Proyecciones Journal of Mathematics
Vol. 42, N^o 4, pp. 1093-1104, August 2023.
Universidad Católica del Norte
Antofagasta - Chile

10.22199/issn.0717-6279-5780

A note on free determinantal hypersurface arrangements in $\mathbf{P}^{14}_{\mathbf{C}}$

Marek Janasz
Pedagogical University of Krakow, Poland
and
Paulina Wiśniewska
Pedagogical University of Krakow, Poland
Received : November 2022. Accepted : April 2023

Abstract

In the present note we study determinantal arrangements constructed with use of the 3*-minors of a* 3×5 *generic matrix of indeterminates. In particular, we show that certain naturally constructed hypersurface arrangements in* $\mathbf{P}^{14}_{\mathbf{C}}$ *are free.*

Keywords: *hypersurface arrangements, freeness, determinantal arrangements.*

Subclass: *14N20, 14C20.*

1. Introduction

The main aim of the present note is to find new examples of free hypersurfaces arrangements constructed as the so-called determinantal arrangements. These arrangements possess many interesting homological properties and some of them will be outlined. On the other side, computations related to these arrangements are very involving and this is probably the main reason why these objects are not well-studied yet. In the note we focus on determinantal arrangements constructed via the 3 minors of a 3×5 generic matrix. Before we present our main results, let us summarize briefly the basic concepts (see [4,5] for more details).

Let $\mathcal{C} \subset \mathbf{P}^n$ be an arrangement of reduced and irreducible hypersurfaces and let $\mathcal{C} = V(F)$, where $F = f_1 \cdots f_d$ with $\mathrm{GCD}(f_i, f_j) = 1$. In the note by $\mathrm{Der}(S) = S \cdot \partial_{x_0} \oplus ... \oplus S \cdot \partial_{x_n}$ the ring of polynomial derivations, where $S = \mathbf{K}[x_0, ..., x_n]$ and $\mathbf{K}$ is a field of characteristic zero. If we take $\theta \in \mathrm{Der}(S)$, then by Leibniz formula

$$\theta(f_1 \cdots f_d) = f_1 \cdot \theta(f_2 \cdots f_d) + f_2 \cdots f_d \cdot \theta(f_1).$$

Now we can define the ring of polynomial derivations tangent to $\mathcal{C}$ as

$$D(\mathcal{C}) = \{\theta \in \mathrm{Der}(S) \, : \, \theta(F) \in F \cdot S\}.$$

An inductive application of the Leibniz formula leads us to the following characterization of $D(\mathcal{C})$, namely

$$D(\mathcal{C}) = \{\theta \in \mathrm{Der}(S) \, : \, \theta(f_i) \in f_i \cdot S \text{ for } i \in \{1, ..., d\}\}.$$

We have the following (automatic) decomposition

$$D(\mathcal{C}) \simeq E \oplus D_0(\mathcal{C}),$$

where E is the Euler derivation and $D_0(\mathcal{C}) = \mathrm{syz}(J_F)$ is the module of syzygies for the Jacobian ideal $J_F = \langle \partial_{x_0} F, ..., \partial_{x_n} F \rangle$ of the defining polynomial F. The freeness of $\mathcal{C}$ boils down to a question of whether $\mathrm{pdim}(S/J_F) = 2$, which is equivalent to J_F being Cohen-Macaulay. One can show that a reduced hypersurfaces $\mathcal{C} \subset \mathbf{P}^n$ given by a homogeneous polynomial F is free if the following condition holds: the minimal resolution of the Milnor algebra $M(F) = S/J_F$ has the following short form

$$0 \to \bigoplus_{i=1}^{n} S(-d_i - (d-1)) \to S^{n+1}(-d+1) \to S,$$

where d is the deegre of F and the multiset of integers $(d_1, ..., d_n)$ with $d_1 \leq ... \leq d_n$ is called the set of exponents of $D_0(\mathcal{C})$, and we will denote it by $\exp(\mathcal{C})$.

