Proyecciones Journal of Mathematics
Vol. 42, N^o 4, pp. 983-1003, August 2023.
Universidad Católica del Norte
Antofagasta - Chile

10.22199/issn.0717-6279-5790

Ergodicity of commuting multioperators and holomorphic multioperators of multiplication

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Received : December 2022. Accepted : March 2023

Abstract

In this paper, the strong ergodic theorems are extended from the case of one bounded operator to the case of commuting multioperators. The authors show that in Grothendieck space with the Dunford-Pettis property, mean ergodic operator, and uniform ergodic operator are the same. We study when multioperators of multiplication on a weighted Banach space of holomorphic multi-functions are power bounded, mean ergodic, or uniformly mean ergodic.

Mathematics Subject Classification: *47A35; 47B38; 32A12.*

Keywords: *Ergodic theorem, holomorphic vector-valued functions, commuting multioperators, power bounded, multioperators of multiplication.*

1. Introduction and Preliminaries

Let $\mathbf{B}_n$ be the open unit ball of $\mathbf{C}^n$, $n \geq 1$, with respect to the euclidean norm, i.e.

$$\mathbf{B}_n := \left\{ z = (z_1, \ldots, z_n) \in \mathbf{C}^n ; \|z\|_2^2 := \sum_{k=1}^{n} |z_k|^2 < 1 \right\}.$$

We simply write $\mathbf{D}$ for the unit disk in the complex plane. We denote $H(\mathbf{B}_n, \mathbf{C}^d)$ the set of all analytic functions from $\mathbf{B}_n$ to $\mathbf{C}^d$, and $H^\infty(\mathbf{B}_n, \mathbf{C}^d) = \left\{ f \in H(\mathbf{B}_n, \mathbf{C}^d) \ : \ \|f\|_\infty < \infty \right\}$ the set of bounded analytic functions, where $\|.\|_\infty$ is the supremum norm. If $d = 1$ we denote $H(\mathbf{B}_n)$ for $H(\mathbf{B}_n, \mathbf{C}^1)$ and $H^\infty(\mathbf{B}_n)$ for $H^\infty(\mathbf{B}_n, \mathbf{C}^1)$. Let $v : \mathbf{B}_n \longrightarrow (0; \infty)$ be a positive continuous and bounded function on $\mathbf{B}_n$ (weight function). A weight v is called typical if it is radial. We shall consider weights of the form $v(z) := v(|z|)$ for every $z \in \mathbf{B}_n$ and satisfying $\lim_{|z| \to 1^-} v(z) = 0$. In [13], W. Lusky studied the corresponding function spaces on the open unit disc $\mathbf{D}$ of the complex plane $\mathbf{C}$ and introduced a large class (B) of radial weight functions v. In [14], W. Lusky and J. Taskinen have generalized the weight class (B) to the case of several variables. Let $\varphi \in H(\mathbf{B}_n, \mathbf{C}^d)$, $\varphi \not\equiv 0$, The linear operator

$$\begin{array}{rcl} M_\varphi : H(\mathbf{B}_n, \mathbf{C}^d) & \longrightarrow & H(\mathbf{B}_n, \mathbf{C}^d) \\ f = (f_1, \ldots, f_d) & \longmapsto & \varphi f = (\varphi_1 f_1, \ldots, \varphi_d f_d) \end{array}$$

is called a pointwise multiplication operator.

The study of pointwise multiplication operators between different spaces of analytic functions have quite a long and rich history. Thus, many properties of multiplication operators have been investigated, see, e.g., [4, 5, 17].

Throughout the following, we will study multiplication operators that act on the weighted Banach spaces of holomorphic vector-valued functions given by

$$H_v^\infty(\mathbf{B}_n, \mathbf{C}^d) := \left\{ f \in H(\mathbf{B}_n, \mathbf{C}^d) \ : \ \|f\|_v = \sup_{z \in \mathbf{B}_n} v(z) \|f(z)\|_2 < \infty \right\},$$

and

$$H_v^0(\mathbf{B}_n, \mathbf{C}^d) := \left\{ f \in H(\mathbf{B}_n, \mathbf{C}^d) \ : \ \|f\|_v = \lim_{\|z_i\|_2 \to 1^-} v(z) \|f(z)\|_2 = 0 \right\}$$

endowed with the norm $\|.\|_v$. Spaces of this type appear in the study of growth conditions of analytic functions and have been studied in various articles, see, e.g., [8, 17]. The space $H_v^0(\mathbf{B}_n, \mathbf{C}^d)$ is a closed subspace of $H_v^\infty(\mathbf{B}_n, \mathbf{C}^d)$.

Let $z \in \mathbf{B}_n$, the evaluation function $\delta_z \ : H_v^\infty(\mathbf{B}_n) \longrightarrow \mathbf{C}$ defined by $\delta_z(f) = f(z)$ is linear and continuous $(\delta_z \in (H_v^\infty(\mathbf{B}_n))'$. Moreover, one can show that $|\delta_z(f)| \leq \frac{\|f\|_v}{v(z)}$. Also, $\delta_z(f) \in (H_v^0(\mathbf{B}_n))'$.

Let $\mathcal{X}$ be a locally convex Hausdorff space. The space of all continuous linear operators on $\mathcal{X}$ by $\mathcal{L}(\mathcal{X})$. The weak topology of $\mathcal{X}$ will be denoted by $\sigma(\mathcal{X}, \mathcal{X}')$, where $\mathcal{X}'$ is the topological dual space of $\mathcal{X}$. If $\mathcal{L}(\mathcal{X})$ is endowed with its strong operator topology (respectively with the topology of uniform convergence on bounded sets of $\mathcal{X}$) we denote $\mathcal{L}_s(\mathcal{X})$ (respectively $\mathcal{L}_b(\mathcal{X})$).

Given $T \in \mathcal{L}(\mathcal{X})$, we denote the Cesàro means of T by

$$T_{[n]} := \frac{1}{n}\sum_{k=1}^{n} T^k, \quad n \in \mathbf{N}^*.$$

The following well-known equality can be checked easily

$$\frac{1}{n}T^n = T_{[n]} - \frac{n-1}{n}T_{[n-1]}, \quad n \in \mathbf{N}^*, \tag{1.1}$$

where $T_{[0]} = I$ is the identity operator on $\mathcal{X}$.

