Proyecciones Journal of Mathematics
Vol. 43, N^o 1, pp. 275-292, February 2024.
Universidad Católica del Norte
Antofagasta - Chile

10.22199/issn.0717-6279-5805

Monophonic-triangular distance in graphs

P. Titus
Anna University, India
J. Ajitha Fancy
Scott Christian College (Autonomous), India
and
A. P. Santhakumaran
Hindustan Institute of Technology and Science, India
Received : December 2022. Accepted : May 2023

Abstract

A path $u_1, u_2, ..., u_n$ in a connected graph G such that for i, j with $j \geq i + 3$, there does not exist an edge $u_i u_j$, is called a monophonic-triangular path or mt-path. The monophonic-triangular distance or mt-distance $d_{mt}(u, v)$ from u to v is defined as the length of a longest $u - v$ mt-path in G. The mt-eccentricity $e_{mt}(v)$ of a vertex v in G is defined as the maximum mt-distance between v and other vertices in G. The mt-radius $rad_{mt}(G)$ is defined as the minimum mt-eccentricity among the vertices of G and the mt-diameter $diam_{mt}(G)$ is defined as the maximum mt-eccentricity among the vertices of G. It is shown that $rad_{mt}(G) \leq diam_{mt}(G)$ for every connected graph G. Some realization and characterization results are given based on mt-radius, mt-diameter, mt-center and mt-periphery of a connected graph.

Key Words: *Distance, detour distance, monophonic distance, mt-distance, mt-radius, mt-diameter, mt-center.*

AMS Subject Classification: *05C12.*

1. Introduction

In this paper, a non-trivial simple finite undirected connected graph G with vertex set V and edge set E is considered. Let p and q be the *order* and *size* of G, respectively. We refer [1, 3] for basic definitions and results. The *distance* between two vertices u and v in G is defined as the minimum length of a $u - v$ path in G and it is denoted by $d(u, v)$. It is known that the distance d is a metric on V.

The *eccentricity* $e(v)$ of a vertex v in a connected graph G is defined as the distance between v and a vertex farthest from v in G. The *radius* $rad(G)$ is defined as the minimum eccentricity among the vertices of G and the *diameter* $diam(G)$ is defined as the maximum eccentricity among the vertices of G. If $e(v) = rad(G)$, then v is a *central vertex* and if $e(v) = diam(G)$, then v is a *peripheral vertex*. The *center* $C(G)$ of G is the subgraph induced by the central vertices of G and *periphery* $P(G)$ of G is the subgraph induced by the peripheral vertices of G.

In 2005, Chartrand et al. [2] introduded a new distance viz. detour distance based on a longest path joining any two vertices in a connected graph. The *detour distance* between two vertices u and v in G is defined as the maximum length of a $u - v$ path in G and it is denoted by $D(u, v)$. A longest $u - v$ path is called a $u - v$ detour. It is also known that the detour distance D is a metric on V. The *detour eccentricity* $e_D(v)$ of a vertex v in a connected graph G is defined as the maximum detour distance between u and other vertices in G. The *detour radius* $rad_D(G)$ is defined as the minimum detour eccentricity among the vertices of G and the *detour diameter* $diam_D(G)$ is defined as the maximum detour eccentricity among the vertices of G. If $e_D(v) = rad_D(G)$, then v is a *detour central vertex* and if $e_D(v) = diam_D(G)$, then v is a *detour peripheral vertex*. The *detour center* $C_D(G)$ of G is defined as the subgraph induced by the detour central vertices of G and *detour periphery* $P_D(G)$ of G is defined as the subgraph induced by the detour peripheral vertices of G.

A *chord* of a path P is an edge joining two non-adjacent vertices of P. A chordless path is called a *monophonic path*. In 2012, Santhakumaran et al. [5] introduced a new distance based on a longest monophonic path joining any two vertices in a connected graph and further investigated in [6]. The *monophonic distance* between any two vertices u and v in G is defined as the maximum length of a $u - v$ monophonic path in G and it is denoted by $d_m(u, v)$. The usual distance and the detour distance are metrics on the vertex set of a connected graph, whereas the monophonic distance is not a metric on the vertex set of a connected graph. The *monophonic eccentricity* $e_m(v)$ of a vertex v in a connected graph G is defined as the maximum monophonic distance between v and other vertices in G. The *monophonic radius* $rad_m(G)$ is defined as the minimum monophonic eccentricity among the vertices of G and the *monophonic diameter* $diam_m(G)$ is defined as the maximum monophonic eccentricity among the vertices of G. If $e_m(v) = rad_m(G)$, then v is a *monophonic central vertex* and if $e_m(v) = diam_m(G)$, then v is a *monophonic peripheral vertex*. The *monophonic center* $C_m(G)$ of G is defined as the subgraph induced by the monophonic central vertices of G and *monophonic periphery* $P_m(G)$ of G is defined as the subgraph induced by the monophonic peripheral vertices of G.

The concept of distance (usual distance) in graphs is a major component in graph theory with its centrality and convexity concepts having numerous applications to real life problems. There are several interesting applications of these concepts to facility location in real life situations. The paths introduced here are monophonic-triangular so that intervention by hackers or rioters is not possible to the respective facilities provided. In fact, the two major applications provided by this path with security and protection are service facility and emergency facility in real life situations of a large city network. Further, as monophonic-triangular paths are secured and longer than geodesics, it is advantageous to more customers in providing protected service of facility locations.

In this article, we introduce the monophonic-triangular distance in a connected graph and based on this new distance, two new graph invariants known as mt-radius and mt-diameter of a graph are introduced and investigated. Also, the relation between mt-radius and mt-diameter with (detour or monophonic) radius and (detour or monophonic) diameter are given. Moreover, realization theorems for these graph parameters are presented. Throughout this article, G denotes a non-trivial simple connected graph.

2. Monophonic-triangular Distance

Definition 2.1. *A path $u_1, u_2, ..., u_n$ in G such that for i, j with $j \geq i + 3$, there does not exist an edge u_iu_j, is called a monophonic-triangular path* *or mt-path. The monophonic-triangular distance or mt-distance* $d_{mt}(u, v)$ *from* u *to* v *is defined as the length of a longest* $u - v$ *mt-path in* G.

