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Note on modified generalized Bessel function

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Abstract

An attempt is made to define Modified Generalized Bessel Function, and Modified Generalized Bessel Matrix Function. Some properties have also been discussed.

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1. Introduction

The Bessel functions, also well-known as the circular cylinder function, is the most frequently used special function in the field of Mathematical Physics. No other special functions have received such detailed treatment in willingly available treaties [11] as have the Bessel functions. In fact a Bessel function is generally defined as a particular solution of a linear differential equation of the second order known as Bessel's equation [11].

We are motivated by the works of Jódar et al., who established and deliberate the Bessel Matrix function of first kind and Hypergeometric Matrix functions in [4, 5, 6, 7]. Shehata et al. [10] defined and studied the Extension of Bessel matrix functions. In sequel to the study, we introduce Generalized Bessel matrix function.

The Bessel function of first kind $J_\nu(z)$ [8, P. 109] is represented as

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(1+\nu+k)k!} \left(\frac{z}{2}\right)^{2k+\nu}, \tag{1.1}$$

where $|z| < \infty$, $|\arg z| < \pi$.

Galue [2, P. 395] generalized Bessel Function as

$${}_hJ_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(1+\nu+hk)k!} \left(\frac{z}{2}\right)^{2k+\nu}, \tag{1.2}$$

where $h > 0$, $|z| < \infty$, $|\arg z| < \pi$.

Some important facts are listed below and also useful in our study.

The well known Generalized hypergeometric function [3, P. 360] is defined by

$${}_pF_q\left[\, \alpha_1, \alpha_2, ..., \alpha_p \beta_1, \beta_2, ..., \beta_q \middle|\, z\right] = 1 + \sum_{k=1}^{\infty} \frac{(\alpha_1)_k (\alpha_2)_k ... (\alpha_p)_k}{(\beta_1)_k (\beta_2)_k ... (\beta_q)_k} \frac{z^k}{k!}, \quad |z| < 1, \tag{1.3}$$

where p and q are nonnegative integers and no β_j $(j = 1, 2, \ldots, q)$ is zero or a negative integer. Here, $(\alpha)_k$ is a Pochhammer symbol [3, P. 360].

Wright function [3, P. 361] is defined and denoted as,

$$W\left(z,\alpha,\beta\right)=\sum_{k=0}^{\infty}\frac{z^{k}}{k!\Gamma\left(\alpha k+\beta\right)}, \tag{1.4}$$

where $z\in C$, $\beta\in C$, $\alpha>-1$.

Jódar and Cortés investigated the Hypergeometric Matrix Function [5, P. 210], which is defined as

$$F\left(A,B;C;z\right)=\sum_{n=0}^{\infty}\frac{(A)_{n}(B)_{n}[(C)_{n}]^{-1}}{n!}z^{n},\ |z|<1, \tag{1.5}$$

where A,B,C be matrices in $C^{N\times N}$, $C+nI$ is invertable for all integers $n\geq 0$.

We also use stirling's formula [10] which is defined below,

$$k!\approx\sqrt{2\pi k}\left(\frac{k}{e}\right)^{k} \tag{1.6}$$

2. Main Results

3. Modified Generalized Bessel Function

In this section, we define the Modified Generalized Bessel Function.

Definition 1. *The Modified Generalized Bessel Function as*

$${}_{h}I_{v}\left(z\right)=i^{-v}{}_{h}J_{v}\left(iz\right) \tag{3.1}$$

From (1.2) and (3.1), we obtain,

$${}_{h}I_{v}\left(z\right)=\sum_{k=0}^{\infty}\frac{1}{\Gamma\left(1+v+hk\right)k!}\left(\frac{z}{2}\right)^{2k+v}, \tag{3.2}$$

which can be easily represented in the term of Wright function (1.4) as,

$${}_{h}I_{v}\left(z\right)=\left(\frac{z}{2}\right)^{2k+v}W\left(\frac{z^{2}}{4},h,v+1\right), \tag{3.3}$$

where $\Re\left(v\right)>-1$, $h>0$, $v\in C$.