We say that the operator T is mean ergodic (respectively uniformly mean ergodic) if the sequence $\left\{T_{[n]}\right\}_{n=1}^{\infty}$ converges in $\mathcal{L}_s(\mathcal{X})$ (respectively in $\mathcal{L}_b(\mathcal{X})$).
We say that the operator T is power bounded if there is $C > 0$ such that

$$\sup_{n \in \mathbf{N}} \|T\| \leq C.$$

For more information on the ergodic theory, we refer the reader to the monograph [9]. For other interesting articles related to this topic see [5, 7, 10, 11, 12].
In [9, Ch II., Theorem 1.1.], U. Krengel characterized the mean ergodic operators. In the present paper we will extend this result to the case of

a d-tuple of commuting multioperators acting on a Banach space. Also, [9, Ch II., Theorem 1.1.] will be extended to the d-tuple of commuting multioperators case.
In [4], the authors characterized when the multiplication operator is Fredholm or is an isomorphism. In [5] J. Bonet and W. Ricker investigated the connection between power boundedness, mean ergodicity, and uniform mean ergodicity of multiplication operators acting on weighted spaces of holomorphic functions on the complex unit disc. Also, they characterized when multiplication operators are power bounded or (uniformly) mean ergodic on these spaces. Multiplication operators on weighted spaces of vector-valued functions have been studied, [17], vector-valued holomorphic functions in [1], weighted spaces of vector-valued functions in [2, 3] and weighted spaces of holomorphic functions of several variables in[14]. In the present paper, our goal is to study when holomorphic multiplication operators on a weighted Banach space of holomorphic vector-valued functions are power bounded, mean ergodic, or uniformly mean ergodic.

If now $T = (T_1, T_2, \ldots, T_d) \in \mathcal{L}(\mathcal{X})^d$ is a commuting multioperator (briefly, a c. m.) we also set

$$T_{[\alpha]} = T_{1[\alpha_1]} T_{2[\alpha_2]} \ldots T_{d[\alpha_d]}, \quad \alpha \in Z_+^d, \ \alpha \geq e, \tag{1.2}$$

where Z_+^d is the family of multi-indices of length d (i.e. d-tuples of non-negative integers) and $e := (1, 1, \ldots, 1) \in Z_+^d$. In other words, (1.2) defines the averages associated with T.

Given a d-tuple $T = (T_1, \ldots, T_d)$ of operators on a Hilbert space $\mathcal{H}$, the joint operator norm of T is defined in [6] as:

$$\|T\| := \sup \left\{ \left(\sum_{k=1}^{d} \|T_k x\|^2 \right)^{\frac{1}{2}} ; x \in \mathcal{H}, \|x\| = 1 \right\} = \left\| \sum_{k=1}^{d} T_k^* T_k \right\|^{\frac{1}{2}}.$$

Definition 1. *[15] A c.m. $T \in \mathcal{L}(\mathcal{X})^d$ is said to be Cesàro quasi-bounded if the sequences*

$$\left(\prod_{i \neq j} T_{i[\alpha_i]} \right)_{\alpha_1 \geq 1, \ldots, \alpha_{j-1} \geq 1, \alpha_{j+1} \geq 1, \ldots \alpha_d \geq 1} \qquad (j = 1, \ldots, d)$$

are bounded in $\mathcal{L}(\mathcal{X})$. If in addition the limit

$$\lim_{v \to \infty} T_{[\alpha]}$$

exists in the uniform (resp. strong) topology of $\mathcal{L}(\mathcal{X})$, then T is said to be uniformly mean ergodic (resp. mean ergodic).

A c.m. $T = (T_1, T_2, \ldots, T_d) \in \mathcal{L}(\mathcal{X})^d$ is said to be power bounded multioperators if there exists a constant M such that

$$\left\| T^k \right\| = \left\| T_1^{k_1} T_2^{k_2} \ldots T_d^{k_d} \right\| \leq M, \tag{1.3}$$

for each $k = (k_1, k_2, \ldots, k_d) \in \mathbf{N}^d$.

Remark 1. *1. If $T_1, T_2, \ldots, T_d$ are power bounded commuting operators, then $T = (T_1, T_2, \ldots, T_d)$ is power bounded multioperators. The converse is not true, in general. Indeed, if $T_1 = 0$ and T_2 is bounded but not power bounded, then $T = (T_1, T_2)$ is, though, power bounded.*

2. If $T = (I, \ldots, I, T_j, I, \ldots, I)$. Then T is power bounded multioperators if and only if T_j is power bounded.

2. Main results

We will start this section by proving the following lemma which extend the formula (1.1) to a commuting multioperator.

Lemma 1. *Let $T = (T_1, T_2, \ldots, T_d) \in \mathcal{L}(\mathcal{X})^d$ be a c. m., then*

$$\frac{1}{\alpha_j} T_j^{\alpha_j} = T_{[\alpha]} - \frac{\alpha_j - 1}{\alpha_j} T_{[\alpha - e_j]}, \tag{2.1}$$

for all $j = 1, \ldots, d$, where

$$e_j = (\underbrace{0, \ldots, 0, 1}_{j}, 0, \ldots, 0).$$

Proof. Let $j \in \{1, \ldots, d\}$ be fixed, then

$$\begin{aligned}
T_{[\alpha]} - \frac{\alpha_j - 1}{\alpha_j} T_{[\alpha - e_j]} &= \prod_{k=1}^{d} T_{k[\alpha_k]} - \frac{\alpha_j - 1}{\alpha_j} \left(\prod_{k \neq j} T_{k[\alpha_k]} \right) \cdot T_{j\,[\alpha_j - 1]} \\
&= \prod_{k=1}^{d} T_{k[\alpha_k]} - \frac{\alpha_j - 1}{\alpha_j} \left(\prod_{k \neq j} T_{k[\alpha_k]} \right) \\
&\quad \cdot \frac{1}{\alpha_j - 1} \left(\sum_{k=0}^{\alpha_j} T_j^k - T_j^{\alpha_j} \right) \\
&= \frac{1}{\alpha_j} T_j^{\alpha_j}.
\end{aligned}$$

□

Lemma 2. *Let* $T = (T_1, T_2, \ldots, T_d) \in \mathcal{L}(\mathcal{X})^d$ *be a c. m. such that* $\ker(I - T_j) = 0$ *for each* $j = 1, \ldots, d$, *and*

$$\lim_{\alpha_j \longrightarrow \infty} \left\| \frac{1}{\alpha_j} T_j^{\alpha_j} \right\| = 0, \text{ for all } j = 1, \ldots, d.$$

Then, the following assertions are equivalent

1. $I - T_{[\alpha]}$ *is surjective for some* $\alpha \in \mathbf{N}^d$;
2. $I - T_j$ *is surjective for all* $j = 1, \ldots, d$;
3. $\lim_{\alpha \longrightarrow \infty} \left\| T_{[\alpha]} \right\| = 0$ *(T is uniformly mean ergodic).*

Proof. 3. ⇒ 1. Since $\lim_{\alpha \longrightarrow \infty} \left\| T_{[\alpha]} \right\| = 0$, thus there exists $\alpha \in \mathbf{N}^d$ such that $\left\| T_{[\alpha]} \right\| < 1$. Hence, $I - T_{[\alpha]}$ is an isomorphism and, in particular, it is surjective.

