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Fuzzy S_β-compactness and fuzzy S_β-closed spaces

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Abstract

The aim of this paper is to introduce the notion of fuzzy S_β - compactness. Some of the basic properties and characterization theorems would be investigated of this newly defined compactness in fuzzy setting. We would also introduce and study fuzzy S_β - closed spaces.

Keywords: *Fuzzy set, fuzzy topology, fuzzy S_β-compact space, fuzzy S_β-open set, fuzzy S_β - closed space.*

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1. Introduction

After introduction of the notion of fuzzy sets by Zadeh [22], researchers contributed for investigation on fuzzy sets in different aspects and successfully applied it for further investigations in all the branches of science and technology. Chang [6] introduced the notion of fuzzy topology. Abd El-Monsef et al. [1] introduced the concepts of β-open sets and β - continuous functions in general topology and Fath Alla [3] introduced these concepts in fuzzy setting. Khalaf and Ahmed [15] introduced and studied a new class of semiopen sets, called S_β - open sets, they then introduced and investigated S_β - continuous functions in general topological spaces. Dhar [9] introduced the notion of fuzzy S_β-open sets, fuzzy S_β-continuous mappings and fuzzy S_β-open mappings. Compactness contributes a very important role in fuzzy topology and so do some of its forms. Balasubramanian [4] introduced and investigated some interesting properties of fuzzy β- compactness. Hanafy [14] introduced and studied the concepts of β - compactness and β-closed spaces in fuzzy setting. Besides them, many researchers [2, 5, 7, 8, 10, 11, 12, 13, 16, 17, 18, 19, 20, 21] contributed themselves to the study of fuzzy setting. These works lead us to this paper with the purpose of defining fuzzy S_β-compactness and fuzzy S_β-closed spaces and to study their basic properties in fuzzy setting. In section 2, the different known concepts and results in fuzzy setting would be procured as ready reference. In section 3, the notion of fuzzy S_β-compact spaces would be introduced and investigated some of their basic properties in fuzzy topological spaces. In section 4, the concept of fuzzy S_β-closed spaces would be introduced and studied in fuzzy setting. Throughout the article, X and Y represent fuzzy topological spaces.

2. Preliminaries

In this section, some preliminary results and definitions have been procured as ready reference.

Definition 2.1. [22] Let A and B be two fuzzy sets in a crisp set X and the membership functions of them be μ_A and μ_B respectively. Then

1. A is equal to B, i.e., $A = B$ if and only if $\mu_A(x) = \mu_B(x)$, for all $x \in X$,

2. A is called a subset of B if and only if $\mu_A(x) \leq \mu_B(x)$, for all $x \in X$,

3. the Union of two fuzzy sets A and B is denoted by $A \vee B$ and its membership function is given by $\mu_{A \vee B} = \max(\mu_A, \mu_B)$,

4. the Intersection of two fuzzy sets A and B is denoted by $A \wedge B$ and its membership function is given by $\mu_{A \wedge B} = \min(\mu_A, \mu_B)$,

5. the Complement of a fuzzy set A is defined as the negation of the specified membership function. Symbolically it can be written as $\mu_A^c = 1 - \mu_A$.

Definition 2.2. [16] A fuzzy point x_p in X is a fuzzy set in X defined by

$$\begin{aligned} x_p(y) \quad &= p\,(0 < p \leq 1),\ \text{for } y = x \\ &= 0,\ \text{for } y \neq x (y \in X), \end{aligned}$$

x and p are respectively the support and the value of x_p. A fuzzy point x_p is said to belong to a fuzzy set A of X if and only if $p \leq A(x)$. A fuzzy set A in X is the union of all fuzzy points which belong to A.

Definition 2.3. [6] Suppose τ is a family of fuzzy subsets in X which satisfies the following axioms :

1. $0_X, 1_X \in \tau$.