Theorem 2. *If ${}_hI_v(z)$ is defined as (3.2) and for $\Re(v) > -1, h > 0, v \in C$, then*

$$D^n\left[z^{-v}{}_hI_v(z)\right] = \frac{z^{-n}}{\Gamma(1+\nu)\Gamma(1-n)2^\nu} \tag{3.4}$$
$$\times {}_1F_{h+2}\left[\begin{array}{c} \frac{1}{2} \\ \frac{\nu+1}{h}, \frac{\nu+2}{h}, \frac{\nu+3}{h}, ..., \frac{\nu+h}{h}, \frac{1-n}{2}, \frac{1-n+1}{2} \end{array} \middle| z^2\right]$$

Proof. From (3.2),

$$D^n\left[z^{-v}{}_hI_v(z)\right] = D^n\left[\sum_{k=0}^{\infty}\frac{z^{2k+v-v}}{2^{2k+v}\Gamma(1+v+hk)\ k!}\right].$$

On differentiating term by term, we get

$$D^n\left[z^{-v}{}_hI_v(z)\right] = \sum_{k=0}^{\infty}\frac{z^{2k-n}(2k)!}{\Gamma(1+v+hk)\ 2^{2k+v}k!\,(2k-n)!}.$$

On using Legendre duplication formula [8, P. 23, section 19], we get

$$D^n\left[z^{-v}{}_hI_v(z)\right] = \sum_{k=0}^{\infty}\frac{2^{2k}\left(\frac{1}{2}\right)_k z^{2k-n}}{\Gamma(1+v+hk)\ 2^{2k+v}\Gamma(2k-n+1)}.$$

Or

$$D^n\left[z^{-v}{}_hI_v(z)\right] = \frac{z^{-n}}{\Gamma(1+\nu)\Gamma(1-n)2^\nu}\sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}\right)_k z^{2k}}{(1+v)_{hk}(1-n)_{2k}},$$

which can be expressed as generalized hypergeometric function (1.3), this leads to assertion (3.4). □

Next, we study Generalized Bessel Matrix Function ${}_HJ_A(z)$.

4. Modified Generalized Bessel Matrix Function

Definition 1. *The Generalized Bessel Matrix function denoted by ${}_HJ_A(z)$, and define as*

$$ {}_HJ_A(z) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\Gamma^{-1}(A+Hk+I)\left(\frac{z}{2}\right)^{2kI+A}, \tag{4.1}$$

which can be written as,

$$(4.2)\qquad {}_HJ_A(z) = \left(\frac{z}{2}\right)^A \Gamma^{-1}(A+I)\,{}_0F_1^H\left(-;A+I;\frac{-z^2}{4}\right).$$

Since

$$(4.3)\qquad {}_0F_1^H\left(-;A+I;\frac{-z^2}{4}\right) = \Gamma(A+I)\sum_{k=0}^{\infty}\frac{\Gamma^{-1}(A+Hk+I)}{k!}\left(\frac{-z^2}{4}\right)^k,$$

where H and A are matrices in $C^{N\times N}$ satisfy the conditions $\Re(h)>0$ for all eigenvalues $h\in\sigma(H)$, and $\Re(a)>-1$ for all eigenvalues $a\in\sigma(A)$ respectively. Also $(Hk+A+I)$ is matrix in $C^{N\times N}$ such that $(Hk+A+I)$ is an invertable matrix for every integer $k\geq 0$.