Example 2.2. *Consider the graph* G *given in Figure 2.1. For the vertices* v_1 *and* v_7, $P_1 : v_1, v_5, v_7$ *is a* $v_1 - v_7$ *geodesic,* $P_2 : v_1, v_5, v_2, v_3, v_4, v_8, v_7$ *is a* $v_1 - v_7$ *detour,* $P_3 : v_1, v_2, v_8, v_7$ *is a longest* $v_1 - v_7$ *monophonic path,* $P_4 : v_1, v_2, v_3, v_8, v_7$ *is a longest* $v_1 - v_7$ *mt-path and so* $d(v_1, v_7) = 2$, $D(v_1, v_7) = 6$, $d_m(v_1, v_7) = 3$, *and* $d_{mt}(v_1, v_7) = 4$, *respectively. Thus, the mt-distance* d_{mt} *is different from the known distances such as* d, D *and* d_m *in graphs.*

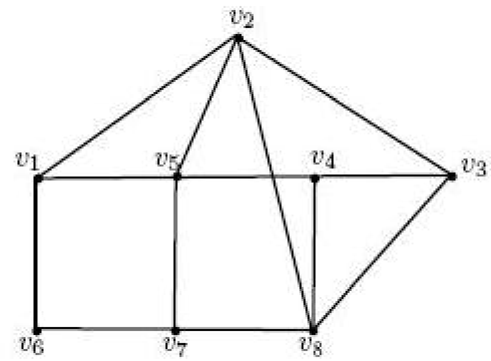

Figure 2.1: G

The usual distance d *and the detour distance* D *are metrics on* V, *and the monophonic distance* d_m *is not a metric on* V. *Now, it is seen that the mt-distance* d_{mt} *is also not a metric on* V. *For the cycle* $C_5 : v_1, v_2, v_3, v_4, v_5, v_1$, $d_{mt}(v_1, v_2) = 1$, $d_{mt}(v_2, v_3) = 1$ *and* $d_{mt}(v_1, v_3) = 3$. *Hence* $d_{mt}(v_1, v_3) > d_{mt}(v_1, v_2) + d_{mt}(v_2, v_3)$ *and so the triangle inequality is not satisfied for the mt-distance.*

Note A mt-path P is either a monophonic path or the subgraph induced by P contains only triangles. Hence a monophonic path is obviously a mt-path and the converse need not be true. For the graph G given in Figure 2.1, $P_4 : v_1, v_2, v_3, v_8, v_7$ is a $v_1 - v_7$ mt-path, but P_4 is not a $v_1 - v_7$ monophonic path.

The following result is trivial from the respective definitions.

Remark 2.3. *For any two vertices* x *and* y *in a connected graph* G *of order* p, $0 \leq d(x, y) \leq d_m(x, y) \leq d_{mt}(x, y) \leq D(x, y) \leq p - 1$. *The* *bounds in this chain inequalities are sharp. In any non-trivial connected graph G, if $x = y$, then $d(x,y) = d_m(x,y) = d_{mt}(x,y) = D(x,y) = 0$. In a non-trivial path P on p vertices, if x and y are end vertices of P, then $d(x,y) = d_m(x,y) = d_{mt}(x,y) = D(x,y) = p-1$. Also, all the inequalities in this chain are strict. For the graph G given in Figure 2.1, $d(v_1,v_7) = 2$, $d_m(v_1,v_7) = 3$, $d_{mt}(v_1,v_7) = 4$, $D(v_1,v_7) = 6$, $p = 8$ and so $0 < d(v_1,v_7) < d_m(v_1,v_7) < d_{mt}(v_1,v_7) < D(v_1,v_7) < p-1$.*

Result 2.4. *Let x and y be any two vertices in a connected graph G of order p. Then*
$(i) d_{mt}(x,y) = 0$ if and only if $x = y$.
$(ii) d_{mt}(x,y) = 1$ if and only if xy is either a cut edge or an edge in a smallest cycle of order at least 4.

Result 2.5. *For every pair of distinct vertices x and y in G, $d_{mt}(x,y) = 2$ if and only if $G = K_p$ $(p \geq 3)$. For any two vertices x and y in G, $d(x,y) = d_m(x,y) = d_{mt}(x,y) = D(x,y)$ if and only if G is a tree. It is possible, however, that for a connected graph, which is not a tree, there exists a pair of vertices x and y such that $d(x,y) = d_m(x,y) = d_{mt}(x,y) = D(x,y)$. For example, if x and y are antipodal vertices in an even cycle $C_{2n}(n \geq 2)$, $d(x,y) = d_m(x,y) = d_{mt}(x,y) = D(x,y) = n$.*

Definition 2.6. *Let G be a connected graph. The mt-eccentricity $e_{mt}(v)$ of a vertex v in G is $e_{mt}(v) = max\{d_{mt}(v,x) : x \in V\}$. The $mt-radius$, $rad_{mt}(G)$ of G is $rad_{mt}(G) = min\{e_{mt}(v) : v \in V\}$ and the $mt-diameter$, $diam_{mt}(G)$ of G is $diam_{mt}(G) = max\{e_{mt}(v) : v \in V\}$. A vertex y in G is a $mt-eccentric$ vertex of a vertex x in G if $e_{mt}(x) = d_{mt}(x,y)$.*

Example 2.7. *For the graph G given in Figure 2.1, the eccentricity, monophonic eccentricity, mt-eccentricity, detour eccentricity and the set of all mt-eccentric vertices of every vertex of G is given in Table 2.1.*

Table 2.1.

vertex v	$e(v)$	$e_m(v)$	$e_{mt}(v)$	$e_D(v)$	mt-eccentric vertices
v_1	2	4	5	7	$\{v_3, v_4\}$
v_2	2	3	4	7	$\{v_6\}$
v_3	3	4	5	7	$\{v_1, v_6\}$
v_4	3	5	5	7	$\{v_1, v_6, v_7\}$
v_5	2	3	4	7	$\{v_3, v_8\}$
v_6	3	4	5	7	$\{v_3, v_4, v_8\}$
v_7	2	5	5	7	$\{v_4\}$
v_8	2	4	5	7	$\{v_6\}$

Note Since $d(u,v) = d_m(u,v) = d_{mt}(u,v) = D(u,v)$ for any two vertices u and v in a tree T, it follows that $rad(T) = rad_m(T) = rad_{mt}(T) = rad_D(T)$ and $diam(T) = diam_m(T) = diam_{mt}(T) = diam_D(T)$. Also, since $d_{mt}(u,v) = 2$ for any two distinct vertices of a complete graph K_p, $rad_{mt}(K_p) = diam_{mt}(K_p) = 2$. Also, Table 2.2 shows the mt-radius and the mt-diameter of some standard graphs.