The literature devoted to determinantal arrangements is not robust. In this context it is worth recalling a general result by Yim [6,Theorem 3.3], where he focuses on determinantal arrangements in $\mathbf{P}^{2n-1}_{\mathbf{C}}$ defined by the products of the 2-minors. For $i < j$ we denote the 2-minor of the matrix

$$N = \begin{pmatrix} x_1 & x_2 & x_3 & ... & x_n \\ y_1 & y_2 & y_3 & ... & y_n \end{pmatrix}$$

by $\triangle_{ij} = x_i y_j - x_j y_i$. Consider the arrangement $\mathcal{A}$ defined by the polynomial $F = \prod_{1 \leq i < j \leq n} \triangle_{ij}$ with $n \geq 3$. Then the arrangement $\mathcal{A}$ is free and a basis of $D(\mathcal{A})$ can be very explicitly described.

Our research is motivated by the following question [6,Question 3.4].

Question 1.1. *Let M be the $m \times n$ matrix of indeterminates, and let F be the product of all maximal minors of M. Is the arrangement defined by F free for any $n > m > 2$?*

Remark 1.2. *First of all, if $\mathcal{C} : F = 0$ is the hypersurface defined by the determinant of a generic 3×3 matrix of indeterminates, then $\mathcal{C}$ is far away from being free. Buchweitz and Mond in [1] showed that the arrangement defined by the product of the maximal minors of a generic $n \times (n+1)$ matrix of indeterminates is free (and it means that we have the freeness property when $m = 3$ and $n = 4$), so the first non-trivial and unsolved case (to the best of our knowledge) is when $m = 3$ and $n = 5$.*

Let us consider the 3×5 matrix of indeterminates

$$M = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ y_1 & y_2 & y_3 & y_4 & y_5 \\ z_1 & z_2 & z_3 & z_4 & z_5 \end{pmatrix}.$$

Now for a triple $\{i, j, k\}$ with $i < j < k$ we construct the 3-minor of M by taking i-th, j-th, and k-th column. Using the 3-minors we can get 10 hypersurfaces $H_l \subset \mathbf{P}^{14}$ which are given by the following defining polynomials:

$$f_1 = -x_3y_2z_1 + x_2y_3z_1 + x_3y_1z_2 - x_1y_3z_2 - x_2y_1z_3 + x_1y_2z_3,$$

$$f_2 = -x_4y_2z_1 + x_2y_4z_1 + x_4y_1z_2 - x_1y_4z_2 - x_2y_1z_4 + x_1y_2z_4,$$

$$f_3 = -x_4y_3z_1 + x_3y_4z_1 + x_4y_1z_3 - x_1y_4z_3 - x_3y_1z_4 + x_1y_3z_4,$$
$$f_4 = -x_4y_3z_2 + x_3y_4z_2 + x_4y_2z_3 - x_2y_4z_3 - x_3y_2z_4 + x_2y_3z_4,$$
$$f_5 = -x_5y_2z_1 + x_2y_5z_1 + x_5y_1z_2 - x_1y_5z_2 - x_2y_1z_5 + x_1y_2z_5,$$
$$f_6 = -x_5y_3z_1 + x_3y_5z_1 + x_5y_1z_3 - x_1y_5z_3 - x_3y_1z_5 + x_1y_3z_5,$$
$$f_7 = -x_5y_3z_2 + x_3y_5z_2 + x_5y_2z_3 - x_2y_5z_3 - x_3y_2z_5 + x_2y_3z_5,$$
$$f_8 = -x_5y_4z_1 + x_4y_5z_1 + x_5y_1z_4 - x_1y_5z_4 - x_4y_1z_5 + x_1y_4z_5,$$
$$f_9 = -x_5y_4z_2 + x_4y_5z_2 + x_5y_2z_4 - x_2y_5z_4 - x_4y_2z_5 + x_2y_4z_5,$$
$$f_{10} = -x_5y_4z_3 + x_4y_5z_3 + x_5y_3z_4 - x_3y_5z_4 - x_4y_3z_5 + x_3y_4z_5.$$

Using these 3-minors we would like to explore new examples of free divisors constructed as determinantal arrangements of hypersurfaces.

