Proyecciones Journal of Mathematics
Vol. 42, N^o 6, pp. 1549-1566, December 2023.
Universidad Católica del Norte
Antofagasta - Chile

10.22199/issn.0717-6279-5873

On edge irregularity strength of different families of graphs

Muhammad Imran
Concordia College Kasur Campus, Pakistan
Murat Cancan
Yuzuncu Yil University, Turkey
Yasir Ali
Concordia College Kasur Campus, Pakistan
Anum
Concordia College Kasur Campus, Pakistan
Jamila Aslam
Concordia College Kasur Campus, Pakistan
Received : February 2023. Accepted : May 2023

Abstract

Edge irregular mapping or vertex mapping $h : V(G) \longrightarrow \{1, 2, 3, ..., s\}$ *is a mapping of vertices in such a way that all edges have distinct weights. We evaluate weight of any edge by using equation* $wt_h(cd) = h(c) + h(d)$, $\forall c, d \in V(G)$ *and* $\forall cd \in E(G)$. *Edge irregularity strength denoted by* $es(G)$ *is a minimum positive integer used to label vertices to form edge irregular labeling. In this paper, we find exact value of edge irregularity strength of linear phenylene graph* PH_n, B_n *graph and different families of snake graph.*

Keywords: *Irregular assignment; irregularity strength; edge irregularity strength; pendant edges; snake graphs; linear phenylene graph* PH_n; B_n *graph.*

1. Introduction

In this paper, we consider finite, simple and undirected graphs.
The procedure of assignment of non-negative integers to the elements of a graph G is termed as labeling. Vertex set $V(G)$ and edge set $E(G)$ are the elements of a graph G. If we label vertices or edges, then this labeling is categorized as vertex labeling or edge labeling respectively. If we label both vertices and edges, then this labeling is termed as total labeling.
Chartrand et al.[15] had introduced edge labeling for a graph G. We call this labeling as irregular assignments because all vertices have distinct weights. Irregularity strength $s(G)$ is a minimum positive integer which is used to form irregular labeling. Results regarding irregularity strength can be seen in [7, 14, 16, 21].
Vertex irregular mapping or edge mapping $h : E(G) \longrightarrow \{1, 2, 3, ..., s\}$ is a mapping of edges in such a way that all vertices have distinct weights. Weight of any vertex can be calculated by using equation $wt_h(c) = \Sigma h(cd)$, $\forall c, d \in V(G)$ and $cd \in E(G)$.
Motivated by Chartrand's work, Bača at [12] introduced new labeling named as vertex irregular total labeling.
Vertex irregular total labeling $h : E(G) \cup V(G) \longrightarrow \{1, 2, 3, ..., s\}$ is a mapping of edges and vertices of G in such a way that the total vertex weight is different for all vertices. We can evaluate total vertex weight by using the relation $wt_h(c) = h(c)+\Sigma h(cd)$, $\forall c, d \in V(G)$ and $cd \in E(G)$. Total vertex irregularity strength denoted by $tvs(G)$ is a minimum positive integer used to label vertices to form vertex irregular total labeling. Inspired by this, more results were developed in [4, 1, 2, 9, 26, 8, 11, 13, 17, 18, 20, 22, 24].
Edge irregularity and vertex irregularity were both new labels developed by Marzuki based on the previously improved motivation in [12], which were categorized as total labels with complete irregularity. Total irregularity strength for a graph G is denoted as $ts(G)$. Results related to irregular total labeling were developed in [12, 23].
Because of the motivation of previous results, Ahmed et al. developed a new concept of edge irregularity strength denoted by es(G) in [3], which was a minimum positive integer used to label vertices to form edge irregular labeling. Edge irregular mapping or vertex mapping $h : V(G) \longrightarrow \{1, 2, 3, ..., s\}$ is a mapping of vertices in such a way that all edges have distinct weights. We evaluate weight of any edge by using equation $wt_h(cd) = h(c) + h(d)$, $\forall c, d \in V(G)$ and $\forall cd \in E(G)$.
In 2012, Siddiqui [25] calculated edge irregularity strength of subdivision

of star S_n. In 2016, Ahmad et al. [5] obtained exact value of edge irregularity strength of different classes of Toeplitz graphs. In 2016, Tarawneh et al. [27, 28] found edge irregularity strength of different families of graphs. Tarawneh et al. [29, 30] found many useful results regarding edge irregularity strength of disjoint union of star graph, subdivision of star graph and grid graphs. In 2017, Mushayt et al. [10] computed edge irregularity strength of products of certain families of graphs with path $P-2$. In 2017, Imran et al. [19] computed edge irregularity strength of caterpillars, n-star graphs, (n,t)-kite graphs, cycle chains and friendship graphs. In 2020, Zhang et al. [31] introduced some new families of comb graph and calculated exact value of edge irregularity strength of these graphs. In 2020, Ahmad et al. [6] performed computer based experiment dealing with the edge irregularity strength of complete bipartite graphs, they also gave bounds on this parameter for wheel related graphs.

Theorem 1.1

Let G be a simple graph with maximum degree $=(G)$, then
$es(G) \geq max\{\lceil\frac{|E(G)|+1}{2}\rceil, (G)\}$ [3].