Table 2.2.

Graph G	K_p	C_p $(p \geq 4)$	$W_{1,p-1}$ $(p \geq 5)$	$K_{1,p-1}$ $(p \geq 3)$	$K_{m,n}$ $(m, n \geq 2)$	P_n	Petersen graph
$rad_{mt}(G)$	2	$p-2$	2	1	2	$\lfloor \frac{n}{2} \rfloor$	4
$diam_{mt}(G)$	2	$p-2$	$p-3$	2	2	$n-1$	4

Since $0 \leq d(x,y) \leq d_m(x,y) \leq d_{mt}(x,y) \leq D(x,y) \leq p-1$, the following proposition is trivial.

Proposition 2.8. *Let G be a connected graph. Then*
(i)$e(x) \leq e_m(x) \leq e_{mt}(x) \leq e_D(x)$ for any vertex x in G.
(ii)$rad(G) \leq rad_m(G) \leq rad_{mt}(G) \leq rad_D(G)$.
(iii)$diam(G) \leq diam_m(G) \leq diam_{mt}(G) \leq diam_D(G)$.

Theorem 2.9. *For any two vertices x and y in G, $d_m(x,y) \leq d_{mt}(x,y) \leq 2d_m(x,y)$.*

Proof. Since any monophonic path is a mt-path and $d_{mt}(x,y)$ is the length of a longest mt-path, we have $d_m(x,y) \leq d_{mt}(x,y)$. Now, claim that $d_{mt}(x,y) \leq 2d_m(x,y)$. If not, there is an $x-y$ mt-path, say P, of length $l > 2d_m(x,y)$. Then by the definition of mt-path, the induced subgraph $\langle V(P) \rangle$ of P contains at most $\frac{l}{2}$ triangles. Form a new path Q from P by replacing the common edges of both P and the triangles in $\langle V(P) \rangle$ by the remaining edge of the triangles in $\langle V(P) \rangle$. It is clear that Q is an $x-y$ monophonic path of length at least $l - \frac{l}{2} = \frac{l}{2} > d_m(x,y)$, which is a contradiction. Hence $d_{mt}(x,y) \leq 2d_m(x,y)$. $\square$

Theorem 2.10. *(a) For integers a, b, c and d with $3 \leq a < b < c \leq d$ and $c \leq 2b$, there is a connected graph G such that $rad(G) = a$, $rad_m(G) = b$, $rad_{mt}(G) = c$ and $rad_D(G) = d$.*
(b) For integers a, b, c and d with $3 \leq a < b < c \leq d$ and $c \leq 2b$, there is a connected graph G such that $diam(G) = a$, $diam_m(G) = b$, $diam_{mt}(G) = c$ and $diam_D(G) = d$.

Proof. (a) This part is proved by considering two cases.

Case 1. $b+1 \leq c \leq 2b-a+3$.

Let $R_1 : x_1, x_2, \ldots, x_{a-1}$ and $R_2 : x_1', x_2', \ldots, x_{a-1}'$ be two copies of the path P_{a-1} of order $a-1$, let $R_3 : y_1, y_2, \ldots, y_{b-a+3}$ and $R_4 : y_1', y_2', \ldots, y_{b-a+3}'$ be two copies of the path P_{b-a+3} of order $b-a+3$, and let R_5 be the complete graph of order $d-c+3$ with $V(R_5) = \{z_1, z_2, \ldots, z_{d-c+3}\}$. Let H be the graph obtained from R_1, R_2, R_3, R_4 and R_5 by (i) identifying the vertices z_1 in R_5 and y_1 in R_3; also identifying the vertices z_{d-c+3} in R_5 and y_1' in R_4, (ii) identifying the vertices y_{b-a+3} in R_3 and x_2 in R_1; and identifying the vertices y_{b-a+3}' in R_4 and x_2' in R_2, and (iii) joining each vertex y_i $(2 \leq i \leq b-a+2)$ in R_3 and x_1 in R_1; and joining each vertex y_i' $(2 \leq i \leq b-a+2)$ in R_4 and x_1' in R_2. Let G be the graph obtained from H by adding $2(c-b-1)$ new vertices $u_1, u_2, \ldots, u_{c-b-1}, u_1', u_2', \ldots, u_{c-b-1}'$ and joining each u_i with the vertices x_1, y_i and y_{i+1} $(1 \leq i \leq c-b-1)$ and joining each u_i' with the vertices x_1', y_i' and y_{i+1}' $(1 \leq i \leq c-b-1)$. The graph G is shown in Figure 2.3.

It is clear that
$P_1 : z_2, z_1, x_1, x_2, \ldots, x_{a-1}$; $P_2 : z_2, z_1, y_2, y_3, \ldots, y_{b-a+2}, x_2, x_3, \ldots, x_{a-1}$;

$P_3 : z_2, z_{d-c+3}, z_1, u_1, y_2, u_2, y_3, u_3, y_4, \ldots, y_{c-b-1}, u_{c-b-1}, y_{c-b}, y_{c-b+1}, \ldots, y_{b-a+2}, x_2, x_3, \ldots, x_{a-1}$ and $P_4 : z_2, z_3, z_4, \ldots, z_{d-c+3}, z_1, u_1, y_2, u_2, y_3, u_3, y_4, \ldots, y_{c-b-1}, u_{c-b-1}, y_{c-b}, y_{c-b+1}, \ldots, y_{b-a+2}, x_2, x_3, \ldots, x_{a-1}$
are a $z_2 - x_{a-1}$ geodesic, a longest $z_2 - x_{a-1}$ monophonic, a longest $z_2 - x_{a-1}$ mt-path, and a $z_2 - x_{a-1}$ detour path, respectively. Hence $d(z_2, x_{a-1}) = a$, $d_m(z_2, x_{a-1}) = b$, $d_{mt}(z_2, x_{a-1}) = c$ and $D(z_2, x_{a-1}) = d$. Also, it is easily verified that $d(z_2, t) \leq a$, $d_m(z_2, t) \leq b$, $d_{mt}(z_2, t) \leq c$, $D(z_2, t) \leq d$ for any vertex t in G and so $e(z_2) = a$, $e_m(z_2) = b$, $e_{mt}(z_2) = c$ and $e_D(z_2) = d$. In a similar way we can verify that $e(v) = a$ if $v \in V(R_5)$; $e(v) > a$ if $v \in V(G) - V(R_5)$, $e_m(v) = b$ if $v \in V(R_5)$; $e_m(v) > b$ if $v \in V(G) - V(R_5)$, $e_{mt}(v) = c$ if $v \in V(R_5)$; $e_{mt}(v) > c$ if $v \in V(G) - V(R_5)$, $e_D(v) = d$ if $v \in V(R_5)$; $e_D(v) > d$ if $v \in V(G) - V(R_5)$. It follows that $rad(G) = a$, $rad_m(G) = b$, $rad_{mt}(G) = c$ and $rad_D(G) = d$.