2. If $A, B \in \tau$, then $A \wedge B \in \tau$.

3. If $A_j \in \tau$ for all j from the index set $J, \vee_{j \in J} A_j \in \tau$.

Then τ is called a fuzzy topology for X and the pair (X, τ) is called a fuzzy topological space (fts, in short). The elements of τ are called fuzzy open subsets. The complement of each member in τ is defined as a fuzzy closed set in X (with respect to τ) or simply a fuzzy closed set in X.

Throughout the paper, the spaces X and Y always represent fuzzy topological spaces (X, τ) and (Y, σ) respectively.

Definition 2.4. [3] A fuzzy set A in a fuzzy topological space X is called fuzzy β-open set if A $\leq$ *clintcl* A.

From definition it follows that each fuzzy semiopen and fuzzy preopen set imply fuzzy β-open set.

Definition 2.5. [3] Let A be a fuzzy set in a fuzzy topological space X. The fuzzy β-closure (βcl) and β-interior (β int) of A are defined as follows :

$$\beta clA = \wedge\{B : A \leq B, B \text{ is fuzzy } \beta\text{-closed }\},$$
$$\beta intA = \vee\{B : A \geq B, B \text{ is fuzzy } \beta\text{-open }\}.$$

It is obvious that $\beta cl(A)^c = (\beta intA)^c$ and $\beta int(A)^c = (\beta clA)^c$.

Definition 2.6. [3] A function $f : X \to Y$ is said to be fuzzy β-continuous (respectively $M\beta$ - continuous) if the inverse image of every fuzzy open (respectively fuzzy β - open) set in Y is fuzzy β - open (respectively fuzzy β-open) set in X.

Definition 2.7. [9] A fuzzy semiopen subset A of a fuzzy topological space (X, τ) is said to be fuzzy S_β-open if for each fuzzy point $x_p \in A$ there exists a fuzzy β-closed set F such that $x_p \in F \leq A$. A fuzzy subset B of a fuzzy topological space X is fuzzy S_β-closed if its complement is fuzzy S_β-open.

The family of all fuzzy S_β-open subsets of X is denoted by $S_\beta O(X)$.

Definition 2.8. [9] A fuzzy point x_p is said to be a fuzzy S_β-interior point of A if there exists a fuzzy S_β-open set U containing x_p such that $x_p \in U \leq A$. The set of all fuzzy S_β-interior points of A is said to be fuzzy S_β-interior of A and it is denoted by $S_\beta intA$.

Definition 2.9. [9] Intersection of all fuzzy S_β-closed sets containing F is called the fuzzy S_β - closure of F and is denoted by $S_\beta clF$.

Theorem 2.10. [22] A function $f : X \to Y$ is fuzzy open function, then $f^{-1}(cl(u)) \leq cl\left(f^{-1}(u)\right)$ for every fuzzy set u in Y.

3. Fuzzy S_β-compact spaces

In this section, we would introduce and study a new class of fuzzy topological spaces, called fuzzy S_β- compact spaces.

Definition 3.1. A function $f : X \to Y$ is said to be fuzzy S_β-continuous (respectively $M\, S_\beta$ - continuous) if the inverse image of every fuzzy open (respectively fuzzy S_β-open) set in Y is fuzzy S_β-open (respectively fuzzy S_β-open) set in X.

Lemma 3.2. Let $f : X \to Y$ be a function. Then the following are equivalent :

1. f is fuzzy MS_β-continuous.

2. $f\left(S_\beta clu\right) \leq S_\beta clf(u)$, for every fuzzy set u in X.

Proof. $1 \Rightarrow 2$. Let u be a fuzzy set of X. Then $S_\beta clf(u)$ is fuzzy S_β-closed. By 1, $f^{-1}\left(S_\beta clf(u)\right)$ is fuzzy S_β-closed and so $f^{-1}\left(S_\beta clf(u)\right) = S_\beta clf^{-1}\left(S_\beta clf(u)\right)$. Hence $f\left(S_\beta clu\right) \leq S_\beta clf(u)$.