2. Main Results

2.1. Quadrilateral Snake Graph

To obtain a quadrilateral snake graph let us consider a path graph P_n, $(n>1)$, if we replace each edge of path graph by a quadrilateral C_4, we get Quadrilateral snake graph QS_n. It has $(3n-2)$ vertices, u_i have n vertices,v_i have $2n-2$ vertices and its edge set can be given as $E(QS_n) = \{U_iU_{i+1}; 1 \leq i \leq n-1\} \bigcup \{V_{2i-1}V_{2i}; 1 \leq i \leq n-1\} \bigcup \{U_iV_{2i-1}; 1 \leq i \leq n-1\} \bigcup \{U_{i+1}V_{2i}; 1 \leq i \leq n-1\}$.

Theorem 2.1. *Let QS_n be a quadrilateral snake graph, then $es(QS_n) = 2n-1$, for $n>1$.*

Proof: Let QS_n be a quadrilateral snake graph. We have to show that $es(QS_n) = 2n-1$. From Theorem 1.1 we get lower bound $es(QS_n) \geq 2n-1$. For converse, we have to prove that $es(QS_n) \leq 2n-1$. For this, define a vertex labeling $h: V(QS_n) \to \{1, 2, 3, ..., 2n-1\}$ such that

$$h(V_i) = i, 1 \leq i \leq 2n-2$$

$$h(U_i) = 2i-1, 1 \leq i \leq n$$

Now we evaluate weights for all edges as:

$$W_t(U_iU_{i+1}) = 4i, 1 \leq i \leq n-1$$

$$W_t(V_{2i-1}V_{2i}) = 4i-1, 1 \leq i \leq n-1$$

$$W_t(U_iV_{2i-1}) = 4i-2, 1 \leq i \leq n-1$$

$$W_t(U_{i+1}V_{2i}) = 4i+1, 1 \leq i \leq n-1$$

On the basis of above calculations we see that all edges have distinct weights.
Hence we can say that $es(QS_n) = 2n-1$, for $n > 1$. □

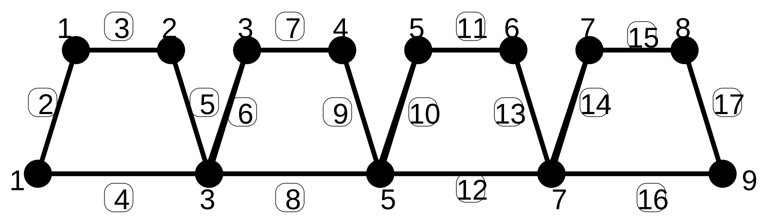

Figure 1. Irregular Labeling on Quadrilateral Snake Graph QS_5.

2.2. Quadrilateral Snake Graph with Pendant Edges

Quadrilateral snake graph PQS_n with pendant edges is formed by vertex set $V(PQS_n)$ consists of $6n-4$ vertices where $n > 1, U_i$ and V_i have $2n-2$ vertices, W_i and X_i have n vertices, and edge set can be given as $E(PQS_n) = \{U_iV_i; 1 \leq i \leq 2n-2\} \bigcup \{V_{2i-1}W_i; 1 \leq i \leq n-1\} \bigcup \{W_iX_i; 1 \leq i \leq n\} \bigcup \{W_{i+1}V_{2i}; 1 \leq i \leq n-1\} \bigcup \{W_iW_{i+1}; 1 \leq i \leq n-1\} \bigcup \{V_{2i-1}V_{2i}; 1 \leq i \leq n-1\}$.

Theorem 2.2. *Let PQS_n be a quadrilateral snake graph with pendant edges, then $es(PQS_n) = \lfloor \frac{7n-4}{2} \rfloor$, for $n > 1$.*

Proof: Let PQS_n be a quadrilateral snake graph with pendant edges. We have to show that $es(PQS_n) = \lfloor \frac{7n-4}{2} \rfloor$. From Theorem 1.1 we get lower bound $es(PQS_n) \geq \lfloor \frac{7n-4}{2} \rfloor$. For converse, we have to prove that $es(PQS_n) \leq \lfloor \frac{7n-4}{2} \rfloor$. For this, define a vertex labeling $h : V(PQS_n) \rightarrow \{1, 2, 3, ..., \lfloor \frac{7n-4}{2} \rfloor\}$ such that

$$h(U_i) = \begin{cases} \frac{7i+1}{4}, & \text{if } i \equiv 1(mod4) \\ \frac{7i-2}{4}, & \text{if } i \equiv 2(mod4) \\ \frac{7i-1}{4}, & \text{if } i \equiv 3(mod4) \\ \frac{7i}{4}, & \text{if } i \equiv 4(mod4) \end{cases}$$

$$h(V_i) = \begin{cases} \frac{7i+1}{4}, & \text{if } i \equiv 1(mod4) \\ \frac{7i+2}{4}, & \text{if } i \equiv 2(mod4) \\ \frac{7i+3}{4}, & \text{if } i \equiv 3(mod4) \\ \frac{7i}{4}, & \text{if } i \equiv 4(mod4) \end{cases}$$

$$h(W_i) = \begin{cases} \frac{7i-5}{2}, & 1 \leq i \leq n, odd \\ \frac{7i-6}{2}, & 2 \leq i \leq n, even \end{cases}$$

$$h(X_i) = \begin{cases} \frac{7i-5}{2}, & 1 \leq i \leq n, odd \\ \frac{7i-4}{2}, & 2 \leq i \leq n, even \end{cases}$$