Theorem 2. *Let H and A be matrices in $C^{N\times N}$ satisfy the conditions $\Re(h)>0$ for all eigenvalues $h\in\sigma(H)$, and $\Re(a)>-1$ for all eigenvalues $a\in\sigma(A)$ respectively. Also $(Hk+A+I)$ is matrix in $C^{N\times N}$ such that $(Hk+A+I)$ is an invertable matrix for every integer $k\geq 0$. Then the Generalized Bessel Matrix Function is an entire function.*

Proof. From (4.1), we have

$$(4.4)\qquad {}_HJ_A(z) = \; = \sum_{k=0}^{\infty} U_K\left(\frac{z}{2}\right)^{2kI+A},$$

where $U_K = \frac{(-1)^k}{k!}\Gamma^{-1}(Hk+A+I)$

The radius of regularity R [1, P. 10 lemma(2.1)] of the function ${}_HJ_A(z)$ is given as,

$$\frac{1}{R} = \lim_{k\to\infty}\sup\left\|(U_K)^{\frac{1}{k}}\right\|$$

$$(4.5)\qquad \frac{1}{R} = \lim_{k\to\infty}\sup\left\|\left(\frac{(-1)^k}{k!}\frac{\Gamma^{-1}(Hk+A+I)}{2^{2kI+A}}\right)^{\frac{1}{k}}\right\|.$$

On using stirling's formula (1.6), we obtain

$$\frac{1}{R}=\lim_{k\to\infty}\sup\left\|\left(\frac{(-1)^{k}}{\sqrt{2\pi k}\left(\frac{k}{e}\right)^{k}\sqrt{2\pi\left(Hk+A+2I\right)}\left(\frac{(Hk+A+2I)}{e}\right)^{(Hk+A+2I)}2^{2kI+A}}\right)^{\frac{1}{k}}\right\|,$$

and further simplification yields,

$$\frac{1}{R}=\lim_{k\to\infty}\sup\left\|\left(\frac{(-1)^{\frac{k}{k}}}{(2\pi k)^{\frac{1}{2k}}\left(\frac{k}{e}\right)\left(2\pi\left(Hk+A+2I\right)\right)^{\frac{1}{2k}}\left(\frac{(Hk+A+2I)}{e}\right)^{\frac{(Hk+A+2I)}{k}}2^{\frac{2kI+A}{k}}}\right)\right\|=0,$$

Therefore,

$$\frac{1}{R}=0. \tag{4.6}$$

Thus, the Generalized Bessel Matrix Function is an entire function. □

Definition 3. *If A and H are matrix in $C^{N\times N}$ satisfy the conditions $\Re(a)>-1$ for all eigenvalues $a\in\sigma(A)$, and $\Re(h)>0$ for all eigenvalues $h\in\sigma(H)$ respectively. Also $A+Hk+I$ is a matrix in $C^{N\times N}$ such that $A+Hk+I$ is an invertable matrix for every integer $k\geq 0$, then the Modified Generalized Bessel Matrix Function is given by*

$${}_{H}I_{A}\left(z\right)=i^{-A}{}_{H}J_{A}\left(iz\right) \tag{4.7}$$

From (4.1), we have

$${}_{H}I_{A}\left(z\right)=\sum_{k=0}^{\infty}\frac{\Gamma^{-1}\left(A+Hk+I\right)}{k!}\left(\frac{z}{2}\right)^{2kI+A}, \tag{4.8}$$

this can be written as,

$${}_{H}I_{A}\left(z\right)=\Gamma\left(A+I\right){}_{0}F_{1}^{H}\left(-;\left(A+I\right);\frac{z^{2}}{4}\right). \tag{4.9}$$

Theorem 4. *If ${}_{H}I_{A}\left(z\right)$ is defined as (4.8), then for $|z|<\infty$, $|\arg z|<\pi$, $\nu\in C$, $h>0$,*

$$\frac{d}{dz}\left[z^{-A}{}_{H}I_{A}\left(z\right)\right]=2^{H-I}z^{I-A-H}{}_{H}I_{A+H}\left(z\right) \tag{4.10}$$

Proof. From (4.8), we have

$$(4.11)\frac{d}{dz}\left[z^{-A}{}_{H}I_{A}\left(z\right)\right]=\frac{d}{dz}\left[z^{-A}\sum_{k=0}^{\infty}\frac{\Gamma^{-1}\left(A+Hk+I\right)}{k!}\left(\frac{z}{2}\right)^{2kI+A}\right].$$