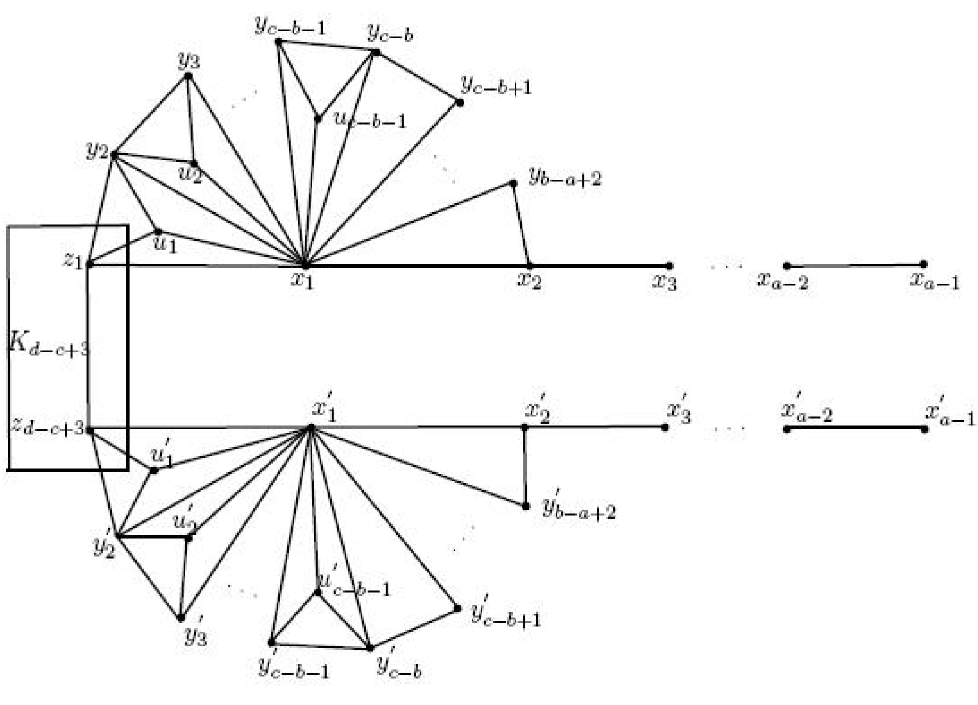

Figure 2.3: G

Case 2. $2b - a + 4 \leq c \leq 2b$.

Let G be the graph obtained from H by adding $2(c-b-1)$ new vertices $u_1, u_2, \ldots, u_{c-b-1}, -u_1^{'}, u_2^{'}, \ldots, u_{c-b-1}^{'}$ and (i) joining each u_i $(1 \leq i \leq b-a+2)$ with the vertices x_1, y_i and y_{i+1} $(1 \leq i \leq b-a+2)$, (ii) joining each u_i $(b-a+3 \leq i \leq c-b-1)$ with the vertices $x_{i-b+a-1}$ and x_{i-b+a} $(b-a+3 \leq i \leq c-b-1)$, (iii) joining each $u_i^{'}$ $(1 \leq i \leq b-a+2)$ with the vertices $x_1^{'}, y_i^{'}$ and $y_{i+1}^{'}$ $(1 \leq i \leq b-a+2)$, (iv) joining each $u_i^{'}$ $(b-a+3 \leq i \leq c-b-1)$ with the vertices $x_{i-b+a-1}^{'}$ and $x_{i-b+a}^{'}$ $(b-a+3 \leq i \leq c-b-1)$. The graph G is shown in Figure 2.4.

It is clear that
$P_1 : z_2, z_1, x_1, x_2, \ldots, x_{a-1}$; $P_2 : z_2, z_1, y_2, y_3, \ldots, y_{b-a+2}, x_2, x_3, \ldots, x_{a-1}$;
$P_3 : z_2, z_{d-c+3}, z_1, u_1, y_2, u_2, y_3, u_3, y_4, \ldots, y_{b-a+2}, u_{b-a+2}, x_2, u_{b-a+3}, x_3,$
$u_{b-a+4}, x_4, \ldots, x_{c-2b+a-2}, u_{c-b-1}, x_{c-2b+a-1}, x_{c-2b+a}, x_{c-2b+a+1}, \ldots, x_{a-1}$, and
$P_4 : z_2, z_3, z_4, \ldots, z_{d-c+3}, z_1, u_1, y_2, u_2, y_3, u_3, y_4, \ldots, y_{b-a+2}, u_{b-a+2}, x_2, u_{b-a+3},$
$x_3, u_{b-a+4}, x_4, \ldots, x_{c-2b+a-2}, u_{c-b-1}, x_{c-2b+a-1}, x_{c-2b+a}, x_{c-2b+a+1}, \ldots, x_{a-1}$
are a $z_2 - x_{a-1}$ geodesic, a longest $z_2 - x_{a-1}$ monophonic, a longest $z_2 - x_{a-1}$ *mt*-path, and a $z_2 - x_{a-1}$ detour path, respectively. Hence $d(z_2, x_{a-1}) = a$, $d_m(z_2, x_{a-1}) = b$, $d_{mt}(z_2, x_{a-1}) = c$ and $D(z_2, x_{a-1}) = d$. Also, it is easily verified that $d(z_2, t) \leq a$, $d_m(z_2, t) \leq b$, $d_{mt}(z_2, t) \leq c$, $D(z_2, t) \leq d$ for any vertex t in G and so $e(z_2) = a$, $e_m(z_2) = b$, $e_{mt}(z_2) = c$ and $e_D(z_2) = d$. We can similarly verify that $e(v) = a$ if $v \in V(R_5)$; $e(v) > a$ if $v \in V(G) - V(R_5)$, $e_m(v) = b$ if $v \in V(R_5)$; $e_m(v) > b$ if $v \in V(G) - V(R_5)$, $e_{mt}(v) = c$ if $v \in V(R_5)$; $e_{mt}(v) > c$ if $v \in V(G) - V(R_5)$, $e_D(v) = d$ if $v \in V(R_5)$; $e_D(v) > d$ if $v \in V(G) - V(R_5)$. It follows that $rad(G) = a$, $rad_m(G) = b$, $rad_{mt}(G) = c$ and $rad_D(G) = d$.