In order to show the freeness of such arrangements, we are going to use the following criterion due to Saito (see for instance [4, Theorem 8.1]). Let $\mathcal{C} \subset \mathbf{P}^n$ be a reduced effective divisor defined by a homogeneous equation $f = 0$. Now we define the graded module of all Jacobian syzygies as

$$\mathrm{AR}(f) := \left\{ r = (a_0, ..., a_n) \in S^{n+1} \,:\, a_0 \cdot \partial_{x_0}(f) + ... + a_n \cdot \partial_{x_n}(f) = 0 \right\}.$$

To each Jacobian relation $r \in \mathrm{AR}(f)$ one can associate a derivation

$$D(r) = a_0 \cdot \partial_{x_0} + ... + a_n \cdot \partial_{x_n}$$

that kills f, i.e., $D(r)(f) = 0$. One can additionally show that in fact $\mathrm{AR}(f)$ is isomorphic, as a graded S-module, with $D_0(\mathcal{C})$.

Theorem 1.3 ([4). *, Theorem 8.1] The homogeneous Jacobian syzygies $r_i = (a_{0i}, \ldots, a_{ni}) \in \mathrm{AR}(f)$ for $i \in \{1, ..., n\}$ form a basis of this S-module if and only if*

$$\phi(f) = c \cdot f,$$

where $\phi(f)$ is the determinant of the $(n+1) \times (n+1)$ matrix $\Phi(f) = (a_{ij})_{0 \le i,j \le n}$, where $r_0 = (a_{00}, ..., a_{n0}) := (x_0, ..., x_n)$ is the first column of the matrix and c is a non-zero constant.

Saito's criterion is a very powerful tool under the assumption that we have a set of potential candidates that might form a basis of $\mathrm{AR}(f)$, so our work boils down to finding appropriate sets of Jacobian relations that will lead us to a basis of $\mathrm{AR}(f)$ for a given arrangement $\mathcal{C} :\, f = 0$.

Here is our first result of the note.

Theorem 1.4. *Let us consider the following hypersurfaces arrangements*

$$\mathcal{C}_j : F_j = f_1 f_2 f_3 f_4 f_j \quad \text{for } j \in \{5, ..., 10\}.$$

Then $\mathcal{C}_j$ *is free with the exponents* $(\underbrace{1, ..., 1}_{14 \text{ times}})$.

Corollary 1.5. *In the setting of the above theorem, one has*

$$\operatorname{reg}(S/J_{F_j}) = 13$$

for each $j \in \{5, ..., 10\}$, *so we reach an upper bound for the regularity according to the content of [2, Proposition 2.7 (the regularity bound in the proof)], where by* $\operatorname{reg}(\cdot)$ *we mean the Castelnuovo-Mumford regularity*

Remark 1.6. *Of course, not every combination of* 5 *defining equations* f_i, f_j, f_k, f_l, f_m *leads to an example of a free determinantal arrangement. Consider* $\mathcal{A} : f_1 f_2 f_3 f_5 f_{10} = 0$, *then the minimal free resolution of the Milnor algebra* $M(F) = S/J_F$ *with* $F = f_1 f_2 f_3 f_5 f_{10}$ *has the following form:*

$$0 \to S^3(-19) \to S^4(-18) \oplus S^{13}(-15) \to S^{15}(-14) \to S,$$

so the projective dimension is equal to 3.

Moreover, not every choice of 5 *consecutive hyperplanes leads to a free arrangement. Consider* $\mathcal{B} : f_6 f_7 f_8 f_9 f_{10} = 0$, *then the minimal free resolution of the Milnor algebra has the following form*

$$0 \to S^3(-16) \to S^1(-18) \oplus S^{16}(-15) \to S^{15}(-14) \to S,$$

so $\mathcal{B}$ *is not free.*

The ultimate goal of the present paper is the understand whether we can expect a positive answer on a (sub)question devoted to the freeness of the full determinantal arrangement in $\mathbf{P}^{14}$.

Question 1.7. *Let us consider the following hypersurfaces arrangements* $\mathcal{H} : F = 0$ *defined by* $F = f_1 f_2 f_3 f_4 f_5 f_6 f_7 f_8 f_9 f_{10}$. *Is it true that* $\mathcal{H}$ *is free?*

Towards approaching the above question, we study mid-step defined arrangements, namely those having the defining equation $Q_k = f_1 f_2 f_3 f_4 f_5 f_k$ with $k \in \{6, 7, 8, 9, 10\}$. In particular, we can show the following results.