1. ⇒ 2. Let $y \in \mathcal{X}$, then, by 1., there exists $x \in \mathcal{X}$ such that $\left(I - T_{[\alpha]} \right) x = y$. A simple computation using the mutual commutativity of $T_1, \ldots, T_j$ shows that

$$y = \left(I - T_{[\alpha]} \right) x = \prod_{j=1}^{d} (I - T_j) \cdot \prod_{j=1}^{d} \frac{1}{\alpha_j} \left(\sum_{r=0}^{\alpha_j - 1} \sum_{i=0}^{r} T_j^i x \right),$$

and $I - T_j$ is surjective for each $j = 1, \ldots, d$.

2. $\Rightarrow$ 3. For each $j = 1, \ldots, d$, we have $I - T_j$ is injective by hypothesis and it is onto by 2., and it is continuous. Applying the open mapping theorem $(I - T_j)$ is continuous for each $j = 1, \ldots, d$. Let $\mathcal{B}$ the closed unit ball of $\mathcal{X}$, then $\Xi = \prod_{j=1}^{d} (I - T_j)^{-1} \mathcal{B}$ is bounded. Thus, by using the mutual commutativity of $T_1, \ldots, T_j$, we get

$$\begin{aligned}\left\|T_{[\alpha]}\right\| &= \sup_{\zeta \in \mathcal{B}} \left\|T_{[\alpha]}\zeta\right\| = \sup_{x \in \Xi} \left\| \prod_{j=1}^{d} (I - T_j)\, T_{[\alpha]} x \right\| = \sup_{x \in \Xi} \left\| \prod_{j=1}^{d} \frac{1}{\alpha_j} \left(T_j - T_j^{\alpha_j + 1} \right) x \right\| \\ &\leq \sup_{x \in \Xi} \|x\| \prod_{j=1}^{d} \left(\frac{1}{\alpha_j} \|T_j\| + \frac{\alpha_j + 1}{\alpha_j} \left\| \frac{1}{\alpha_j + 1} T_j^{\alpha_j + 1} \right\| \right) \\ &\leq C \prod_{j=1}^{d} \left(\frac{1}{\alpha_j} \|T_j\| + \frac{2}{\alpha_j + 1} \left\| T_j^{\alpha_j + 1} \right\| \right),\end{aligned}$$

where $C = \sup_{x \in \Xi} \|x\|$. Therefore, by hypothesis, $\lim_{\alpha \longrightarrow \infty} \left\|T_{[\alpha]}\right\| = 0$. $\square$

In the strong ergodic theorem [9, Ch II., Theorem 1.1.], U. Krengel characterized the mean ergodic operators. In the following theorem, we will extend this result to the case of a d-tuple of commuting multioperators.

Theorem 1. *Let $T = (T_1, T_2, \ldots, T_d) \in \mathcal{L}(\mathcal{X})^d$ be a d-tuple of commuting Cesàro bounded linear operators. For any $x \in \mathcal{X}$ satisfying $\lim_{\alpha_i \longrightarrow \infty} \frac{1}{\alpha_i} T_i^{\alpha_i} x = 0$ for all $i = 1, \ldots, d$, and any $y \in \mathcal{X}$ the following assertions are equivalent*

(i) $T_i y = y$ *for all* $i = 1, \ldots, d$, *and* $y \in \overline{co}\left\{x,\ \prod_{i \in \Lambda} T_i^{\alpha_i} x,\ \alpha_i \in \mathbf{N},\ \Lambda \subseteq \{1, \ldots, d\}\right\}$;

(ii) $y = \lim_{\alpha} T_{[\alpha]} x$;

(iii) $y = w - T_{[\alpha]} x$;

(iv) y *is a weak cluster point of the sequence* $\left(T_{[\alpha]} x\right)$.

Proof. One can show that $(ii) \Longrightarrow (iii) \Longrightarrow (iv)$. For $(i) \Longrightarrow (ii)$, since $T_1, T_2, \ldots, T_d$ are commuting Cesàro bounded linear operators thus $(T_{[\alpha]})$ is a bounded sequence. Set $M = \sup_{\alpha} \left\|T_{[\alpha]}\right\|$. For $\epsilon > 0$, (i) implies that there exists an operator $S \in co\left\{\prod_{i \in \Lambda} T_i^{\alpha_i},\ \alpha_i \in \mathbf{N},\ \Lambda \subseteq \{1, \ldots, d\}\right\}$ such that

$$\|y - Sx\| < \epsilon. \tag{2.2}$$

For each $k = (k_1, \dots, k_d) \in \mathbf{N}^d$, each $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbf{N}^d$ and each $\Lambda \subseteq \{1, \dots, d\}$ we have

$$T_{[\alpha]} \prod_{i \in \Lambda} (T_i)^{k_i} x - T_{[\alpha]} x = \prod_{i \notin \Lambda} (T_i)_{[\alpha_i]} \prod_{i \in \Lambda} \frac{1}{\alpha_i} \sum_{j=0}^{\alpha_i - 1} T_i^{k_i + j} x - \prod_{i=1}^{d} \frac{1}{\alpha_i} \sum_{j=0}^{\alpha_i - 1} T_i^j x.$$

Since $\lim_{\alpha_i \longrightarrow \infty} \frac{1}{\alpha_i} T_i^{\alpha_i} = 0$ for all $i = 1, \dots, d$ and $\frac{\alpha_i}{\alpha_i + j} \longrightarrow 1$. Hence, for a large enough α, we get

$$\left\| T_{[\alpha]} \prod_{i \in \Lambda} (T_i)^{k_i} x - T_{[\alpha]} x \right\| < \epsilon.$$

As S is a convex combination of finitely many $\prod_{i \in \Lambda} T_i^{\alpha_i}$ and the set $\{z \in \mathcal{X} : \ \|z\| < \epsilon\}$ is convex, thus there exists $\beta = (\beta_1, \dots, \beta_d) \in \mathbf{N}^d$ such that

$$\left\| T_{[\alpha]} Sx - T_{[\alpha]} x \right\| < \epsilon, \ \text{ for all } \alpha \geq \beta. \tag{2.3}$$

Since $T_i y = y$ for all $i = 1, \dots, d$, hence $T_{[\alpha]} y = y$ for all α. Thus, for $\alpha \geq \beta$ and by using (2.2) and (2.3), we obtain

$$\left\| y - T_{[\alpha]} x \right\| \leq \left\| T_{[\alpha]} (y - Sx) \right\| \leq + \left\| T_{[\alpha]} Sx - T_{[\alpha]} x \right\| \leq M\epsilon + \epsilon. \tag{2.4}$$