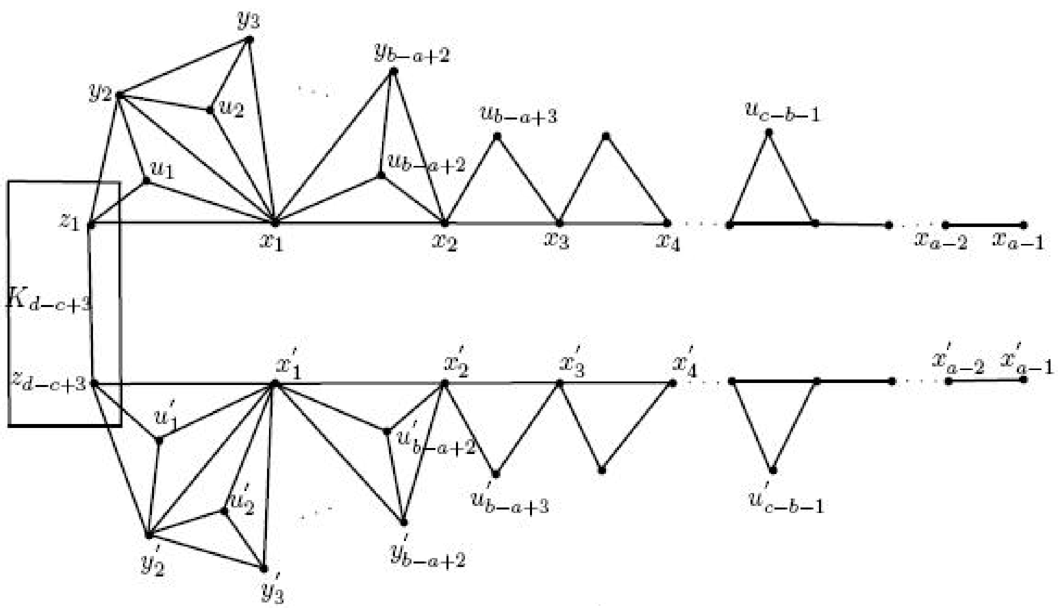

Figure 2.4: G

(b) This part is proved by considering two cases.

Let $R_1 : x_1, x_2, \ldots, x_{a-1}$ be a path of order $a-1$, let $R_2 : y_1, y_2, \ldots, y_{b-a+3}$ be a path of order $b-a+3$ and let R_3 be the complete graph of order $d-c+3$ with $V(R_3) = \{z_1, z_2, \ldots, z_{d-c+3}\}$. Let H be the graph obtained from R_1, R_2 and R_3 by (i) identifying the vertices z_1 in R_3 and y_1 in R_2; (ii) identifying the vertices y_{b-a+3} in R_2 and x_2 in R_1; and (iii) joining each vertex y_i $(2 \leq i \leq b-a+2)$ in R_2 and x_1 in R_1. Now, the graph G is constructed as in the following two cases.

Case 1. $b+1 \leq c \leq 2b-a+3$.

Let G be the graph obtained from H by adding $c-b-1$ new vertices $u_1, u_2, \ldots, u_{c-b-1}$ and joining each u_i $(1 \leq i \leq c-b-1)$ with the vertices x_1, y_i and y_{i+1} $(1 \leq i \leq c-b-1)$. The graph G is shown in Figure 2.5.

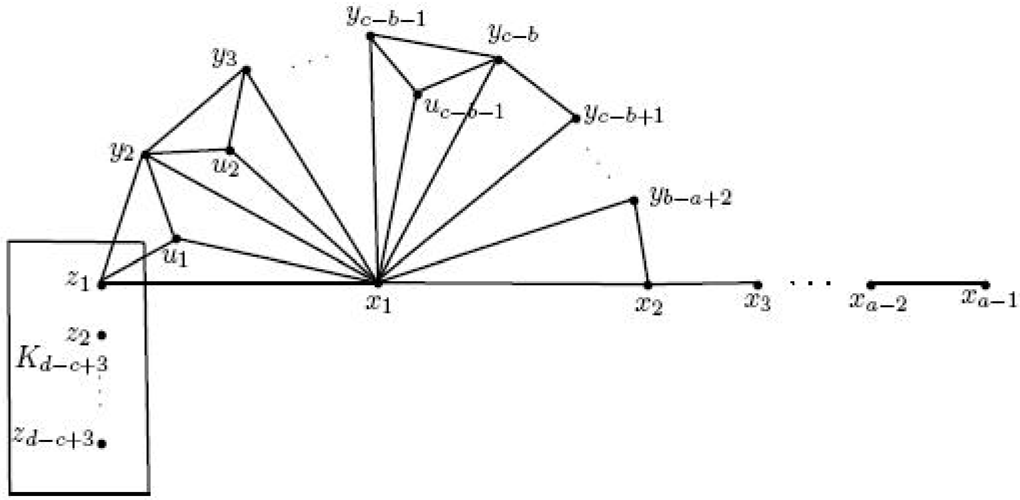

Figure 2.5: G

Case 2. $2b - a + 4 \leq c \leq 2b$.

Let G be the graph obtained from H by adding $c - b - 1$ new vertices $u_1, u_2, \ldots, u_{c-b-1}$ and joining each u_i $(1 \leq i \leq b - a + 2)$ with the vertices x_1, y_i and y_{i+1} $(1 \leq i \leq b-a+2)$ and joining each u_i $(b-a+3 \leq i \leq c-b-1)$ with the vertices $x_{i-b+a-1}$ and x_{i-b+a} $(b-a+3 \leq i \leq c-b-1)$. The graph G is shown in Figure 2.6.