$2 \Rightarrow 1$. Let v be a fuzzy S_β-closed set in Y. By 2, if $u = f^{-1}(v)$, then $S_\beta clf^{-1}(v) \leq f^{-1}\left(S_\beta clf\left(f^{-1}(v)\right)\right) \leq f^{-1}\left(S_\beta clv\right) = f^{-1}(v)$. Since $f^{-1}(v) \leq S_\beta clf^{-1}(v)$, then $f^{-1}(v) = S_\beta clf^{-1}(v)$. Hence $f^{-1}(v)$ is fuzzy S_β-closed set in X. Hence f is fuzzy MS_β-continuous. $\square$

Lemma 3.3. Let $f : X \to Y$ be a function. Then the following are equivalent :

1. f is fuzzy S_β-continuous.

2. $f\left(S_\beta clu\right) \leq clf(u)$, for every fuzzy set u in X.

Proof. Obvious.

Definition 3.4. A fuzzy topological space X is said to be fuzzy S_β-compact if and only if for every family μ of fuzzy S_β-open sets where $\bigvee_{A\in\mu} A = 1_X$, there is a finite subfamily $\eta \subseteq \mu$ such that $\bigvee_{A\in\eta} A = 1_X$.

Definition 3.5. A fuzzy set u in a fuzzy topological space X is said to be fuzzy S_β-compact relative to X if and only if for every family μ of fuzzy S_β-open sets such that $\bigvee_{A\in\mu} A \geq u(x)$, there is a finite subfamily $\eta \subseteq \mu$ such that $\bigvee_{A\in\eta} A \geq u(x)$ for every $x \in S(u)$.

Theorem 3.6. A fuzzy topological space X is fuzzy S_β-compact if and only if for every collection $\{A_j : j \in J\}$ of fuzzy S_β - closed sets of X having the finite intersection property, $\bigwedge_{j\in J} A_j \neq 0_X$.

Proof. Let $\{A_j : j \in J\}$ be a collection of fuzzy S_β-closed sets with the finite intersection property. Suppose that $\bigwedge_{j\in J} A_j = 0_X$. Then $\bigvee_{j\in J} A_j^c = 1_X$. Since $\left\{A_j^c : j \in J\right\}$ is a collection of fuzzy S_β-open sets cover of X, then from the fuzzy S_β-compactness of X it follows that there exists a finite subset $F \subseteq J$ such that $\bigvee_{j\in F} A_j^c = 1_X$. Then $\bigwedge_{j\in F} A_j = 0_X$ which gives a contradiction. Therefore, $\bigwedge_{j\in J} A_j \neq 0_X$

Conversely, let $\{A_j : j \in J\}$ be a collection of fuzzy S_β-open sets cover of X. Suppose that for every finite subset $F \subseteq J$, we have $\bigvee_{j\in F} A_j \neq 1_X$. Then $\bigwedge_{j\in F} A_j^c \neq 0_X$. Hence $\left\{A_j^c : j \in J\right\}$ satisfies the finite intersection property. Then from the hypothesis, we have $\bigwedge_{j\in J} A_j^c \neq 0_X$ which implies $\bigwedge_{j\in F} A_j \neq 1_X$ and this contradicting that $\{A_j : j \in J\}$ is a fuzzy S_β-open cover of X. Thus X is fuzzy S_β-compact. □

Theorem 3.7. A fuzzy topological space X is fuzzy S_β-compact if and only if every fuzzy filter base ξ in $X, \wedge_{G\in\xi} S_\beta clG \neq 0_X$.

Proof. Let μ be a fuzzy S_β-open cover of X and μ has no a finite subcover. Then for every finite subcollection $\{A_1, A_2, \ldots, A_n\}$ of μ, there exists $x \in X$ such that $A_j(x) < 1$ for every $j = 1, 2, \ldots.$, n. Then $A_j^c(x) > 0$, so that $\bigwedge_{j=1}^n A_j^c \neq 0_X$. Thus $\left\{A_j^c(x) : j \in \mu\right\}$ forms a fuzzy filter base in X. Since μ is fuzzy S_β-open cover of X, then $\bigvee_{A_j\in\mu} A_j)(x) = 1_X$ for every $x \in X$ and hence $\bigwedge_{A_j\in\mu} S_\beta clA_j^c(x) = \bigwedge_{A_j\in\mu} A_j^c(x) = 0_X$, which is a contradiction. Then every fuzzy S_β-open cover of X has a finite subcover and hence X is fuzzy S_β-compact.