$(iv) \Longrightarrow (i)$ By Mazur's theorem, any closed convex subset of $\mathcal{X}$ is also weakly closed. The weak cluster point y of the sequence $\left(T_{[\alpha]} x\right)$ of convex combinations of $x, \prod_{i \in \Lambda} T_i^{\alpha_i} x,\ \alpha_i \in \mathbf{N},\ \Lambda \subseteq \{1, \dots, d\}$. Therefore, $y \in \overline{co}\, \{x,\ \prod_{i \in \Lambda} T_i^{\alpha_i} x,\ \alpha_i \in \mathbf{N},\ \Lambda \subseteq \{1, \dots, d\}\}$.
We will show that $T_i y = y$ for all $i = 1, \dots, d$. Let $h \in \mathcal{X}^*$ be arbitrary and let $\epsilon > 0$. By the above part of the Proof we have $T_i T_{[\alpha]} x - T_{[\alpha]} x \longrightarrow 0$ as $n \longrightarrow \infty$ for each $i = 1, \dots, d$. Then, for large enough $\alpha \in \mathbf{N}^d$, we get

$$\left| \langle T_i T_{[\alpha]} x - T_{[\alpha]} x,\ h \rangle \right| < \epsilon. \tag{2.5}$$

As y is a weak cluster point of the sequence $\left(T_{[\alpha]}x\right)$, there exist arbitrary large values of α such that

$$(2.6)\left|\langle y,\ h\rangle - \langle T_{[\alpha]}x,\ h\rangle\right| < \epsilon \quad \text{and} \quad \left|\langle y,\ T_i^*h\rangle - \langle T_{[\alpha]}x,\ T_i^*h\rangle\right| < \epsilon.$$

Using the estimates (2.5) and (2.6), one can see that

$$\begin{aligned}|\langle y,\ h\rangle - \langle T_i y,\ h\rangle| \le \left|\langle y,\ h\rangle - \langle T_{[\alpha]}x,\ h\rangle\right| + \left|\langle T_{[\alpha]}x,\ h\rangle - \langle T_i T_{[\alpha]}x,\ h\rangle\right| \\ + \left|\langle T_{[\alpha]}x,\ T_i^*h\rangle - \langle y,\ T_i^*h\rangle\right| < 3\epsilon.\end{aligned}$$

The result derive from the fact that ϵ and h are arbitrary. □

We shall introduce some notations which we will use to extend [9, Ch II., Theorem 1.3.] to the case of d-tuple of commuting multioperators (see Theorem 2 below).

$$\mathcal{X}_{me} = \mathcal{X}_{me}(T) := \left\{x \in \mathcal{X} : \lim T_{[\alpha]}x \text{ exists } \right\}$$

Clearly, if T_i is Cesàro bounded for each $i = 1, \ldots, d$, $\mathcal{X}_{me}$ is a closed linear subspace of $\mathcal{X}$. T is called mean ergodic if $\mathcal{X} = \mathcal{X}_{me}$. Set

$$\begin{aligned}&F_i = F(T_i) := \{x \in \mathcal{X} : T_i x = x\}, \quad N_i := \{x - T_i x : x \in \mathcal{X}\} = (I - T_i)\mathcal{X} \\ &F_i^* = F(T_i^*) := \{h \in \mathcal{X}^* : T_i^* h = h\}, \quad N_i^* := (I - T_i^*)\,\mathcal{X}^*\end{aligned}$$

Theorem 2. *Let $T = (T_1, T_2, \ldots, T_d) \in \mathcal{L}(\mathcal{X})^d$ be a d-tuple of commuting Cesàro bounded linear operators, and assume that $\lim_{\alpha_i \longrightarrow \infty} \frac{1}{\alpha_i} T_i^{\alpha_i} x = 0$ holds for all $x \in \mathcal{X}$ and each $i = 1, \ldots, d$. Then $\mathcal{X}_{me} = cl\,(\sum N_i) \oplus \cap F_i$. The operator E assigning to $x \in \mathcal{X}_{me}$ the limit $Ex := \lim T_{[\alpha]}x$ is the projection of $\mathcal{X}_{me}$ onto $\cap F_i$. We have $E = E^2 = T_i E = E T_i$ for each $i = 1, \ldots, d$. For any $z \in \mathcal{X}$ the following assertions are equivalent*

(i) $\lim T_{[\alpha]}z = 0$;

(ii) $\langle z, h\rangle = 0$ *for all* $h \in \cap F_i^*$;

(iii) $z \in cl\,(\sum N_i)$.

Proof. One can show that F_i and N_i are linear subspaces. then, also $cl\,(\sum N_i)$ and $\cap F_i$ are linear subspaces. We start by verifying $cl\,(\sum N_i)\bigcap \cap F_i = \{0\}$: For $\varepsilon > 0$ and $z \in cl\,(\sum N_i)\bigcap(\cap F_i)$ there exists u such that $\|z - \sum_{i=1}^{d}(u - T_i u)\| < \varepsilon$. Thus

$$\left\| T_{[\alpha]}\left(z - \sum_{i=1}^{d}(u - T_i u)\right)\right\| < M\varepsilon.$$

Now, by using $T_{[\alpha]}z = z$ and $T_{[\alpha]}\sum_{i=1}^{d}(u - T_i u)) \longrightarrow 0$, we get $\|z\| < M\varepsilon + \varepsilon$. By the same technical, we find $cl\,(\sum N_i^*)\bigcap \cap F_i^* = \{0\}$.
Let $x \in \mathcal{X}_{me}$. Then, by Theorem 1, $Ex \in \cap F_i$. Thus $z = x - Ex$ satisfies (i).
$(i) \Longrightarrow (ii)$ Let $h \in \cap F_i^*$, then $h = T_{[\alpha]}^* h$ for all α, thus

$$\langle z, h\rangle = \left\langle z, T_{[\alpha]}^* h\right\rangle = \left\langle T_{[\alpha]}x, h\right\rangle \longrightarrow 0.$$

$(ii) \Longrightarrow (iii)$ Assume that $z \notin cl\,(\sum N_i)$, then there exist, by Hahn-Banach theorem, $h \in \mathcal{X}^*$ such that $\langle z, h\rangle \neq 0$ and $\langle y, h\rangle = 0$ for all $y \in cl\,(\sum N_i)$. In particular, $\langle u - T_i u, h\rangle = 0$ for $u \in \mathcal{X}$ and $i = 1, \ldots, d$. Hence $\langle u, h - T_i^* h\rangle = 0$ for all u and $i = 1, \ldots, d$. This implies $h \in \cap F_i^*$, which is a contradiction to $\langle z, h\rangle \neq 0$.
We have proved $\mathcal{X}_{me} \subset \cap F_i \oplus cl\,(\sum N_i)$. It is clear that $\cap F_i \subset \mathcal{X}_{me}$, then the opposite inclusion will follow from $(iii) \Longrightarrow (i)$: For any $u \in \mathcal{X}$ we have $T_{[\alpha]}(u - T_i u) = {\alpha_i}^{-1}\prod_{k\neq i}^{d} T_{[\alpha_k]}\,(u - T_i^{\alpha_i}u)$ and this tends to 0, $i = 1, \ldots, d$. Thus, all $z \in cl\,(\sum N_i)$ satisfy $T_{[\alpha]}z \to 0$. But the set of z with this property is closed because T_i is Cesáro bounded for all $i = 1, \ldots, d$. As E is the projection of $\cap F_i \oplus cl\,(\sum N_i)$ on $\cap F_i$ and the elements of $\cap F_i$ are fixed under T_i the identities $E = E^2 = T_i E$ are clear. $E = ET_i$ follows from $T_{[\alpha]}(x - T_i x) \to 0$. □

In [12, Theorem 5.], H. P. Lotz proved that for a Grothendieck space with the Dunford-Pettis property one can replace the strong by the uniform operator topology. In the following theorem, we will extend this result to the case of d-tuple of commuting multioperators by using many tools like the Taylor spectrum, [16, Ch. IV., Definition 1.].