Theorem 1.8. *Let us consider the hypersurfaces arrangement*

$$\mathcal{H}_k \, : \, Q_k = 0$$

given by $Q_k = f_1 f_2 f_3 f_4 f_5 f_k$ *with* $k \in \{6, 7, 8, 9\}$. *Then* $\mathcal{H}_k$ *is free with the exponents* $(\underbrace{1, ..., 1}_{13 \text{ times}}, 4)$.

Corollary 1.9. *In the setting of the above theorem, one has*

$$\operatorname{reg}(S/J_{Q_k}) = 19$$

for each $k \in \{6, 7, 8, 9\}$, *so we reach an upper bound for the regularity according to the content of [2, Proposition 2.7 (the regularity bound in the proof)].*

Remark 1.10. *If we consider the arrangement* $\mathcal{H}_{10}$ *defined by* Q_{10}, *then it is not free since the minimal free resolution of the Milnor algebra has the following form:*

$$0 \to S^3(-22) \to S^5(-21) \oplus S^{12}(-18) \to S^{15}(-17) \to S,$$

which is quite surprising.

Our very ample numerical experiments suggest that the full determinantal arrangement $\mathcal{H} \, : \, f_1 \cdots f_{10} = 0$ should be free with the exponents $(\underbrace{1, ..., 1}_{9 \text{ times}}, \underbrace{4, ..., 4}_{5 \text{ times}})$. In order to verify our claim we also checked other larger arrangements of hyperplanes, for instance we can verify that $\mathcal{C} \, : \, f_1 f_2 f_3 f_4 f_7 f_8 f_9 = 0$ is free with the exponents $(\underbrace{1, ..., 1}_{12 \text{ times}}, 4, 4)$. However, the derivations of degree 4 seem to have no natural geometric or algebraic explanation, so it is very hard to find the basis of derivations for $\mathcal{H}$. We hope to solve this problem in the nearest future.

2. Proofs

We start with our proof of Theorem 1.4.

Proof. We are going to apply Saito's Criterion directly. In order to do so, we need to find a basis of the S-modules $\mathrm{AR}(F_j)$ for each $j \in \{5, ..., 10\}$. This means that in each case, we need to find 14 derivations for $\mathrm{AR}(F_j)$. Since for each choice of F_j the procedure goes along the same lines, let us focus on the first case $F_5 = f_1 f_2 f_3 f_4 f_5$.

We start with a group of (obvious to see) derivations, namely

$$\begin{aligned}
&\theta_1 = z_1 \cdot \partial_{x_1} + z_2 \cdot \partial_{x_2} + z_3 \cdot \partial_{x_3} + z_4 \cdot \partial_{x_4} + z_5 \cdot \partial_{x_5},\\
&\theta_2 = z_1 \cdot \partial_{y_1} + z_2 \cdot \partial_{y_2} + z_3 \cdot \partial_{y_3} + z_4 \cdot \partial_{y_4} + z_5 \cdot \partial_{y_5},\\
&\theta_3 = y_1 \cdot \partial_{x_1} + y_2 \cdot \partial_{x_2} + y_3 \cdot \partial_{x_3} + y_4 \cdot \partial_{x_4} + y_5 \cdot \partial_{x_5},\\
&\theta_4 = y_1 \cdot \partial_{z_1} + y_2 \cdot \partial_{z_2} + y_3 \cdot \partial_{z_3} + y_4 \cdot \partial_{z_4} + y_5 \cdot \partial_{z_5},\\
&\theta_5 = x_1 \cdot \partial_{y_1} + x_2 \cdot \partial_{y_2} + x_3 \cdot \partial_{y_3} + x_4 \cdot \partial_{y_4} + x_5 \cdot \partial_{y_5},\\
&\theta_6 = x_1 \cdot \partial_{z_1} + x_2 \cdot \partial_{z_2} + x_3 \cdot \partial_{z_3} + x_4 \cdot \partial_{z_4} + x_5 \cdot \partial_{z_5},\\
&\theta_7 = x_2 \cdot \partial_{x_5} + y_2 \cdot \partial_{y_5} + z_2 \cdot \partial_{z_5},\\
&\theta_8 = x_1 \cdot \partial_{x_5} + y_1 \cdot \partial_{y_5} + z_1 \cdot \partial_{z_5},\\
&\theta_9 = y_1 \cdot \partial_{y_1} + y_2 \cdot \partial_{y_2} + y_3 \cdot \partial_{y_3} + y_4 \cdot \partial_{y_4} + y_5 \cdot \partial_{y_5}\\
&-z_1 \partial_{z_1} - z_2 \partial_{z_2} - z_3 \partial_{z_3} - z_4 \partial_{z_4} - z_5 \partial_{z_5}.
\end{aligned}$$