Now we evaluate weights for all edges as:

$$w_t(U_iV_i) = \begin{cases} \frac{7i+1}{2}, & 1 \leq i \leq 2n-2, odd \\ \frac{7i}{2}, & 2 \leq i \leq 2n-2, even \end{cases}$$

$$w_t(V_{2i-1}V_{2i}) = 7i-1, 1 \leq i \leq n-1$$

$$w_t(W_iV_{2i-1}) = 7i-4, 1 \leq i \leq n-1$$

$$w_t(V_{2i}W_{i+1}) = 7i+1, 1 \leq i \leq n-1$$

$$w_t(W_iX_i) = 7i-5, 1 \leq i \leq n$$

$$w_t(W_iW_{i+1}) = 7i-2, 1 \leq i \leq n-1$$

On the basis of above calculations we see that all edges have distinct weights.
Hence we can say that $es(PQS_n) = \lfloor \frac{7n-4}{2} \rfloor$, for $n > 1$. $\square$

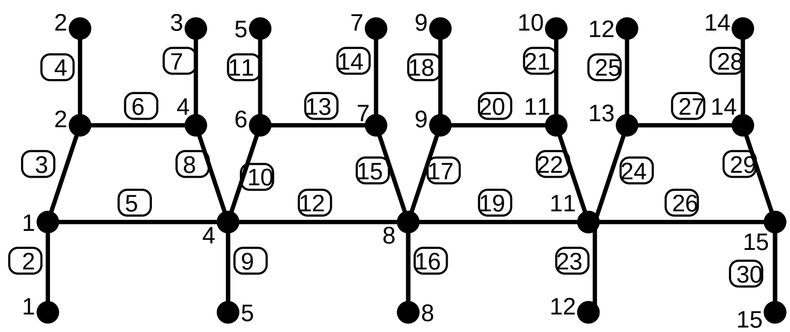

Figure 2. Irregular Labeling on Quadrilateral Snake Graph PQS_5 With Pendant Edges.

2.3. Alternate Quadrilateral Snake Graph

To obtain an alternate quadrilateral snake graph AQS_n, let us consider a path graph P_n having $(n > 1)$ vertices, if we join P_i and P_{i+1} (alternatively) to a new vertex a_i in such a way that every alternate edge of a path is replaced by triangle C_4, we get alternate quadrilateral snake graph AQS_n. It has $2n$ vertices where $n > 1, U_i$ and V_i have n vertices. Its edge set can be defined as $E(AQS_n) = \{U_iU_{i+1}; 1 \leq i \leq n-1\} \bigcup \{V_{2i-1}V_{2i}; 1 \leq i \leq \frac{n}{2}\} \bigcup \{U_{2i}V_{2i}; 1 \leq i \leq \frac{n}{2}\} \bigcup \{U_{2i-1}V_{2i-1}; 1 \leq i \leq \frac{n}{2}\}$.

Theorem 2.3. *Let AQS_n be an alternate quadrilateral snake graph, then $es(AQS_n) = \lfloor \frac{5n+2}{4} \rfloor$, for even n.*

Proof: Let AQS_n be an alternate quadrilateral snake graph. We have to show that $es(AQS_n) = \lfloor \frac{5n+2}{4} \rfloor$. From Theorem 1.1 we get lower bound $es(AQS_n) \geq \lfloor \frac{5n+2}{4} \rfloor$. For converse, we have to prove that $es(AQS_n) \leq \lfloor \frac{5n+2}{4} \rfloor$. For this, define a vertex labeling $h : V(AQS_n) \rightarrow \{1, 2, 3, ..., \lfloor \frac{5n+2}{4} \rfloor\}$ such that

$$h(V_i) = \begin{cases} \frac{5i+3}{4}, & \text{if } i \equiv 1(mod4) \\ \frac{5i+2}{4}, & \text{if } i \equiv 2(mod4) \\ \frac{5i+1}{4}, & \text{if } i \equiv 3(mod4) \\ \frac{5i-8}{4}, & \text{if } i \equiv 4(mod4) \end{cases}$$

$$h(U_i) = \begin{cases} \frac{5i-1}{4}, & \text{if } i \equiv 1(mod4) \\ \frac{5i-6}{4}, & \text{if } i \equiv 2(mod4) \\ \frac{5i+5}{4}, & \text{if } i \equiv 3(mod4) \\ \frac{5i}{4}, & \text{if } i \equiv 4(mod4) \end{cases}$$

Now we evaluate weights for all edges as:

$$W_t(U_iU_{i+1}) = \begin{cases} \frac{5i-1}{2}, & \text{if } i \equiv 1(mod4) \\ \frac{5i+2}{2}, & 2 \leq i \leq n-1, even \\ \frac{5i+5}{2}, & \text{if } i \equiv 3(mod4) \end{cases}$$

$$W_t(V_{2i-1}V_{2i}) = \begin{cases} 5i, & 1 \leq i \leq \frac{n}{2}, odd \\ 5i-3, & 2 \leq i \leq \frac{n}{2}, even \end{cases}$$

$$W_t(U_{2i}V_{2i}) = \begin{cases} 5i-1, & \text{if } i \equiv 1(mod4) and i \equiv 3(mod4) \\ 5i-2, & \text{if } i \equiv 2(mod4) and i \equiv 4(mod4) \end{cases}$$

$$W_t(U_{2i-1}V_{2i-1}) = \begin{cases} 5i-2, & \text{if } i \equiv 1(mod4) and i \equiv 3(mod4) \\ 5i-1, & \text{if } i \equiv 2(mod4) and i \equiv 4(mod4) \end{cases}$$

On the basis of above calculations we see that all edges have distinct weights.