Theorem 3. *Let $\mathcal{X}$ be a Grothendieck space with the Dunford-Pettis property, let $T = (T_1, T_2, \ldots, T_d) \in \mathcal{L}(\mathcal{X})^d$ be Cesàro quasi-bounded with $\lim_{\alpha_i \longrightarrow \infty}\left\|\frac{1}{\alpha_j}T_j^{\alpha_j}\right\| = 0$, for all $j = 1, \ldots, d$. If for every $x \in \mathcal{X}$, the sequence $(T_{[\alpha]}x)$ is relatively weakly compact, then the means $T_{[\alpha]}$ of T converge in the uniform operator topology.*

Proof. By the strong ergodic theorem (Theorem 1), the sequence $\left(T_{[\alpha]}\right)$ converges in the strong operator topology to a projection E with $T_{[\alpha]}E = E = ET_{[\alpha]}$ for all multi-index $\alpha \in \mathbf{N}^d$. Let

$$f_\alpha(z) = \prod_{k=1}^{d} \frac{1}{\alpha_k} \sum_{i=0}^{\alpha_k - 1} z_k^i,$$

then $T_{[\alpha]} = f_\alpha(T)$. Set $S = (T_1 - E, \ldots, T_d - E)$ and $S_{[\alpha]} = f_\alpha(S)$. Then

$$S_{[\alpha]} = T_{[\alpha]} - E \text{ and } \lim_\alpha \frac{\|S^\alpha\|}{\prod_{k=1}^d \alpha_k} = \lim_\alpha \frac{\|T^\alpha - E\|}{\prod_{k=1}^d \alpha_k} = 0.$$

Since $S_{[\alpha]}$ tends to zero in the strong operator topology, it follows from [12, Lemma 1.] that (ii) of [12, Theorem 2] holds and that $\left(S'_{[\alpha]}\right) = \left(f_\alpha\left(S'\right)\right)$ tends to zero in the weak operator topology. Then, by applying the strong ergodic theorem, $\left(S'_{[\alpha]}\right)$ tends to zero in the strong operator topology. Thus also (i) of [12, Theorem 2] holds and we deduce that $\lim r\left(S_{[\alpha]}\right) = 0$. Hence $1 \notin \sigma\left(S_{[\alpha]}\right) = \sigma\left(f_\alpha(S)\right)$ for a multi-index α sufficiently large. Since $f_\alpha(e) = 1$ for all $\alpha \in \mathbf{N}^d$, thus, by the spectral mapping theorem, $e \notin \sigma(S)$. Hence, by [16, Ch. IV., Definition 1.] (see also [18]), there exist $L_1, \ldots, L_d \in \mathcal{L}(\mathcal{X})$ such that

$$\sum_{i=1}^{d} \left(I - S_i\right) L_i = I.$$

Thus

$$S_{[\alpha]} = S_{[\alpha]} \sum_{i=1}^{d} \left(I - S_i\right) L_i = \sum_{i=1}^{d} \frac{1}{\alpha_i} \left(I - S_i^{\alpha_i}\right) \prod_{k \neq i}^{d} S_{[\alpha_k]} L_i.$$

Therefore, the operator $S_{[\alpha]}$ converges in norm to zero, which of course means that $\left(T_{[\alpha]}\right)$ converges to E in the uniform operator topology. $\square$

In the following, our aim is to study the holomorphic multiplication operators and giving under which conditions these operators are power bounded, mean ergodic, uniformly mean ergodic, respectively.

Let $M_\varphi = (M_{\varphi_1}, M_{\varphi_2}, \ldots, M_{\varphi_d}) \in \mathcal{L}(H_v^\infty(\mathbf{B}_n))^d$ be a multioperator of multiplication. It is known that the adjoint of M_φ is
$M_\varphi^t = \left(M_{\varphi_1}^t, M_{\varphi_2}^t, \ldots, M_{\varphi_d}^t\right)$.

In the following, the multiplication operators $M_\varphi = (M_{\varphi_1}, M_{\varphi_2}, \ldots, M_{\varphi_d})$ on $H_v^\infty(\mathbf{B}_n, \mathbf{C}^d)$ and $H_v^0(\mathbf{B}_n, \mathbf{C}^d)$ will be denoted by $T_\varphi = (T_{\varphi_1}, T_{\varphi_2}, \ldots, T_{\varphi_d})$ and $S_\varphi = (S_{\varphi_1}, S_{\varphi_2}, \ldots, S_{\varphi_d})$, respectively, such that $T_{\varphi_i} := M_{\varphi_i|H_v^\infty(\mathbf{B}_n)}$ and $S_{\varphi_i} := M_{\varphi_i|H_v^0(\mathbf{B}_n)}$, $i = 1, \ldots, d$.

First, we need some auxiliary results. In case $n = 1$ the following two Lemmas have been proved in [5]. We will show them here for n-tuple of operators.

Lemma 3. *Let* $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_d) \in H(\mathbf{B}_n, \mathbf{C}^d)$. *If* $T_\varphi = (T_{\varphi_1}, T_{\varphi_2}, \ldots, T_{\varphi_d}) \in \mathcal{L}(H_v^\infty(\mathbf{B}_n, \mathbf{C}^d))$, *then* $\varphi \in H^\infty(\mathbf{B}_n, \mathbf{C}^d)$. *The same holds if* $S_\varphi \in \mathcal{L}(H_v^0(\mathbf{B}_n, \mathbf{C}^d))$.