In both cases, it is easily verified that $e(v) = a$ if $v \in (V(R_3) - \{z_1\}) \cup \{x_{a-1}\}$; $e(v) < a$ if $v \in V(R_2) \cup (V(R_1) - \{x_{a-1}\})$, $e_m(v) = b$ if $v \in (V(R_3) - \{z_1\}) \cup \{x_{a-1}\}$; $e(v) < b$ if $v \in V(R_2) \cup (V(R_1) - \{x_{a-1}\})$, $e_{mt}(v) = c$ if $v \in (V(R_3) - \{z_1\}) \cup \{x_{a-1}\}$; $e_m(v) < c$ if $v \in V(R_2) \cup (V(R_1) - \{x_{a-1}\})$, and $e_D(v) = d$ if $v \in (V(R_3) - \{z_1\}) \cup \{x_{a-1}\}$; $e(v) < d$ if $v \in V(R_2) \cup (V(R_1) - \{x_{a-1}\})$. It follows that $diam(G) = a$, $diam_m(G) = b$, $diam_{mt}(G) = c$ and $diam_D(G) = d$. $\square$

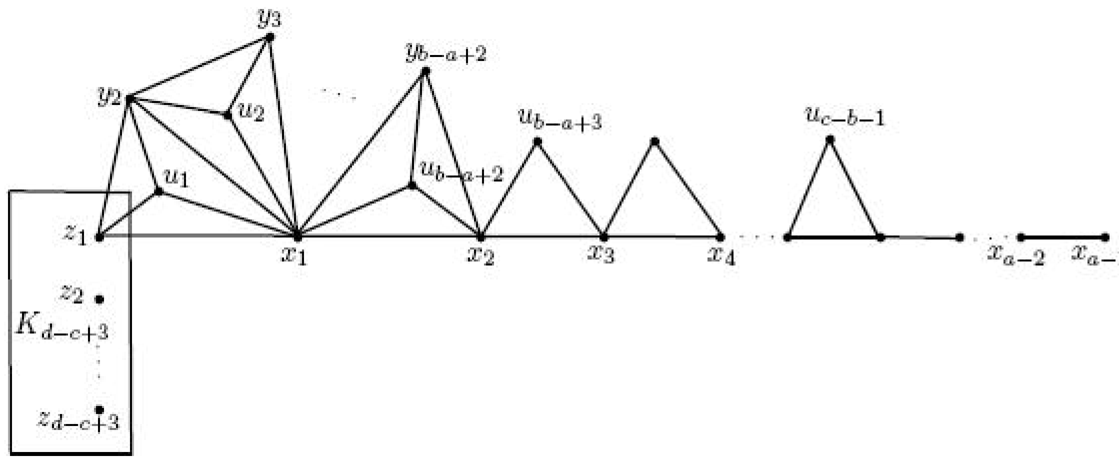

Figure 2.6: G

In any connected graph G, the radius and diameter are related by the inequality $rad(G) \leq diam(G) \leq 2rad(G)$, and the detour radius and detour diameter are related by the inequality $rad_D(G) \leq diam_D(G) \leq 2rad_D(G)$. But Santhakumaran et. al. [5] showed that this inequality is not true in the case of monophonic distance. Similar to monophonic distance, this inequality is not true in the case of mt-distance. For the graph G given in Figure 2.7, it is clear that for any vertex v in G, $2 \leq e_{mt}(v) \leq 5$, $e_{mt}(x_1) = 2$ and $e_{mt}(x_2) = 5$. It follows that $rad_{mt}(G) = 2$ and $diam_{mt}(G) = 5$ and so $diam_{mt}(G) > 2rad_{mt}(G)$.

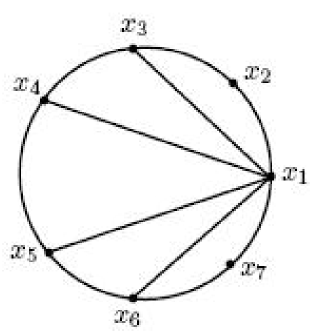

Figure 2.7: G

Ostrand [4] showed that every two positive integers a and b with $a \leq b \leq 2a$ are realizable as the radius and diameter, respectively, of some connected graph. Similarly, Chartrand et al. [2] showed that every two positive integers a and b with $a \leq b \leq 2a$ are realizable as the detour radius and detour diameter, respectively, of some connected graph. Also, Santhakumaran et al. [5] showed that every two positive integers a and b with $a \leq b$ are realizable as the monophonic radius and monophonic diameter, respectively, of some connected graph. Now we have a realization theorem for $rad_{mt}(G) \leq diam_{mt}(G)$.

Theorem 2.11. *For each pair a, b of positive integers with $2 \leq a \leq b$, there exists a connected graph G with $rad_{mt}(G) = a$, $diam_{mt}(G) = b$.*

Proof. For $a = b \geq 2$, the cycle C_{a+2} has the desired property. For $2 \leq a < b$, let $C : x_1, x_2, \ldots, x_{b-a+3}, x_1$ be a cycle of order $b-a+3$ and let $P : y_1, y_2, \ldots, y_{a-1}$ be a path of order $a-1$. Let G be the graph obtained from C and P by joining the vertex y_1 of P with the vertices x_1 and x_2 of C, and joining the vertex x_1 with every vertex x_i $(3 \leq i \leq b-a+2)$ in C. The graph G is shown in Figure 2.8.

It is clear that $d_{mt}(x_1, y_{a-1}) = a$ and $d_{mt}(x_1, x) \leq a$ for any vertex x in G and so $e_{mt}(x_1) = a$. Similarly, it is clear that $d_{mt}(x_{b-a+3}, y_{a-1}) = b$ and $d_{mt}(x_{b-a+3}, x) \leq b$ for any vertex x in G and so $e_{mt}(x_{b-a+3}) = b$. Also, it is clear that $a \leq e_{mt}(x) \leq b$ for any vertex x in G. Hence $rad_{mt}(G) = a$ and $diam_{mt}(G) = b$. □