Hence we can say that $es(AQS_n) = \lfloor \frac{5n+2}{4} \rfloor$, for even n. □

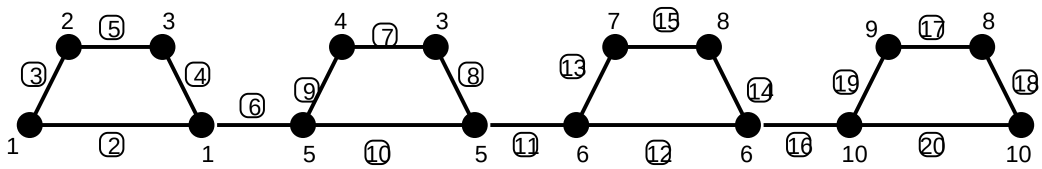

Figure 3. Irregular Labeling on Alternate Quadrilateral Snake Graph AQS_8.

2.4. Alternate Quadrilateral Snake Graph with Pendant Edges

Alternate Quadrilateral snake graph $PAQS_n$ with pendant edges consists of $4n$ vertices where $n > 1, U_i, V_i, W_i$ and X_i have n vertices. Its edge set can be defined as $E(PAQS_n) = \{U_iV_i; 1 \leq i \leq n\} \bigcup \{V_iW_i; 1 \leq i \leq n\} \bigcup \{W_iX_i; 1 \leq i \leq n\} \bigcup \{V_{2i-1}V_{2i}; 1 \leq i \leq \frac{n}{2}\} \bigcup \{W_iW_{i+1}; 1 \leq i \leq n-1\}$. It has $\frac{9n-2}{2}$ edges.

Theorem 2.4. *Let $PAQS_n$ be an alternate quadrilateral snake graph with pendant edges, then $es(PAQS_n) = \lfloor \frac{9n+2}{4} \rfloor$, for even n.*

Proof: Let $PAQS_n$ be a alternate quadrilateral snake graph with pendant edges. We have to show that $es(PAQS_n) = \lfloor \frac{9n+2}{4} \rfloor$ From Theorem 1.1 we get lower bound $es(PAQS_n) \geq \lfloor \frac{9n+2}{4} \rfloor$. For converse, we have to prove that $es(PAQS_n) \leq \lfloor \frac{9n+2}{4} \rfloor$. For this, define a vertex labeling $h : V(PAQS_n) \rightarrow \{1, 2, 3, ..., \lfloor \frac{9n+2}{4} \rfloor\}$ such that

$$h(U_i) = \begin{cases} \frac{9i-1}{4}, & \text{if } i \equiv 1(mod4) \\ \frac{9i-2}{4}, & \text{if } i \equiv 2(mod4) \\ \frac{9i-11}{4}, & \text{if } i \equiv 3(mod4) \\ \frac{9i-4}{4}, & \text{if } i \equiv 4(mod4) \end{cases}$$

$$h(V_i) = \begin{cases} \frac{9i-1}{4}, & \text{if } i \equiv 1(mod4) \\ \frac{9i-6}{4}, & \text{if } i \equiv 2(mod4) \\ \frac{9i+1}{4}, & \text{if } i \equiv 3(mod4) \\ \frac{9i-4}{4}, & \text{if } i \equiv 4(mod4) \end{cases}$$

$$h(W_i) = \begin{cases} \frac{9i-5}{4}, & \text{if } i \equiv 1(mod4) \\ \frac{9i+2}{4}, & \text{if } i \equiv 2(mod4) \\ \frac{9i-7}{4}, & \text{if } i \equiv 3(mod4) \\ \frac{9i}{4}, & \text{if } i \equiv 4(mod4) \end{cases}$$

$$h(X_i) = \begin{cases} \frac{9i-5}{4}, & \text{if } i \equiv 1(mod4) \\ \frac{9i-2}{4}, & \text{if } i \equiv 2(mod4) \\ \frac{9i+5}{4}, & \text{if } i \equiv 3(mod4) \\ \frac{9i}{4}, & \text{if } i \equiv 4(mod4) \end{cases}$$

Now we evaluate weights for all edges as:

$$w_t(U_iV_i) = \begin{cases} \frac{9i-1}{2}, & \text{if } i \equiv 1(mod4) \\ \frac{9i-5}{2}, & \text{if } i \equiv 3(mod4) \\ \frac{9i-4}{2}, & \text{if } 2 \leq i \leq n, even \end{cases}$$

$$w_t(V_iW_i) = \begin{cases} \frac{9i-3}{2} & \text{if } 1 \leq i \leq n, odd \\ \frac{9i-2}{2} & \text{if } 2 \leq i \leq n, even \end{cases}$$

$$w_t(W_iX_i) = \begin{cases} \frac{9i-1}{2}, & \text{if } i \equiv 3(mod4) \\ \frac{9i-5}{2}, & \text{if } i \equiv 1(mod4) \\ \frac{9i}{2}, & \text{if } 2 \leq i \leq n, even \end{cases}$$

$$w_t(W_iW_{i+1}) = \begin{cases} \frac{9i+1}{2}, & \text{if } i \equiv 3(mod4) \\ \frac{9i+3}{2}, & \text{if } i \equiv 1(mod4) \\ \frac{9i+2}{2}, & \text{if } 2 \leq i \leq n-1, even \end{cases}$$

$$w_t(V_{2i-1}V_{2i}) = \begin{cases} 9i-3, & \text{if } 2 \leq i \leq \frac{n}{2}, even \\ 9i-2, & \text{if } 1 \leq i \leq \frac{n}{2}, odd \end{cases}$$

On the basis of above calculations we see that all edges have distinct weights.