Proof. We will use the adjoint operator. Let $z = (z_1, \ldots, z_n) \in \mathbf{B}_n$ fixed, then for each $f \in H_v^\infty(\mathbf{B}_n)$, we have

$$\left\langle T_{\varphi_i}^t(\delta_z), f\right\rangle = \langle \delta_z, \varphi_i f\rangle = \varphi_i(z)f(z) = (\varphi_i(z)\delta_z)(f),$$

for all $i = 1, \ldots, d$. Then we obtain $T_{\varphi_i}^t(\delta_z) = \varphi_i(z)\delta_z$, $i = 1, \ldots, d$. Since T_{φ_i} is continuous, thus $T_{\varphi_i}^t$ is also continuous. Hence,

$$\begin{aligned}
|\varphi_1(z)|^2\|\delta_z\|^2 + \ldots + |\varphi_d(z)|^2\|\delta_z\|^2 &= \|\varphi_1(z)\delta_z\|^2 + \ldots + \|\varphi_d(z)\delta_z\|^2 \\
&= \|T_{\varphi_1}^t(\delta_z)\|^2 + \ldots + \|T_{\varphi_d}^t(\delta_z)\|^2 \\
&\leq \|T_{\varphi_1}^t\|^2\|\delta_z\|^2 + \ldots + \|T_{\varphi_d}^t\|^2\|\delta_z\|^2,
\end{aligned}$$

hence, $\|\varphi(z)\|_2 \leq \|T_\varphi^t\|$ for all $z = (z_1, \ldots, z_n) \in \mathbf{B}_n$. Therefore, $\varphi \in H^\infty(\mathbf{B}_n, \mathbf{C}^d)$ for all $i = 1, \ldots, d$.
By the same technical we prove that the result holds for S_φ. $\square$

Lemma 4. *If* $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_d) \in H^\infty(\mathbf{B}_n, \mathbf{C}^d)$, *then* $T_\varphi = (T_{\varphi_1}, T_{\varphi_2}, \ldots, T_{\varphi_d}) \in \mathcal{L}(H_v^\infty(\mathbf{B}_n, \mathbf{C}^d))$ *and* $S_\varphi \in \mathcal{L}(H_v^0(\mathbf{B}_n, \mathbf{C}^d))$. *Moreover*

$$\|T_\varphi\| = \|\varphi\|_\infty = \|S_\varphi\|.$$

Proof. We will show that $\|S_\varphi\| \le \|T_\varphi\| \le \|\varphi\|_\infty \le \|S_\varphi\|$. It is easy to show that

$$\|S_\varphi\| = \sup_{f \in H_v^0(\mathbf{B}_n, \mathbf{C}^d)} \frac{\|T_\varphi f\|_v}{\|f\|_v} \le \sup_{f \in H_v^\infty(\mathbf{B}_n, \mathbf{C}^d)} \frac{\|T_\varphi f\|_v}{\|f\|_v} = \|T_\varphi\|.$$

Then, $\|S_\varphi\| \le \|T_\varphi\|$. Now, let us show that $\|T_\varphi\| \le \|\varphi\|_\infty$. We have

$$\begin{aligned}
\|T_\varphi\| &= \sup_{\|f\|_v=1} \|T_\varphi f\|_v = \sup_{\|f\|_v=1} \|\varphi f\|_v = \sup_{\|f\|_v=1} \sup_{z \in \mathbf{B}_n} v(z)|\varphi(z) f(z)| \\
&\le \sup_{z \in \mathbf{B}_n} |\varphi(z)| \sup_{\|f\|_v=1} \sup_{z \in \mathbf{B}_n} v(z)|f(z)| \\
&= \sup_{z \in \mathbf{B}_n} |\varphi(z)| \sup_{\|f\|_v=1} \|f\|_v = \|\varphi\|_\infty.
\end{aligned}$$

Thus, the continuity of the operators. Finally, we check the last inequality. Let $z = (z_1, \ldots, z_N) \in \mathbf{B}_n$, and using the adjoint operator S_φ^t of S_φ, we have, by the same technical of the second part of the Proof of [5, Lemma 2.1.], $\|S_{\varphi_i}\| \ge |\varphi_i(z)|$ for every $z = (z_1, \ldots, z_N) \in \mathbf{B}_n$. Thus, $\|S_\varphi\| \ge \|\varphi(z)\|_2$ for every $z = (z_1, \ldots, z_N) \in \mathbf{B}_n$. Hence $\|\varphi\|_\infty \le \|S_\varphi\|$, which complete the proof. $\square$

In the following, our main interest is to know the properties of iterated operators.

Remark 2. *From the Lemma 4 and the identities*

$$T_\varphi^m = T_{\varphi_1}^{m_1} \ldots T_{\varphi_d}^{m_d} = T_{\varphi_1^{m_1}} \ldots T_{\varphi_d^{m_d}} = T_{\varphi_1^{m_1} \ldots \varphi_d^{m_d}} = T_{\varphi^m}, \tag{2.7}$$

and

$$S_\varphi^m = S_{\varphi_1}^{m_1} \ldots S_{\varphi_d}^{m_d} = S_{\varphi_1^{m_1} \ldots \varphi_d^{m_d}} = S_{\varphi^m}, \tag{2.8}$$

for every $m = (m_1, \ldots, m_d) \in \mathbf{N}^d$, where $\varphi^m = \varphi_1^{m_1} \ldots \varphi_d^{m_d}$ and for $\varphi_i \not\equiv 0$ we will use the notation $\varphi_i^0 = 1$. It follows that, for every $m = (m_1, \ldots, m_d) \in \mathbf{N}^d$, we have

$$\|T_\varphi^m\| = \|\varphi^m\|_\infty = \prod_{i=1}^d \|\varphi_i\|_\infty^{m_i} = \|S_\varphi^m\|.$$

For $\varphi = (\varphi_1, \varphi_2, \dots, \varphi_d) \in H^\infty(\mathbf{B}_n)^d$. It is easy to show that if there exists $i \in \{1, \dots, d\}$ such that $\varphi_i \equiv 0$ then T_φ (respectively S_φ) is power bounded even if the T_{φ_i} (respectively S_{φ_i}) are not all power bounded operators. In the Proposition below, we will characterize the power boundedness of multioperators of multiplication acting on $H_v^\infty(\mathbf{B}_n)$ and $H_v^0(\mathbf{B}_n)$.

Proposition 1. *Let $\varphi = (\varphi_1, \varphi_2, \dots, \varphi_d) \in H^\infty(\mathbf{B}_n)^d$ such that $\varphi_i \not\equiv 0$ for each $i = 1, \dots, d$. Then, the following assertions are equivalent*

(1) $\|\varphi_i\|_\infty \leq 1$ *for each* $i = 1, \dots, d$;

(2) $T_\varphi \in \mathcal{L}(H_v^\infty(\mathbf{B}_n))^d$ *is power bounded;*

(3) $S_\varphi \in \mathcal{L}(H_v^0(\mathbf{B}_n))^d$ *is power bounded;*

(4) $T_{\varphi_i} \in \mathcal{L}(H_v^\infty(\mathbf{B}_n))$ *is power bounded for each* $i = 1, \dots, d$;

(5) $S_{\varphi_i} \in \mathcal{L}(H_v^0(\mathbf{B}_n))$ *is power bounded for each* $i = 1, \dots, d$.