Conversely, suppose there exists a fuzzy filter base ξ such that $\bigwedge_{G\in\xi} S_\beta clG = 0_X$, so that $\left(\bigvee_{G\in\xi} (S_\beta clG)^c\right)(x) = 1_X$ for every $x \in X$ and hence $\mu = \{(S_\beta clG)^c : G \in \xi\}$ is a fuzzy S_β-open set cover of X. Since X is fuzzy S_β-compact, then μ has a finite subcover. Then $\left(\bigvee_{j=1}^n (S_\beta clG_j)^c\right)(x) = 1_X$ and hence $\left(\bigvee_{j=1}^n G_j^c\right)(x) = 1_X$, so that $\bigwedge_{j=1}^n G_j = 0_X$ which is a contradiction, since the G_j are members of filter base ξ. Therefore $\bigwedge_{G\in\xi} S_\beta clG \neq 0_X$ for every filter base ξ. □

Theorem 3.8. A fuzzy set A in a fuzzy topological space X is fuzzy S_β-compact relative to X if and only if for every fuzzy filter base ξ such that every finite members of ξ is quasi-coincident with A, that is $(\wedge_{G\in\xi} S_\beta clG) \wedge A \neq 0_X$.

Proof. Let A not be fuzzy S_β-compact relative to X. Then there exists a fuzzy S_β-open set μ cover of A such that μ has no finite subcover η. Then $\left(\bigvee_{A_j\in\eta} A_j\right)(x) < A(x)$ for some $x \in S(A)$, so that $\left(\wedge_{A_j\in\eta} A_j^c\right)(x) > A^c(x) \geq 0_X$ and hence $\xi = \left\{A_j^c : A_j \in \mu\right\}$ forms a filterbase and $\bigwedge_{A_j\in\eta} A_j^c qA$. By hypothesis $\left(\wedge_{A_j\in\eta} S_\beta clA_j^c\right) \wedge A \neq 0_X$ and hence $\left(\wedge_{A_j\in\eta} A_j^c\right) \wedge A \neq 0_X$. Then for some $x \in S(A)$, $\left(\wedge_{A_j\in\eta} A_j^c\right)(x) > 0_X$, that is $\left(\vee_{A_j\in\mu} A_j\right)(x) < 1_X$, which is a contradiction. Hence A is fuzzy S_β-compact relative to X.

Conversely, suppose that there exists a filter base ξ such that every finite of members of ξ is quasi-coincident with A and $\left(\bigwedge_{G\in\xi} S_\beta clG\right) \wedge A \neq 0_X$. Then for every $x \in S(A), \left(\bigwedge_{G\in\xi} S_\beta clG\right)(x) = 0_X$ and hence $\left(\bigvee_{G\in\xi} (S_\beta clG)^c\right)(x) = 1_X$ for every $x \in S(A)$. Thus $\mu = \{(S_\beta ClG)^c : G \in \xi\}$ is fuzzy S_β - open cover of A. Since A is fuzzy S_β-compact relative to X, then there exists a finite subcover, say $\{(S_\beta clG_1)^c, \ldots, (S_\beta clG_n)^c\}$, such that $\left(\bigvee_{j=1}^{n} (S_\beta clG_j)^c\right)(x) \geq A(x)$ for every $x \in S(A)$. Hence $\left(\bigwedge_{j=1}^{n} (S_\beta clG_j)\right)(x) \leq A^c(x)$ for every $x \in S(A)$, so that $\bigwedge_{j=1}^{n} (S_\beta clG_j)$ is not quasi - coincident with A, which is a contradiction. Therefore, for every fuzzy filter base ξ such that every finite of members of ξ is quasi-coincident with A, that is $\left(\bigwedge_{G\in\xi} S_\beta clG\right) \wedge A \neq 0_X$. □

Theorem 3.9. Every fuzzy S_β-closed subset of a fuzzy S_β-compact space is fuzzy S_β-compact relative to X.