Hence we can say that $es(PAQS_n) = \lfloor \frac{9n+2}{4} \rfloor$, for even n. □

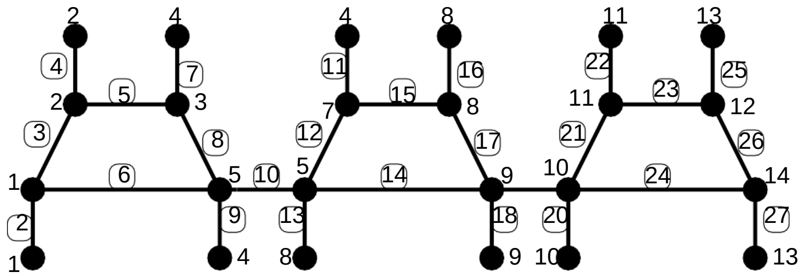

Figure 4. Irregular Labeling on Alternate Quadrilateral Snake Graph $PAQS_6$ With Pendant Edges.

2.5. Double Alternate Quadrilateral Snake Graph

Double alternate quadrilateral snake graph $DAQS_n$ consists of two alternate quadrilateral snake graphs that have common path. $DAQS_n$ consists of $3n$ vertices where $n > 1, U_i, V_i$ and W_i have n vertices. Its edge set can be defined as $E(DAQS_n) = \{U_iU_{i+1}; 1 \leq i \leq n-1\} \bigcup \{V_{2i-1}V_{2i}; 1 \leq i \leq \frac{n}{2}\} \bigcup \{U_{2i}V_{2i}; 1 \leq i \leq \frac{n}{2}\} \bigcup \{U_{2i-1}V_{2i-1}; 1 \leq i \leq \frac{n}{2}\} \bigcup \{W_{2i-1}W_{2i}; 1 \leq i \leq \frac{n}{2}\} \bigcup \{U_{2i}W_{2i}; 1 \leq i \leq \frac{n}{2}\} \bigcup \{U_{2i-1}W_{2i-1}; 1 \leq i \leq \frac{n}{2}\}$.

Theorem 2.5. *Let $DAQS_n$ be a double alternate quadrilateral snake graph, then $es(DAQS_n) = 2n$, for even n.*

Proof: Let $DAQS_n$ be a double alternate quadrilateral snake graph. We have to show that $es(DAQS_n) = 2n$. From Theorem 1.1 we get lower bound $es(DAQS_n) \geq 2n$. For converse, we have to prove that $es(DAQS_n) \leq 2n$. For this, define a vertex labeling $h : V(DAQS_n) \rightarrow \{1, 2, 3, ..., 2n\}$ such that

$$h(V_i) = 2i, 1 \leq i \leq n$$

$$h(U_i) = \begin{cases} 2i, & \text{if } 2 \leq i \leq n, even \\ 2i-1, & \text{if } 1 \leq i \leq n, odd \end{cases}$$

$$h(W_i) = 2i-1, 1 \leq i \leq n$$

Now we evaluate weights for all edges as:

$$w_t(U_iU_{i+1}) = 4i + 1, 1 \leq i \leq n - 1$$

$$w_t(V_{2i-1}V_{2i}) = 8i - 2, 1 \leq i \leq \frac{n}{2}$$

$$w_t(U_{2i}V_{2i}) = 8i, 1 \leq i \leq \frac{n}{2}$$

$$w_t(U_{2i-1}V_{2i-1}) = 8i - 5, 1 \leq i \leq \frac{n}{2}$$

$$w_t(W_{2i-1}W_{2i}) = 8i - 4, 1 \leq i \leq \frac{n}{2}$$

$$w_t(U_{2i}W_{2i}) = 8i - 1, 1 \leq i \leq \frac{n}{2}$$

$$w_t(U_{2i-1}W_{2i-1}) = 8i - 6, 1 \leq i \leq \frac{n}{2}$$

On the basis of above calculations we see that all edges have distinct weights.
Hence we can say that $es(DAQS_n) = 2n$, for even n. □

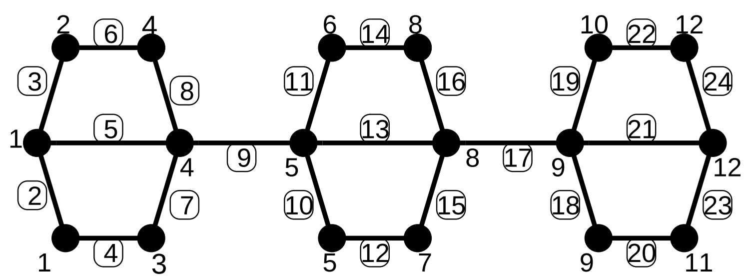

Figure 5. Irregular Labeling on Double Alternate Quadrilateral Snake Graph $DAQS_6$.

2.6. Linear Phenylene Graph

To obtain a Linear phenylene graph, we make a chain of alternate copies of hexagon and squares. Its vertex set and edge set can be defined as $V(PH_n) = \{X_i; 1 \leq i \leq n\} \bigcup \{Y_i; 1 \leq i \leq n\}$ and $E(PH_n) = \{X_iX_{i+1}; 1 \leq i \leq n-1\} \bigcup \{Y_iY_{i+1}; 1 \leq i \leq n-1\} \bigcup \{X_{3i}Y_{3i}; 1 \leq i \leq \frac{n-1}{3}\} \bigcup \{X_{3i+1}Y_{3i+1}; 1 \leq i \leq \frac{n-1}{3}\} \bigcup \{X_1Y_1\}$.

