \leq n-2$ assign labels $f(v_i)$ by arbitrary nonzero element of A such that $\sum_{i=1}^{n-2} f(v_i) = a$, $a \in A \setminus \{0\}$. Now define $f(v_{n-1}) = -a$, $f(v_n) = g$. This gives a group vertex labeling of $B(3, n)$, with magic constant $2g$. $\square$

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