Proof. Let ξ be a fuzzy filter base in X such that $Vq \wedge \{G : G \in \lambda\}$ holds for every finite subcollection λ of ξ and a fuzzy S_β-closed set V. Consider $\xi^* = \{V\} \cup \xi$. For any finite subcollection λ^* of ξ^*, if V does not belong to λ^*, then $\wedge\lambda^* \neq 0_x$. If $V \in \lambda^*$ and since $Vq \wedge \{G : G \in \lambda^* - V\}$, then $\bigwedge\lambda^* \neq 0_X$. Hence λ^* is a fuzzy filter base in X. Since X is fuzzy S_β-compact, then $\bigwedge_{G\in\xi^*} S_\beta clG \neq 0_X$, so that $\left(\bigwedge_{G\in\xi} S_\beta clG\right) \wedge V = \left(\bigwedge_{G\in\xi} S_\beta clG\right) \wedge S_\beta ClV \neq 0_X$. Hence by Theorem 3.8, we have V is fuzzy S_β-compact relative to X.

Theorem 3.10. If a function $f : X \to Y$ is fuzzy MS_β-continuous and V is fuzzy S_β-compact relative to X, then so is $f(V)$.

Proof. Let $\{A_j : j \in J\}$ be a fuzzy S_β-open set cover of $S(f(V))$. For $x \in S(V), f(x) \in f(S(V)) = S(f(V))$. Since f is fuzzy MS_β-continuous, then $\{f^{-1}(A_j) : j \in J\}$ is fuzzy S_β-open cover of $S(V)$. Since V is fuzzy S_β-compact relative to X, there is a finite subfamily $\{f^{-1}(A_j) : j = 1, \ldots, n\}$ such that $S(V) \leq \bigvee_{j=1}^{n} f^{-1}(A_j)$ which implies $S(V) \leq f^{-1}\left(\bigvee_{j=1}^{n} A_j\right)$ and then $S(f(V)) = f(S(V)) \leq ff^{-1}\left(\bigvee_{j=1}^{n} A_j\right) \leq \bigvee_{j=1}^{n} A_j$. Therefore, $f(V)$ is fuzzy S_β-compact relative to Y.

Lemma 3.11. If $f : X \to Y$ is fuzzy open and fuzzy continuous function, then f is fuzzy MS_β - continuous.

Proof. Let v be a fuzzy S_β-open set in Y, then $v \leq clintclv$. So $f^{-1}(v) \leq f^{-1}\,(\text{clintclv}) \leq cl\left(f^{-1}\,(intclv)\right)$. Since f is fuzzy continuous, then $f^{-1}(intclv) = int\left(f^{-1}(clv)\right)$. Also by Theorem 2.10, $f^{-1}(intclv) = int\left(f^{-1}(intclv)\right) \leq int\left(f^{-1}(clv)\right) \leq intcl\left(f^{-1}(v)\right)$. Thus $f^{-1}(v) \leq cl\left(f^{-1}(\text{ intclv })\right) \leq cl\ intcl\left(f^{-1}(v)\right)$. Hence the result. □

Corollary 3.12. Let $f : X \to Y$ be fuzzy open and fuzzy continuous function and X is fuzzy S_β - compact, then $f(X)$ is fuzzy S_β-compact.

Proof. It is followed directly from Lemma 3.11 and Theorem 3.10.

Definition 3.13. A function $f : X \to Y$ is said to be fuzzy MS_β-open if and only if the image of every fuzzy S_β-open set in X is fuzzy S_β-open set in Y.

Theorem 3.14. Let $f : X \to Y$ be a fuzzy MS_β-open bijective function and Y is fuzzy S_β - compact, then X is fuzzy S_β-compact.