Theorem 2.6. *Let PH_n be a linear phenylene graph, then $es(PH_n) = \lfloor \frac{4n}{3} \rfloor$, for $n \equiv 4(mod 3)$.*

Proof: Let PH_n be a linear phenylene graph. We have to show that $es(PH_n) = \lfloor \frac{4n}{3} \rfloor$. From Theorem 1.1 we get lower bound $es(PH_n) \geq \lfloor \frac{4n}{3} \rfloor$. For converse, we have to prove that $es(PH_n) \leq \lfloor \frac{4n}{3} \rfloor$. For this, define a vertex labeling $h : V(PH_n) \rightarrow \{1, 2, 3, ..., \lfloor \frac{4n}{3} \rfloor\}$ such that

$$h(X_i) = \begin{cases} \frac{4i-1}{3}, & \text{if } i \equiv 1(mod3) \\ \frac{4i-2}{3}, & \text{if } i \equiv 2(mod3) \\ \frac{4i-6}{3}, & \text{if } i \equiv 3(mod3) \end{cases}$$

$$h(Y_i) = \begin{cases} \frac{4i-1}{3}, & \text{if } i \equiv 1(mod3) \\ \frac{4i+4}{3}, & \text{if } i \equiv 2(mod3) \\ \frac{4i}{3}, & \text{if } i \equiv 3(mod3) \end{cases}$$

Now we evaluate weights for all edges as:

$$W_t(Y_iY_{i+1}) = \begin{cases} \frac{8i+7}{3}, & \text{if } i \equiv 1(mod3) \\ \frac{8i+8}{3}, & \text{if } i \equiv 2(mod3) \\ \frac{8i+3}{3}, & \text{if } i \equiv 3(mod3) \end{cases}$$

$$W_t(X_iX_{i+1}) = \begin{cases} \frac{8i+1}{3}, & \text{if } i \equiv 1(mod3) \\ \frac{8i-4}{3}, & \text{if } i \equiv 2(mod3) \\ \frac{8i-3}{3}, & \text{if } i \equiv 3(mod3) \end{cases}$$

$$W_t(X_{3i+1}Y_{3i+1}) = 8i + 2, 1 \leq i \leq \frac{n-1}{3}$$

$$W_t(X_{3i}Y_{3i}) = 8i - 2, 1 \leq i \leq \frac{n-1}{3}$$

$$W_t(X_1Y_1) = 2$$

On the basis of above calculations we see that all edges have distinct weights.
Hence we can say that $es(PH_n) = \lfloor \frac{4n}{3} \rfloor$, for $n \equiv 4(mod3)$. □

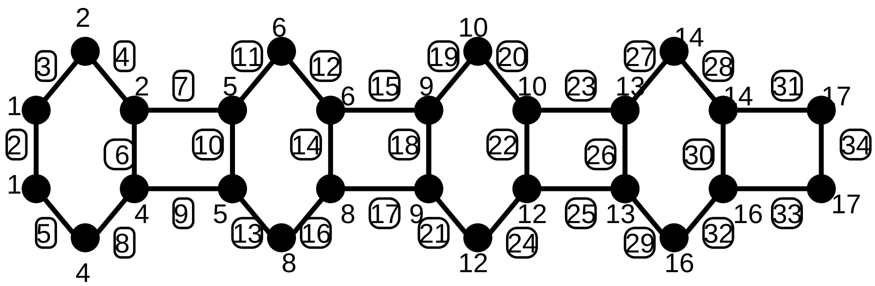

Figure 6. Linear Phenylene Graph PH_{13}.

2.7. B_n Graph

B_n graph is obtained by making a ladder $L_n \simeq P_n \times P_2$. It is a planar graph. Its vertex set is formed by $V(B_n) = \{X_i; 1 \leq i \leq n\} \bigcup \{Y_i; 1 \leq i \leq n\} \bigcup \{Z_i; 1 \leq i \leq n\}$ and edge set can be given as $E(B_n) = \{X_iX_{i+1}; 1 \leq i \leq n-1\} \bigcup \{Y_iY_{i+1}; 1 \leq i \leq n-1\} \bigcup \{Z_iZ_{i+1}; 1 \leq i \leq n-1\} \bigcup \{X_iY_i; 1 \leq i \leq n\} \bigcup \{X_{i+1}Y_i; 1 \leq i \leq n-1\} \bigcup \{Y_iZ_i; 1 \leq i \leq n\}$.