We have additionally 5 non-obvious-to-see relations among the partials derivatives (we have found them with use of **Singular** [3]), namely:

$$\begin{aligned}
&\theta_{10} = 5x_5 \cdot \partial_{x_5} + 5y_5 \cdot \partial_{y_5} - z_1 \cdot \partial_{z_1} - z_2 \cdot \partial_{z_2} - z_3 \cdot \partial_{z_3} - z_4 \cdot \partial_{z_4} + 4z_5 \cdot \partial_{z_5},\\
&\theta_{11} = 5x_4 \cdot \partial_{x_4} + 5y_4 \cdot \partial_{y_4} - 3z_1 \cdot \partial_{z_1} - 3z_2 \cdot \partial_{z_2} - 3z_3 \cdot \partial_{z_3} + 2z_4 \cdot \partial_{z_4} - 3z_5 \cdot \partial_{z_5},\\
&\theta_{12} = 5x_3 \cdot \partial_{x_3} - 3y_1 \cdot \partial_{y_1} - 3y_2 \cdot \partial_{y_2} + 2y_3 \cdot \partial_{y_3} - 3y_4 \cdot \partial_{y_4} - 3y_5 \cdot \partial_{y_5} + 5z_3 \cdot \partial_{z_3},\\
&\theta_{13} = 5x_1 \cdot \partial_{x_1} + 5y_1 \cdot \partial_{y_1} + z_1 \cdot \partial_{z_1} - 4z_2 \cdot \partial_{z_2} - 4z_3 \cdot \partial_{z_3} - 4z_4 \cdot \partial_{z_4} - 4z_5 \cdot \partial_{z_5},
\end{aligned}$$

and

$$\begin{aligned}
\theta_{14} \quad &= 5x_2 \cdot \partial_{x_2} - 3y_1 \cdot \partial_{y_1} + 2y_2 \cdot \partial_{y_2} - 3y_3 \cdot \partial_{y_3} - 3y_4 \cdot \partial_{y_4} - 3y_5 \cdot \partial_{y_5}\\
&\quad -z_1 \cdot \partial_{z_1} + 4z_2 \cdot \partial_{z_2} - z_3 \cdot \partial_{z_3} - z_4 \cdot \partial_{z_4} - z_5 \cdot \partial_{z_5}.
\end{aligned}$$

Now we are going to construct Saito's matrix. In order to do so, let us write the coefficients of all θ_i's as the columns, and for the Euler derivation $E = \sum_{i=1}^{5} x_i \cdot \partial_{x_i} + \sum_{j=1}^{5} y_j \cdot \partial_{y_j} + \sum_{i=k}^{5} z_k \cdot \partial_{z_k}$ we write $r_0 = (x_1, ..., x_5, y_1, ..., y_5, z_1, ..., z_5)^t$.