Proof. Let $\{A_j; j \in J\}$ be a collection of fuzzy S_β-open cover of X. Then $\{f\,(A_j) : j \in J\}$ is a collection of fuzzy S_β-open set covering of Y. Since Y is fuzzy S_β-compact, there is a finite subset $F \subseteq J$ such that $\{f\,(A_j) : j \in F\}$ is a covering of Y. But $1_X = f^{-1}\,(1_Y) = f^{-1}f\left(\bigvee_{j\in F}(A_j) = \bigvee_{j\in F}(A_j)\right.$ and therefore X is fuzzy S_β-compact. □

4. Fuzzy S_β-closed spaces

In this section, we would introduce and study fuzzy S_β-closed spaces.

Definition 4.1. A fuzzy set U in a fuzzy topological space X is said to be a fuzzy $S_\beta q$-nbd of a fuzzy point x_t in X if there exists a fuzzy S_β-open set $A \leq U$ such that $x_t q A$.

Theorem 4.2. Let x_t be a fuzzy point in a fuzzy topological space X and U be any fuzzy set in X. Then $x_t \in S_\beta clU$ if and only if for every $S_\beta q$-nbd H of x_t, HqU.

Proof. Let $x_t \in S_\beta clU$ and there exists a $S_\beta q$-nbd H of x_t, H is not quasi-coincident with U. Then there exists a fuzzy S_β-open set $A \leq H$ in X such that $x_t q A$, which implies that A is not quasi-coincident with U and $U \leq A^c$. Since A^c is fuzzy S_β-closed set, then $S_\beta clU \leq A^c$. Since $x_t \notin A^c$, then $x_t \in S_\beta clU$, which is a contradiction.

Conversely, let $x_t \notin S_\beta clU = \wedge\{A : A$ is fuzzy S_β-closed in $X, A \geq U\}$. Then there exists a fuzzy S_β-closed set $A \geq U$ such that $x_t \notin A$. Hence $x_t q A^c = H$ where H is fuzzy S_β-open set in X and H is not quasi-coincident with U. Then there exists a fuzzy $S_\beta q - nbdH$ of x_t which is not quasi-coincident with U. Hence the result. □

Definition 4.3. A fuzzy topological space X is said to be fuzzy S_β-closed if and only if for every family μ of fuzzy S_β-open sets such that $\bigvee_{A\in\mu} A = 1_X$, there is a finite subfamily $\eta \subseteq \mu$ such that $\left(\bigvee_{A\in\eta} S_\beta clA\right)(x) = 1_X$, for every $x \in X$.

Remark 4.4. From the above definition and other types of fuzzy compactness, one can draw the following diagram :

$$FS_\beta\text{-compact } \to F\beta\text{-compact } \to F\beta\text{-closed}$$

$$FS_\beta\text{-compact } \to FS_\beta\text{-closed } \to F\beta\text{-closed}$$

where $F =$ fuzzy.

Theorem 4.5. A fuzzy topological space X is fuzzy S_β-closed if and only if for every fuzzy S_β- open filter base ξ in $X, \bigwedge_{G\in\xi} S_\beta clG \neq 0_X$.

Proof. Let μ be a fuzzy S_β-open set cover of X and let for every finite subfamily η of μ, $\left(\bigvee_{A\in\eta} S_\beta clA\right)(x) < 1_X$ for some $x \in X$. Then $\left(\bigwedge_{A\in\eta} S_\beta clA^c\right)(x) > 0_X$ for some $x \in X$. Thus $\{(S_\beta clA)^c : A \in \mu\} = \xi$ forms a fuzzy S_β-open filter base in X. Since μ is a fuzzy S_β-open cover of X, then $\bigwedge_{A\in\mu} A^c = 0_X$, which is a contradiction. Then every fuzzy S_β-open μ cover of X has a finite subfamily η such that $\left(\bigvee_{A\in\eta} S_\beta clA\right)(x) = 1_X$ for every $x \in X$. Hence X is fuzzy S_β-closed.