Theorem 2.7. *Show that $es(B_n) = \lfloor \frac{7n-2}{2} \rfloor$, for $n > 1$.*

Proof: Let B_n be a graph. We have to show that $es(B_n) = \lfloor \frac{7n-2}{2} \rfloor$. From Theorem 1.1 we get lower bound $es(B_n) \geq \lfloor \frac{7n-2}{2} \rfloor$. For converse, we have to prove that $es(B_n) \leq \lfloor \frac{7n-2}{2} \rfloor$. For this, define a vertex labeling $h : V(B_n) \rightarrow \{1, 2, 3, ..., \lfloor \frac{7n-2}{2} \rfloor\}$ such that

$$h(X_i) = \begin{cases} \frac{7i-6}{2}, & 2 \leq i \leq n, even \\ \frac{7i-5}{2}, & 1 \leq i \leq n, odd \end{cases}$$

$$h(Y_i) = \begin{cases} \frac{7i-4}{2}, & 2 \leq i \leq n, even \\ \frac{7i-3}{2}, & 1 \leq i \leq n, odd \end{cases}$$

$$h(Z_i) = \begin{cases} \frac{7i-2}{2}, & 2 \leq i \leq n, even \\ \frac{7i-3}{2}, & 1 \leq i \leq n, odd \end{cases}$$

Now we evaluate weights for all edges as:

$$w_t(X_iX_{i+1}) = 7i - 2, 1 \leq i \leq n - 1$$

$$w_t(Y_iY_{i+1}) = 7i, 1 \leq i \leq n - 1$$

$$w_t(Z_iZ_{i+1}) = 7i + 1, 1 \leq i \leq n - 1$$

$$w_t(X_{i+1}Y_i) = 7i - 1, 1 \leq i \leq n - 1$$

$$w_t(X_iY_i) = \begin{cases} 7i - 5, & 2 \leq i \leq n, even \\ 7i - 4, & 1 \leq i \leq n, odd \end{cases}$$

$$w_t(Y_iZ_i) = 7i - 3, 1 \leq i \leq n$$

On the basis of above calculations we see that all edges have distinct weights.

Hence we can say that $es(B_n) = \lfloor \frac{7n-2}{2} \rfloor$, for $n > 1$. □

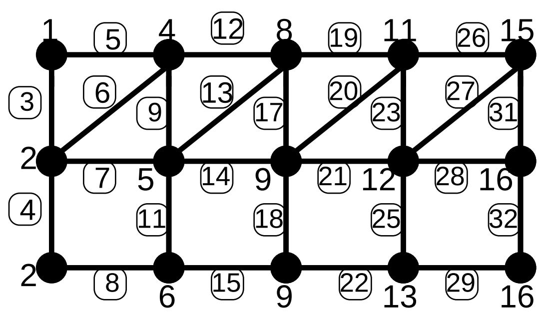

Figure 7. Irregular Labeling on B_5 Graph.

Conflicts of Interest

All the authors declare that they have no conflicts of interest.

3. Conclusion

In this research paper, we obtained exact values of edge irregularity strength of linear phenylene graph PH_n, B_n graph and different families of snake graph.

References

[1] A. Ahmad, M. Baca, Y. Bashir, and M. K. Siddiqui, “Total edge irregularity strength of strong product of two paths”, *Ars Combinatoria*, vol. 106, pp. 449-459, 2012.

[2] A. Ahmad, M. Baca, and M. K. Siddiqui, “On edge irregular total labeling of categorical product of two cycles”. *Theory of Computing Systems*, vol. 54, no. 1, pp. 1-12, 2014. doi: 10.1007/s00224-013-9470-3

[3] A. Ahmad, O. B. S. Al-Mushayt, and M. Baca, “On edge irregularity strength of graphs”, *Appl. Math. Comput*, vol. 243, pp. 607-610, 2014. doi: 10.1016/j.amc.2014.06.028

[4] A. Ahmad, M. Arshad, and G. Iarkov, “Irregular labelings of helm and sun graphs”. *AKCE International Journal of Graphs and combinatorics*, vol. 12, no. 2-3, pp. 161-168, 2015. doi: 10.1016/j.akcej.2015.11.010

[5] A. Ahmad, M. Baca, and M. F. Nadeem, “On edge irregularity strength of Toeplitz graphs”, *U.P.B. Sci. Bull., Series A*, vol. 78, no. 4, pp. 155-162, 2016.

[6] A. Ahmad, M. A. Asim, B. Assiri, and A. Semaničová-Feňovčíková, “Computing the edge irregularity strength of bipartite graphs and wheel related graphs”, *Fundamenta Informaticae*, vol. 174, no. 1, pp. 1-13, 2020. doi: 10.3233/fi-2020-1927

[7] D. Amar, and O. Togni, “Irregularity strength of trees”, *Discrete Math*, vol. 190, pp. 15-38, 1998. doi: 10.1016/s0012-365x(98)00112-5

[8] M. Anholcer, M. Kalkowski, and J. Przybyo, “A new upper bound for the total vertex irregularity strength of graphs”, *Discrete Math.*, vol. 309, no. 21, pp. 6316-6317, 2009. doi: 10.1016/j.disc.2009.05.023

[9] O. B. S. Al-Mushayt, A. Ahmad, and M. K. Siddiqui, “On the total edge irregularity strength of hexagonal grid graphs”, *Australasian J. Combinatorics*, vol. 53, 263-272, 2012.

[10] O. S. Al-Mushayt, “On edge irregularity strength of products of certain families of graphs with path P-2”, *Ars Combinatoria*, vol. 135, pp. 323-334, 2017.

[11] E. T. Baskoro, A. N. M. Salman, and N. N. Gaos, “On the total vertex irregularity strength of trees”, *Discrete Math.*, vol. 310, no. 21, pp. 3043-3048, 2010. doi: 10.1016/j.disc.2010.06.041

[12] M. Baca, and S. Jendrol, M. Miller, and J. Ryan, “On irregular total labellings”, *Discrete Mathematics*, pp. 1378-1388, vol. 307, no. 11-12, 2007. doi: 10.1016/j.disc.2005.11.075

[13] M. Baca, and M. K. Siddiqui, “Total edge irregularity strength of generalized prism”. *Appl. Math. Comput*, vol. 235, pp. 168-173, 2014. doi: 10.1016/j.amc.2014.03.001

[14] T. Bohman, and D. Kravitz, “On the irregularity strength of trees”, *J. Graph Theory*, vol. 45, pp. 241-254, 2004. doi: 10.1002/jgt.10158

[15] G. Chartrand, M. S. Jacobson, J. Lehel, O. R. Oellermann, S. Ruiz, and F. Saba, “Irregular networks”, *Congr. Numer.*, vol. 64, pp. 187-192, 1988.