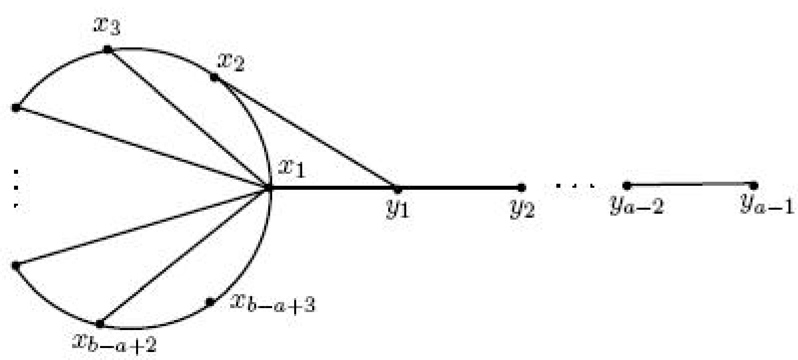

Figure 2.8: G

3. Mt-center and Mt-periphery

Definition 3.1. *A vertex v in a connected graph G is called a mt-central vertex if $e_{mt}(v) = rad_{mt}(G)$ and the subgraph induced by the mt-central vertices of G is the mt-center $C_{mt}(G)$ of G. A vertex v in a connected graph G is called a mt-peripheral vertex if $e_{mt}(v) = diam_{mt}(G)$ and the subgraph induced by the mt-peripheral vertices of G is the mt-periphery $P_{mt}(G)$ of G.*

In [1], it is shown that every graph is the center of some connected graph and Chartrand et al. [2] proved that every graph is the detour center of some connected graph. Also, Santhakumaran et al. [6] proved that every graph is the monophonic center of some connected graph. Now, we have a similar theorem.

Theorem 3.2. *Every graph is the mt-center of some connected graph.*

Proof. Let G be a graph. We prove this theorem by considering two cases.

Case 1. $G = \overline{K_n}$.

Let H be the graph obtained from the graph G by adding the new edges xy and uv, and joining every vertex of G with the vertices y and u. The graph H is shown in Figure 3.1. It is clear that $e_{mt}(z) = 2$ if $z \in V(G)$, $e_{mt}(y) = e_{mt}(u) = 3$ and $e_{mt}(x) = e_{mt}(v) = 4$. Hence $V(G)$ is the set of all mt-central vertices of H and so $C_{mt}(H) = G$.

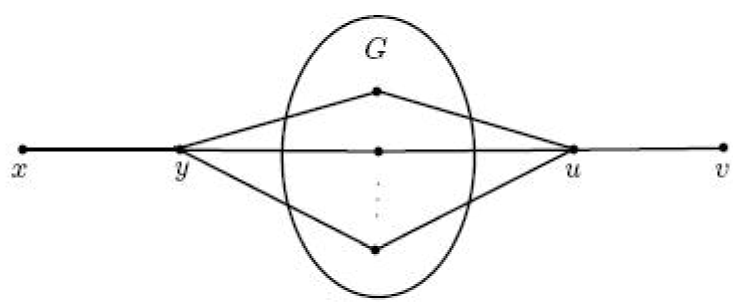

Figure 3.1: H

Case 2. $G \neq \overline{K_n}$.

Let $d = \max\ \{diam_{mt}(G_i) : G_i$ is a component of $G\}$. Let $P_1 : x_1, x_2, \ldots, x_{d+1}$ and $P_2 : y_1, y_2, \ldots, y_{d+1}$ be two copies of the path P of order $d+1$. Let H be the graph obtained from G, P_1 and P_2 by joining every vertex of G with x_1 in P_1 and y_1 in P_2, and if G contains isolated vertices, say $z_1, z_2, \ldots, z_k$, then add two more vertices u and v, and join u with the vertices $z_1, z_2, \ldots, z_k$ and x_1, and join v with the vertices $z_1, z_2, \ldots, z_k$ and y_1. It is clear that $e_{mt}(x) = d+2$ if $x \in V(G)$ and $e_{mt}(x) > d+2$ if $x \in V(H) - V(G)$ in H. Hence $V(G)$ is the set of all mt-central vertices of H and so $C_{mt}(H) = G$. The graph in Figure 3.2 shows the construction of the graph H when $G = \overline{K_2} \cup P_3 \neq \overline{K_5}$. □

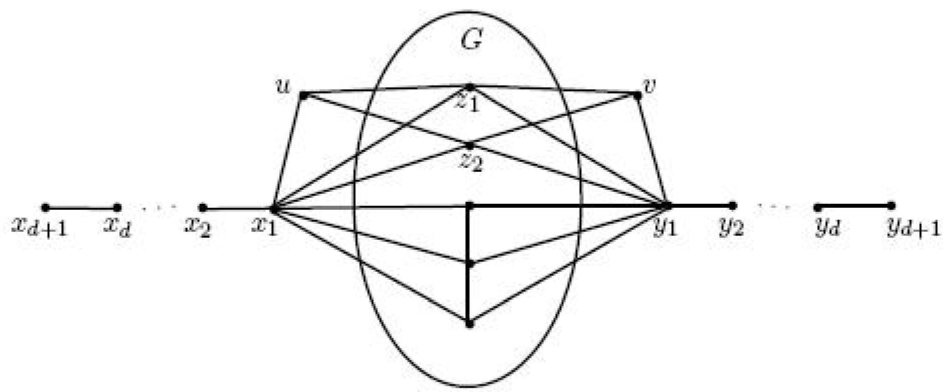

Figure 3.2: H

Now, we have the following observations for the mt-center of a graph which are similar to ordinary center, detour center, and monophonic center of a graph.

Observation 3.3. *(i) The mt-center $C_{mt}(G)$ of every connected graph G is a subgraph of some block of G.*
(ii) The mt-center of every tree is isomorphic to K_1 or K_2.

Definition 3.4. *A connected graph G is mt-$self$ centered if $rad_{mt}(G) = diam_{mt}(G)$.*

Theorem 3.5. *Every connected mt-self centered graph contains no cut-vertex.*

Proof. Since mt-center $C_{mt}(G)$ of any connected graph G is a subgraph of some block of G, no cut-vertex lies in the center $C_{mt}(G)$ of G. Hence, if G contains a cut-vertex, then G is not a mt-self centered graph. Thus, every connected mt-self centered graph contains no cut-vertex. □

Since $1 \leq e_{mt}(x) \leq p-1$ for any vertex $x \in G$, we have $1 \leq rad_{mt}(G) \leq p-1$. The following theorem gives a characterisation result for $rad_{mt}(G) = 1$ or 2 with some conditions.

Theorem 3.6. *Let G be a connected graph. Then*
(i) G is mt-self centered graph with $rad_{mt}(G) = 1$ if and only if $G = K_2$.
(ii) G is mt-self centered graph with $rad_{mt}(G) = 2$ if and only if G is either K_p $(p \geq 3)$ or $K_{m,n}$ $(m, n \geq 2)$.