Conversely, suppose there exists a fuzzy S_β-filter base ξ in X such that $\bigwedge_{G\in\xi} S_\beta clG = 0_X$, so that $\left(\bigvee_{G\in\xi} (S_\beta clG)^c\right)(x) = 1_X$ for every $x \in X$ and hence $\mu = \{(S_\beta clG)^c : G \in \xi\}$ is a fuzzy S_β-open cover of X. Since X is fuzzy S_β-closed, then μ has a finite subfamily η such that $\left(\bigvee_{G\in\eta} S_\beta cl\,(S_\beta clG)^c\right)(x) = 1_X$ for every $x \in X$ and hence $\bigwedge_{G\in\eta} \left(S_\beta cl\,(S_\beta clG)^c\right)^c = 0_X$. Thus $\bigwedge_{G\in\eta} G = 0_X$ which is a contradiction, since all the G are members of filter base.

Definition 4.6. A fuzzy set u in a fuzzy topological space X is said to be fuzzy S_β-closed relative to X if and only if for every family μ of fuzzy

S_β-open sets such that $\bigvee_{A\in\mu} A = u$, there is a finite subfamily $\eta \subseteq \mu$ such that $\left(\bigvee_{A\in\eta} S_\beta clA\right)(x) \geq u(x)$ for every $x \in S(u)$.

Theorem 4.7. A fuzzy subset u in a fuzzy topological space X is fuzzy S_β-closed relative to X if and only if every fuzzy S_β-open filter base ξ in X, $\left(\bigwedge_{G\in\xi} S_\beta clG\right) \wedge u = 0_X$, there exists a finite subfamily $\lambda of \xi$ such that $(\bigwedge_{G\in\lambda} G)$ is not quasi-coincident, with u.

Proof. Let u be a fuzzy S_β-closed relative to X. Suppose ξ is a fuzzy S_β-open filter base in X such that for every finite subfamily λ of ξ, $(\bigwedge_{G\in\lambda} G)\, qu$, but $\left(\bigwedge_{G\in\xi} S_\beta \text{ clG}\right) \wedge u = 0_X$. Then for every $x \in S(u)$, $(\wedge_{G\in\xi} S_\beta clG)(x) = 0_X$ and hence $(\vee_{G\in\xi} (S_\beta clG)^c)(x) = 1_X$ for every $x \in S(u)$. Then $\mu = \{(S_\beta clG)^c : G \in \xi\}$ is a fuzzy S_β-open set cover of u and hence there exists a finite subfamily $\lambda \subseteq \xi$ such that $\bigvee_{G\in\lambda} S_\beta cl\,(S_\beta clG)^c \geq u$, so that $\bigwedge_{G\in\lambda} (S_\beta cl\,(S_\beta clG)^c)^c = \bigwedge_{G\in\lambda} S_\beta int\,(S_\beta clG) \leq u^c$ and hence $\bigwedge_{G\in\lambda} G \leq u^c$. Then $\bigwedge_{G\in\lambda} G$ is not quasi - coincident with u which is a contradiction.

Conversely, let u not be a fuzzy S_β-closed set relative to X. Then there exists a fuzzy S_β-open set μ cover of u such that every finite subfamily $\eta \subseteq \mu$, $\left(\bigvee_{A\in\eta} S_\beta clA\right)(x) \leq u(x)$ for some $x \in S(u)$ and hence $\left(\bigwedge_{A\in\eta} (S_\beta clA)^c\right)(x) > u^c(x) \geq 0_X$ for some $x \in S(u)$. Thus $\xi = \{(S_\beta clA)^c : A \in \mu\}$ forms a fuzzy S_β-open filter base in X. Let there exists a finite subfamily $\{(S_\beta clA)^c : A \in \eta\}$ such that $\left(\bigwedge_{A\in\eta} (S_\beta clA)^c\right)$ is not quasi-coincident with u. Then $u \leq \bigvee_{A\in\eta} S_\beta clA$. So there exists a finite subfamily $\eta \subseteq \mu$ such that $\bigvee_{A\in\eta} S_\beta clA \geq u$ which is a contradiction. Then for each finite subfamily $\lambda = \{(S_\beta clA)^c : A \in \eta\}$ of ξ, we have $\left(\bigwedge_{A\in\eta} (S_\beta clA)^c\right) qu$. Hence by the given condition $\left(\bigwedge_{A\in\mu} S_\beta cl\,(S_\beta clA)^c\right) \wedge u \neq 0_X$. So there exists $x \in S(x)$ such that $\left(\bigwedge_{A\in\mu} S_\beta cl\,(S_\beta clA)^c\right)(x) > 0_X$. Then,