On differentiating term by term on the right hand side of the above equation,

$$(4.12)\qquad \frac{d}{dz}\left[z^{-A}{}_{H}I_{A}\left(z\right)\right]=\sum_{k=1}^{\infty}\frac{\Gamma^{-1}\left(A+Hk+I\right)}{2^{(2k-1)I+A}\left(k-1\right)!}z^{2k-1},$$

afterwards setting $k=r+1$ on equation (4.12), we find that

$$(4.13)\qquad \frac{d}{dz}\left[z^{-A}{}_{H}I_{A}\left(z\right)\right]=\sum_{r=0}^{\infty}\frac{\Gamma^{-1}\left(A+H\left(r+1\right)+I\right)}{2^{(2r+1)I+A}r!}z^{2r+1}.$$

After further simplification,

$$(4.14)\frac{d}{dz}\left[z^{-A}{}_{H}I_{A}\left(z\right)\right]=\frac{z^{I-A-H}}{2^{-H+I}}\sum_{r=0}^{\infty}\frac{\Gamma^{-1}\left(\left(A+H\right)+Hr+I\right)}{2^{2rI+(A+H)}r!}z^{2rI+(A+H)}.$$

In view of definition (4.8), we get assertion (4.10). □

Significance Statement

Bessel functions play a vital role in the theory of Special Functions, Fractional Calculus, Mathematical Sciences including Mathematical Physics, Control Theory, etc. In our opinion, this work can also play pivotal role in the theory of Special Matrix Functions, Statistics, Probability Theory, Engineering Sciences etc.

References

[1] R. S. Batahan and M. S. Metwally, “Differential and Integral Operators on Appell’s Matrix Function”, *Alandalus Journal of Social and Applied Sciences,* vol. 2008, no. 2, pp. 7-25, 2023.

[2] L. Galue, “A Generalized Bessel Function”, *Integral Transforms and Special Functions,* vol. 14, no. 5, pp. 395-401, 2003.

[3] R. Gorenflo, A. A. Kilbas, F. Mainardi and S. V. Rogosin, *Mittage-Leffler Functions, Related Topics and Applications*. Springer Monographs in Mathematics. New York: Springer, 2004.

[4] L. Jódar, R. Company and E. Navarro, "Bessel matrix functions, explicit solution of coupled Bessel type equations", *Utilitas Mathematica,* vol. 46, pp. 129-141, 1994.

[5] L. Jódar and J. C. Cortés, On the Hypergeometric Matrix Function", *Journal of Computational and Applied Mathematics,* vol 99, nos. 1-2, pp. 205-217, 1998. doi: 10.1016/S0377-0427(98)00158-7

[6] L. Jódar and J. C. Cortés, "Some properties of Gamma and Beta Matrix Functions", *Applied Mathematics Letters,* vol. 11, pp. 89-93, 1998. doi: 10.1016/S0893-9659(97)00139-0

[7] L. Jódar and J. C. Cortés, "Closed form general solution of the hypergeometric matrix differential equation", *Mathematical and Computer Modelling,* vol. 32, pp. 1017-1028, 2000. doi: 10.1016/S0895-7177(00)00187-4

[8] E. D. Rainville, *Special functions*. New York: The Macmillan Company, 1960.

[9] J. Sastre and L. Jódar, "Asymptotics of the modified Bessel and incomplete gamma matrix functions", *Applied Mathematics Letters,* vol. 16, no. 6, pp. 815-820, 2003. doi: 10.1016/S0893-9659(03)90001-2

[10] A. Shehata and S. Khan, "On Bessel-Maitland Matrix function", *Mathematica,* vol. 80, pp. 90-103, 2015.

[11] G. N. Watson, *A treatise on the Theory of Bessel Functions,* Cambridge University Press, London, 1958.

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