Proof. (i) Let G be a mt-self centered graph with $rad_{mt}(G) = 1$. If $G \neq K_2$, then there exists a vertex, say x, in G such that $e_{mt}(x) \geq 2$. Since $rad_{mt}(G) = 1$, there exists a vertex, say y, in G such that $e_{mt}(y) = 1$. Hence $e_{mt}(x) \neq e_{mt}(y)$ and so G is not a mt-self centered graph, which is a contradiction.

Conversely, if $G = K_2$, then $e_{mt}(x) = 1$ for any vertex x in K_2 and so $rad_{mt}(G) = 1$ and G is a mt-self centered graph.

(ii) Let G be a mt-self centered graph with $rad_{mt}(G) = 2$. Then by Theorem 3.5, G has no cut-vertices. If $p = 3$, then $G = K_3$ has the desired property. Now, let $p \geq 4$. If $G = K_p$, then G has the desired property. If $G \neq K_p$, then we claim that $G = K_{m,n}$ $(m, n \geq 2)$. If there exists a vertex, say x, in G with $e_{mt}(x) \geq 3$, then $rad_{mt}(G) \geq 3$ or G is not a mt-self centered graph with $rad_{mt}(G) = 2$, which is a contradiction. Similarly, if there exists a vertex, say x, in G with $e_{mt}(x) = 1$, then $rad_{mt}(G) = 1$, which is a contradiction. Hence $e_{mt}(x) = 2$ for any vertex x in G. Let u be a vertex in G and let U be the set of all vertices of G with even distance from u and let $W = V(G) - U$. Let $u \in U$ and $w \in W$. If uw is not an edge in G, then since G is connected with $p \geq 4$, there exists an $u - w$ mt-path with $d_{mt}(u, w) \geq 3$. Hence $e_{mt}(u) \geq 3$, which is a contradiction. Now we claim that no two vertices in U are adjacent and also no two vertices in W are adjacent in G. Let $u_1, u_2 \in U$ and u_1u_2 is an edge in G. Since $G \neq K_p$, there exist two vertices x and y in G with xy not an edge in G. Then either $x, y \in U$ or $x, y \in W$. If $x, y \in U$, then either $x \notin \{u_1, u_2\}$ or $y \notin \{u_1, u_2\}$. If $x \notin \{u_1, u_2\}$, then x, w_i, u_1, u_2 is an $x - u_2$ mt-path, for some w_i in W, and so $e_{mt}(x) = 3$, which is a contradiction. Similarly, if $y \notin \{u_1, u_2\}$, then

y, w_i, u_1, u_2 is an $y - u_2$ mt-path, for some w_i in W, and so $e_{mt}(y) = 3$, which is a contradiction. If $x, y \in W$, then x, u_1, u_2, y is an $x - y$ mt-path and so $e_{mt}(x) = 3$, which is a contradiction. Hence $G = K_{m,n}$ with the partite sets U and W. Since G has no cut-vertices, we have m and n are at least 2. The converse is clear. □

Theorem 3.7. *Let G be a connected graph of order p. Then $rad_{mt}(G) = p - 1$ if and only if $G = K_2$ or K_3.*

Proof. Let $rad_{mt}(G) = p - 1$. Then $diam_{mt}(G) = p - 1$ and so G is a mt-self centered graph. By Theorem 3.5, G has no cut-vertex. If $p = 2$, then $G = K_2$ has the desired property and if $p = 3$, then $G = K_3$ has the desired property. Now, let $p \geq 4$. Since G is a mt-self centered graph with $rad_{mt}(G) = p-1$, $e_{mt}(x) = p-1$ for every vertex x in G. Let y be a vertex in G with $d_{mt}(x, y) = p - 1$, and let P be an $x - y$ mt-path with length $d_{mt}(x, y)$. Since every vertex of G lies on P and G has no cut-vertex, we have xy is an edge in G. Hence $P \cup \{xy\}$ is a hamiltonian cycle of length at least 4 in G and so P is not an $x-y$ mt-path in G, which is a contradiction. Hence G is either K_2 or K_3. Converse is clear. □

Theorem 3.8. *Let G be a non-trivial connected graph. Then $rad_{mt}(G) = 1$ if and only if G is a star.*

Proof. Let $rad_{mt}(G) = 1$. Then there exists a vertex, say x, in G such that $e_{mt}(x) = 1$. If G is not a tree, then G contains a cycle, say C, of order at least 3. If x is a vertex of C, then there exists a vertex y in C such that $d_{mt}(x, y) \geq 2$ and so $e_{mt}(x) \geq 2$, which is a contradiction. Similarly, if x is not a vertex in C, then there exists a vertex y in C such that $d_{mt}(x, y) \geq 2$ and so $e_{mt}(x) \geq 2$, which is a contradiction. Hence G is a tree. If G is not a star, then $e_{mt}(x) \geq 2$ for any vertex x in G and so $rad_{mt}(G) \geq 2$, which is a contradiction. Hence G is a star. Converse is clear. □

References

[1] F. Buckley and F. Harary, Distance in Graphs, Addison-Wesley, Redwood city, CA, 1990.

[2] G. Chartrand, H. Escuadro and P. Zhang, Detour Distance in Graphs, J.Combin. Math. Combin. Comput., Vol. 53, pp. 75-94, 2005.

[3] F. Harary, Graph Theory, Addison-Wesley, 1969.

[4] P. A. Ostrand, Graphs with Specified Radius and Diameter, Discrete Mathematics, Vol. 4, pp. 71-75, 1973.

[5] A. P. Santhakumaran and P. Titus, Monophonic Distance in Graphs, Discrete Mathematics, Algorithms and Applications, Vol. 3, No. 2, pp. 159-169, 2011.

[6] A. P. Santhakumaran and P. Titus, A Note on "Monophonic Distance in Graphs", Discrete Mathematics, Algorithms and Applications, Vol. 4, No. 2, 2012, DOI: 10.1142 s1793830912500188.

P. Titus
Department of Mathematics
University College of Engineering Nagercoil
Anna University,
Tirunelveli Region
Nagercoil-629 004,
India
e-mail: titusvino@yahoo.com
Corresponding author

J. Ajitha Fancy
Department of Mathematics
Scott Christian College (Autonomous)
Nagercoil-629 003,
India
e-mail: ajithafancy@gmail.com

and

A. P. Santhakumaran
Former Professor,
Department of Mathematics
Hindustan Institute of Technology and Science
Chennai-603 103,
India
e-mail: apskumar1953@gmail.com