$$\left(\bigvee_{A\in\mu} (S_\beta cl\,(S_\beta clA)^c)\left(\bigvee_{A\in\mu} (S_\beta cl\,(S_\beta clA)^c)^c(x) = \left(\bigvee_{A\in\mu} S_\beta int\,(S_\beta clA)\right)(x) < 1_X\right.\right.$$

and hence $\left(\bigvee_{A\in\mu} A\right)(x) < 1_X$ which contradicts the fact that μ is a fuzzy S_β-open cover of u. Therefore u is a fuzzy S_β-closed set relative to X. □

Definition 4.8. A fuzzy set u of X is said to be fuzzy S_β-regular if it is both fuzzy S_β-open and fuzzy S_β-closed.

Proposition 4.9. If u is fuzzy S_β-open in X, then $S_\beta\,$clu is fuzzy S_β-regular.

Proof. Since $S_\beta\,$clu is fuzzy S_β-closed, we must show that S_β clu is fuzzy S_β-open. Since u is fuzzy S_β-open in X, v $\leq u \leq clv$ holds for some fuzzy preopen set v in X. Therefore, we have $v \leq S_\beta\, cl$v $\leq S_\beta\,$clu$\leq clv$ and hence S_β clu is fuzzy S_β-open. □

Theorem 4.10. For a fuzzy topological space X, the following are equivalent:

1. X is fuzzy S_β-closed space.
2. Every cover of X by fuzzy S_β-regular sets has a finite subcover.
3. For every collection $\{A_j : j \in J\}$ of fuzzy S_β-regular sets such that $\bigwedge_{j\in J} A_j = 0_X$, there exists a finite subset $F \subseteq J$ such that $\bigwedge_{j\in F} A_j = 0_X$.

Proof. It is obvious from Proposition 4.9 and from the fact that for every collection $\{A_j : j \in J\}$, $\left(\bigvee_{j\in J} A_j\right)^c = \bigwedge_{j\in J} A_j^c$, $\left(\bigwedge_{j\in J} A_j\right)^c = \bigvee_{j\in J} A_j^c$ and A is fuzzy S_β-open set if and only if A^c is fuzzy S_β-closed. □

Theorem 4.11. Let $f : X \to Y$ be a fuzzy S_β-continuous surjection function. If X is fuzzy S_β-closed space, then Y is almost fuzzy compact.

Proof. Let $\{A_j ; j \in J\}$ be a collection of fuzzy open sets cover of Y. Then $\{f^{-1}(A_j) : j \in J\}$ is a fuzzy S_β-open cover of X. By hypothesis, there exists a finite subset $F \subseteq J$ such that $\bigvee_{j\in F} S_\beta cl f^{-1}(A_j) = 1_X$. From the surjective of f and by Lemma 3.3, $1_Y = f(1_X) = f\left(\bigvee_{j\in F} S_\beta cl f^{-1}(A_j)\right) \leq \bigvee_{j\in F} cl f\left(f^{-1}(A_j)\right) = \bigvee_{j\in F} cl A_j$. Hence Y is almost fuzzy compact.

Theorem 4.12. If $f : X \to Y$ is fuzzy MS_β-continuous surjection function and X is fuzzy S_β-closed, then so is Y.

Proof. Using Lemma 3.2, the theorem can be proved similarly to Theorem 4.11. □

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