Proof. For (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3), it suffice the using Remark 2. For (1) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5), by the same technical of the proof of [5, Proposition 2.3.]. Now, we will prove (2) $\Rightarrow$ (1). Let $i \in \{1, \dots, d\}$ be arbitrary and let $m \in \mathbf{N}$ fixed. Consider $s = (s_1, \dots, s_d) \in \mathbf{N}^d$ such that $s_j = 0$ for $j \neq i$ and $s_i = m$. then, by using Remark 2, we have

$$\|\varphi_i\|_\infty^m = \prod_{j=1}^{d} \|\varphi_j\|_\infty^{s_j} = \|\varphi\|_\infty^s = \|T_\varphi^s\| \leq \sup_{k \in \mathbf{N}^d} \|T_\varphi^k\|.$$

Since T_φ is power bounded, $\sup_{k \in \mathbf{N}^d} \|T_\varphi^k\| < \infty$, thus $\|\varphi_i\|_\infty \leq 1$ for each $i = 1, \dots, d$.
A similar argument yields (3) $\Rightarrow$ (1). □

Let $\varphi = (\varphi_1, \varphi_2, \dots, \varphi_d) \in H^\infty(\mathbf{B}_n, \mathbf{C}^d)$. Then for each $f \in H_v^\infty(\mathbf{B}_n)$ and every $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbf{N}^d$, we have

$$\left((T_\varphi)_{[\alpha]} f\right)(z) = \frac{f(z)}{\prod_{i=1}^{d} \alpha_i} \sum_{(k)=(1)}^{(\alpha)} (\varphi(z))^k \text{ for each } z \in \mathbf{B}_n, \tag{2.9}$$

where,

$$\sum_{(k)=(1)}^{(\alpha)} (\varphi(z))^k := \prod_{i=1}^{d} \sum_{k=1}^{\alpha_i} (\varphi_i(z))^k. \tag{2.10}$$

Moreover, if $\varphi_i(z) \neq 1$ for all $i = 1, \ldots, d$, we obtain

$$\left((T_\varphi)_{[\alpha]} f\right)(z) = \prod_{i=1}^{d} \frac{\varphi_i(z) f(z)}{\alpha_i} \prod_{i=1}^{d} \frac{1-(\varphi_i(z))^{\alpha_i}}{1-(\varphi_i(z))}, \text{ for each } z \in \mathbf{B}_n \setminus \ker(1-\varphi_i),\ i = 1, \ldots, d. \tag{2.11}$$

It is well known that mean ergodicity does not imply power boundedness in general [7, §6]. But, it does for multiplication operators in weighted spaces of holomorphic functions, $d = 1$ in our notations, see [5]. In this paper we prove that this result holds for multioperators of multiplication acting on $H_v^\infty(\mathbf{B}_n)$ and $H_v^0(\mathbf{B}_n)$.

Proposition 2. *Let $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_d) \in H^\infty(\mathbf{B}_n, \mathbf{C}^d)$. If $T_\varphi \in \mathcal{L}(H_v^\infty(\mathbf{B}_n, \mathbf{C}^d))$ (respectively $S_\varphi \in \mathcal{L}(H_v^0(\mathbf{B}_n, \mathbf{C}^d))$) is mean ergodic, then T_φ (respectively S_φ) is power bounded.*

Proof. Since T_φ is mean ergodic. Thus, by Lemma 1, we have

$$\lim \frac{1}{\alpha_j} T_{\varphi_j}^{\alpha_j} = 0.$$

Hence, for the constant function 1 we have

$$\lim \left\| \frac{1}{\alpha_j} {\varphi_j}^{\alpha_j} \right\|_v = 0.$$

Let $z = (z_1, \ldots, z_N) \in \mathbf{B}_n$ be fixed, The inequality

$$\left\| \frac{1}{\alpha_j} {\varphi_j}^{\alpha_j} \right\|_v \geq v(z) \left| \frac{(\varphi_j(z))^{\alpha_j}}{\alpha_j} \right|$$

imply

$$\lim \frac{(\varphi_j(z))^{\alpha_j}}{\alpha_j} = 0$$

in $\mathbf{C}$. Evidently, $|\varphi_j(z)| \leq 1$, then $\|\varphi_j\|_\infty \leq 1$ for each $j = 1, \ldots, d$. by using Proposition 1, T_φ is power bounded. □

For the operator S_φ, the converse of Proposition 2 holds.

Proposition 3. *Let* $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_d) \in H^\infty(\mathbf{B}_n, \mathbf{C}^d)$. *Then* $S_\varphi \in \mathcal{L}\left(H_v^0(\mathbf{B}_n, \mathbf{C}^d)\right)$ *is mean ergodic if and only if* S_φ *is power bounded if and only if* $\|\varphi_i\|_\infty \leq 1$ *for each* $i = 1, \ldots, d$.

Proof. In view of Propositions 1 and 2 it suffices to show that $\|\varphi_i\|_\infty \leq 1$ for each $i = 1, \ldots, d$ implies mean ergodicity of S_φ.
First, suppose that there exists $z_0 \in \mathbf{B}_n$ such that $|\varphi_i(z_0)| = 1$ for each $i = 1, \ldots, d$. Then, by the Maximum Principle, for each $i = 1, \ldots, d$ there exists $w_i \in \mathbf{C}$ such that $|w_i| = 1$ and $\varphi_i(z) = w_i$ for all $z \in \mathbf{B}_n$. Then, $S_\varphi = (w_1 I, w_2 I, \ldots, w_d I)$.
If $w_i = 1$ for each $i = 1, \ldots, d$, then $(S_\varphi)_{[\alpha]} = (I, I, \ldots, I)$ for each $\alpha \in \mathbf{N}^d$ and

$$\lim \left\|(S_\varphi)_{[\alpha]} - I\right\| = 0.$$

If there exists $\Lambda \subseteq \{1, 2, \ldots, d\}$ such that $w_i \neq 1$ for all $i \in \Lambda$, then, by using formula (2.11), we obtain

$$(S_\varphi)_{[\alpha]} = \prod_{i \in \Lambda} \frac{w_i}{\alpha_i} \frac{1 - w_i^{\alpha_i}}{1 - w_i} I,$$

for each $\alpha \in \mathbf{N}^d$. Thus

$$\lim \left\|(S_\varphi)_{[\alpha]}\right\| = 0.$$

In each case we find that S_φ is (uniformly) mean ergodic.
Second, If $|\varphi_i(z)| < 1$ for each $i = 1, \ldots, d$ and all $z \in \mathbf{B}_n$. By the same technical of the second part of the proof of [5, Proposition 2.5.], we get that $\lim_{\alpha_i \longrightarrow \infty} (S_{\varphi_i})_{[\alpha_i]} = 0$ in $\mathcal{L}_s\left(H_v^0(\mathbf{B}_n)\right)$ which mean that S_{φ_i} is mean ergodic for each $i = 1, \ldots, d$. Therefore, S_φ is mean ergodic. □

In [5, Proposition 2.6.], J. Bonet characterized uniformly ergodic multiplication operator acting on a weighted space of holomorphic functions. In the following we will extend this result to d-tuple of multiplication operators acting on $H_v^0(\mathbf{B}_n)$. It is easy to show that if there exists $i \in \{1, \ldots, d\}$ such that $\varphi_i \equiv 0$, then S_φ is uniformly ergodic.