[16] A. Frieze, R. J. Gould, M. Karonski, and F. Pfender, “On graph irregularity strength”, *J. Graph Theory*, vol. 41, pp. 120-137, 2002. doi: 10.1002/jgt.10056

[17] K. M. M. Haque, “Irregular total labellings of generalized Petersen graphs”, *Theory of Computing Systems*, vol. 50, no. 3, pp. 537-544, 2012. doi: 10.1007/s00224-011-9350-7

[18] J. Ivano, and S. Jendrol, “Total edge irregularity strength of trees”, *Discussiones Mathematicae Graph Theory*, vol. 26, no. 3, pp. 449-456, 2006. doi: 10.7151/dmgt.1337

[19] M. Imran, A. Aslam, S. Zafar, and W. Nazeer, “Further results on edge irregularity strength of graphs”. *Indonesian Journal of Combinatorics*, vol. 1, no. 2, pp. 82-91, 2017. doi: 10.19184/ijc.2017.1.2.5

[20] S. Jendrol, J. Mikuf, and R. Sotk, “Total edge irregularity strength of complete graphs and complete bipartite graphs”, *Discrete Mathematics*, vol. 310, no. 3, pp. 400-407, 2010. doi: 10.1016/j.disc.2009.03.006

[21] M. Kalkowski, M. Karonski, and F. Pfender, “A new upper bound for the irregularity strength of graphs”, *SIAM J.Discrete Math.*, vol. 25, no. 3, pp. 1319-1321, 2011. doi: 10.1137/090774112

[22] P. Majerski, and J. Przybyo, “Total vertex irregularity strength of dense graphs”. *Journal of Graph Theory*, vol. 76, no. 1, pp. 34-41, 2014. doi: 10.1002/jgt.21748

[23] P. Majerski, and J. Przybylo, “On irregularity strength of dense graphs”, *J. Graph Theory*, vol. 28, no. 1, pp. 197-205, 2014. doi: 10.1137/120886650

[24] J. Przybyo, “Linear bound on the irregularity strength and the total vertex irregularity strength of graphs”, *SIAM Journal on Discrete Mathematics*, vol. 23, no. 1, pp. 511-516, 2009. doi: 10.1137/070707385

[25] M. K. Siddiqui, “On edge irregularity strength of subdivision of star Sn”, *International Journal of Mathematics and Soft Computing*, vol. 2, no. 1, pp. 75-82, 2012. doi: 10.26708/ijmsc.2012.1.2.09

[26] M. K. Siddiqui, A. Ahmad, M. F. Nadeem, and Y. Bashir, “Total edge irregularity strength of the disjoint union of sun graphs”, *International Journal of Mathematics and Soft Computing*, vol. 3, no. 1, pp. 21-27, 2013. doi: 10.26708/ijmsc.2013.1.3.02

[27] I. Tarawneh, R. Hasni, and A. Ahmad, “On the edge irregularity strength of corona product of graphs with paths”, *Appl. Math. E-Notes*, vol. 16, pp. 80-87, 2016. doi: 10.1142/s1793830920500834

[28] I. Tarawneh, R. Hasni, and A. Ahmad, “On the edge irregularity strength of corona product of cycle with isolated vertices”, *AKCE International Journal of Graphs and Combinatorics*, vol. 13, no. 3, pp. 213-217, 2016. doi: 10.1016/j.akcej.2016.06.010

[29] I. Tarawneh, R. Hasni, and M. A. Asim, “On the edge irregularity strength of disjoint union of star graph and subdivision of star graph”, *Ars Combinatoria*, vol. 141, pp. 93-100, 2018.

[30] I. Tarawneh, R. Hasni, and A. Ahmad, “On the edge irregularity strength of grid graphs”, *AKCE International Journal of Graphs and Combinatorics*, vol. 17, no. 1, pp. 414-418, 2020. doi: 10.1016/j.akcej.2018.06.011

[31] X. Zhang, M. Cancan, M. F. Nadeem, and M. Imran, "Edge irregularity strength of certain families of comb graph", *Proyecciones (Antofagasta)*, vol. 39, no. 4, pp. 787-797, 2020. doi: 10.22199/issn.0717-6279-2020-04-0049

Muhammad Imran
Department of Mathematics,
Concordia College Kasur Campus,
Pakistan
e-mail: imranbepakistani@gmail.com
Corresponding Author

Murat Cancan
Faculty of Education,
Yuzuncu Yil University,
Van,
Turkey
e-mail: mcancan@yyu.edu.tr

Yasir Ali
Department of Mathematics,
Concordia College Kasur Campus,
Pakistan
e-mail: yasirbepakistani@gmail.com

Anum
Department of Mathematics,
Concordia College Kasur Campus,
Pakistan
e-mail: anamikram319@gmail.com

and

Jamila Aslam
Department of Mathematics,
Concordia College Kasur Campus,
Pakistan
e-mail: shaynisawera065@gmail.com