Proposition 4. *Let* $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_d) \in H^\infty(\mathbf{B}_n, \mathbf{C}^d)$ *such that* $\varphi_i \not\equiv 0$ *for each* $i = 1, \ldots, d$. *Then* $S_\varphi \in \mathcal{L}(H_v^0(\mathbf{B}_n, \mathbf{C}^d))$ *is uniformly ergodic if and only if* $\|\varphi_i\|_\infty \leq 1$ *for each* $i = 1, \ldots, d$ *and either*

1. *there exists* $\Lambda \subseteq \{1, 2, \ldots, d\}$ *and* $w = (w_1, \ldots, w_d) \in \mathbf{C}^d$ *with* $|w_i| = 1$ *for all* $i \in \Lambda$ *such that* $\varphi(z) = w$ *for all* $z \in \mathbf{B}_n$, *or*

2. $(1-\varphi)^{-1} = \left((1-\varphi_1)^{-1}, \ldots, (1-\varphi_d)^{-1}\right) \in H^\infty(\mathbf{B}_n, \mathbf{C}^d)$.

Proof. Assume that $\|\varphi_i\|_\infty \leq 1$ for each $i = 1, \ldots, d$. If φ satisfies 1., then, by the proof of the Proposition 3, S_φ is uniformly ergodic.
Now suppose that $\|\varphi_i\|_\infty \leq 1$ for each $i = 1, \ldots, d$ and φ satisfies 2.. Then there exists $\epsilon > 0$ such that $|1-\varphi_i(z)| \geq \epsilon$ for all $z \in \mathbf{B}_n$ and all $i = 1, \ldots, d$. Hence, by applying formula (2.11), for each $f \in H_v^0(\mathbf{B}_n)$, each $z \in \mathbf{B}_n$ and each $\alpha \in \mathbf{N}^d$, we get

$$\left|\left((S_\varphi)_{[\alpha]} f\right)(z)\right| = |f(z)| \prod_{i=1}^d \frac{|\varphi_i(z)|}{\alpha_i} \cdot \frac{|1-\varphi_i(z)^{\alpha_i}|}{|1-\varphi_i(z)|} \leq \frac{2^d |f(z)|}{\epsilon^d} \prod_{i=1}^d \frac{\|\varphi_i\|_\infty}{\alpha_i}.$$

Then, by taking suprema over $f \in H_v^0(\mathbf{B}_n)$ and $z \in \mathbf{B}_n$, we obtain

$$\left\|(S_\varphi)_{[\alpha]}\right\| \leq \frac{2^d}{\epsilon^d} \prod_{i=1}^d \frac{\|\varphi_i\|_\infty}{\alpha_i},$$

thus

$$\lim_{\alpha \longrightarrow \infty} \left\|(S_\varphi)_{[\alpha]}\right\| = 0.$$

Therefore, S_φ is uniformly ergodic.
For the converse, assume that S_φ is uniformly ergodic. Thus, by applying Proposition 3, $\|\varphi_i\|_\infty \leq 1$ for each $i = 1, \ldots, d$. Suppose that 1. does not hold. Then, by the Maximum Principle, $|\varphi_i(z)| < 1$ for each $i = 1, \ldots, d$ and all $z \in \mathbf{B}_n$. By the second part of the proof, we have the pointwise limit

$$\lim_{\alpha_i \longrightarrow \infty} (S_{\varphi_i})_{[\alpha_i]} = 0 \text{ for all } i = 1, \ldots, d.$$

It is routine to show that $\ker(I - S_{\varphi_i}) = \ker(S_{1-\varphi_i}) = \{0\}$ for each $i = 1, \ldots, d$. By Proposition 1, S_{φ_i} is power bounded. Thus

$$\lim_{\alpha \longrightarrow \infty} \left\| \frac{1}{\alpha_i} S_{\varphi_i}^{\alpha_i} \right\| = 0 \text{ for all } i = 1, \ldots, d.$$

Applying Lemma 2, S_φ is uniformly ergodic if and only if $I - S_{\varphi_i} = S_{1-\varphi_i}$ is an isomorphism for each $i = 1, \ldots, d$. Therefore, by [4, Lemma 2.3.],

$\frac{1}{1-\varphi_i} \in H^\infty(\mathbf{B}_n)$ for each $i = 1, \ldots, d$. □

The previous results show that results were the same for T_φ and S_φ. However, the last result showed some differences, which are confirmed in the next proposition which improves the one operator case [5, Proposition 2.8.].

Proposition 5. *Let $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_d) \in H^\infty(\mathbf{B}_n, \mathbf{C}^d)$ such that $\varphi_i \not\equiv 0$ and $\|\varphi_i\|_\infty \leq 1$ for each $i = 1, \ldots, d$. Then, the following assertions are equivalent*

(1) $T_\varphi \in \mathcal{L}(H_v^\infty(\mathbf{B}_n, \mathbf{C}^d))$ is mean ergodic;

(2) $T_\varphi \in \mathcal{L}(H_v^\infty(\mathbf{B}_n, \mathbf{C}^d))$ is uniformly mean ergodic;

(3) Either

i. there exists $\Lambda \subseteq \{1, 2, \ldots, d\}$ and $w = (w_1, \ldots, w_d) \in \mathbf{C}^d$ with $|w_i| = 1$ for all $i \in \Lambda$ such that $\varphi(z) = w$ for all $z \in \mathbf{B}_n$, or

ii. $(1-\varphi)^{-1} = \left((1-\varphi_1)^{-1}, \ldots, (1-\varphi_d)^{-1}\right) \in H^\infty(\mathbf{B}_n, \mathbf{C}^d)$.

Proof. (2) $\Leftrightarrow$ (3) Is proved by the same technical as S_φ in Proposition 4.
(2) $\Rightarrow$ (1) By definition.
(1) $\Rightarrow$ (2) By proposition 2, T_φ is power bounded. Then by Proposition 1, T_{φ_i} is power bounded for each $i = 1 \ldots d$, thus

$$\lim_{\alpha_i \longrightarrow \infty} \left\| \frac{1}{\alpha_i} T_{\varphi_i}^{\alpha_i} \right\| = 0 \text{ for all } i = 1, \ldots, d.$$

By Lusky [14, Theorem 1.1], $H_v^\infty(\mathbf{B}_n)$ is isomorphic to the Banach space ℓ^∞ which is Grothendieck space with the Dunford-Pettis property, see [11, p.121]. Fix $f \in H_v^\infty(\mathbf{B}_n)$. Since $\left\{(T_\varphi)_{[\alpha]}\right\}$ is convergent in $\mathcal{L}_s(H_v^\infty(\mathbf{B}_n))$, the sequence $\left\{(T_\varphi)_{[\alpha]} f\right\}$ is relatively $\sigma\left(H_v^\infty(\mathbf{B}_n), (H_v^\infty(\mathbf{B}_n))'\right)$-compact. Then, by Theorem 3, T_φ is uniformly mean ergodic. □

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