Then we get the following matrix

$$
\mathrm{A} = \begin{pmatrix}
x_1 & z_1 & 0 & y_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5x_1 & 0 & 0 & 0 \\
x_2 & z_2 & 0 & y_2 & 0 & 0 & 0 & 0 & 0 & 5x_2 & 0 & 0 & 0 & 0 & 0 \\
x_3 & z_3 & 0 & y_3 & 0 & 0 & 0 & 0 & 5x_3 & 0 & 0 & 0 & 0 & 0 & 0 \\
x_4 & z_4 & 0 & y_4 & 0 & 0 & 0 & 5x_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
x_5 & z_5 & 0 & y_5 & 0 & 0 & 5x_5 & 0 & 0 & 0 & x_2 & 0 & x_1 & 0 & 0 \\
y_1 & 0 & z_1 & 0 & y_1 & 0 & 0 & 0 & -3y_1 & -3y_1 & 0 & 5y_1 & 0 & x_1 & 0 \\
y_2 & 0 & z_2 & 0 & y_2 & 0 & 0 & 0 & -3y_2 & 2y_2 & 0 & 0 & 0 & x_2 & 0 \\
y_3 & 0 & z_3 & 0 & y_3 & 0 & 0 & 0 & 2y_3 & -3y_3 & 0 & 0 & 0 & x_3 & 0 \\
y_4 & 0 & z_4 & 0 & y_4 & 0 & 0 & 5y_4 & -3y_4 & -3y_4 & 0 & 0 & 0 & x_4 & 0 \\
y_5 & 0 & z_5 & 0 & y_5 & 0 & 5y_5 & 0 & -3y_5 & -3y_5 & y_2 & 0 & y_1 & x_5 & 0 \\
z_1 & 0 & 0 & 0 & -z_1 & y_1 & -z_1 & -3z_1 & 0 & -z_1 & 0 & z_1 & 0 & 0 & x_1 \\
z_2 & 0 & 0 & 0 & -z_2 & y_2 & -z_2 & -3z_2 & 0 & 4z_2 & 0 & -4z_2 & 0 & 0 & x_2 \\
z_3 & 0 & 0 & 0 & -z_3 & y_3 & -z_3 & -3z_3 & 5z_3 & -z_3 & 0 & -4z_3 & 0 & 0 & x_3 \\
z_4 & 0 & 0 & 0 & -z_4 & y_4 & -z_4 & 2z_4 & 0 & -z_4 & 0 & -4z_4 & 0 & 0 & x_4 \\
z_5 & 0 & 0 & 0 & -z_5 & y_5 & 4z_5 & -3z_5 & 0 & -z_5 & z_2 & -4z_5 & z_1 & 0 & x_5
\end{pmatrix}.
$$

By straightforward computation, we obtain

$$\mathrm{Det}(A) = 9375 \cdot F_5,$$

which completes the proof. □

Now we are going to sketch the proof of Theorem 1.8.

Proof. Once again, we are going to apply Saito's Criterion. We focus on the case $k = 7$ since the other cases can be shown in an analogical way. The proof is heavily based on **Singular** computations and experiments. We can find polynomial derivations that preserves $\mathcal{H}$, namely

$$
\begin{aligned}
&\theta_1 = z_1 \cdot \partial_{x_1} + z_2 \cdot \partial_{x_2} + z_3 \cdot \partial_{x_3} + z_4 \cdot \partial_{x_4} + z_5 \cdot \partial_{x_5}, \\
&\theta_2 = z_1 \cdot \partial_{y_1} + z_2 \cdot \partial_{y_2} + z_3 \cdot \partial_{y_3} + z_4 \cdot \partial_{y_4} + z_5 \cdot \partial_{y_5}, \\
&\theta_3 = y_1 \cdot \partial_{x_1} + y_2 \cdot \partial_{x_2} + y_3 \cdot \partial_{x_3} + y_4 \cdot \partial_{x_4} + y_5 \cdot \partial_{x_5}, \\
&\theta_4 = y_1 \cdot \partial_{z_1} + y_2 \cdot \partial_{z_2} + y_3 \cdot \partial_{z_3} + y_4 \cdot \partial_{z_4} + y_5 \cdot \partial_{z_5}, \\
&\theta_5 = 3x_5 \cdot \partial_{x_5} + 3y_5 \cdot \partial_{y_5} - z_1 \cdot \partial_{z_1} - z_2 \cdot \partial_{z_2} - z_3 \cdot \partial_{z_3} - z_4 \cdot \partial_{z_4} + 2z_5 \cdot \partial_{z_5}, \\
&\theta_6 = 2x_4 \cdot \partial_{x_4} + 2y_4 \cdot \partial_{y_4} - z_1 \cdot \partial_{z_1} - z_2 \cdot \partial_{z_2} - z_3 \cdot \partial_{z_3} + z_4 \cdot \partial_{z_4} - z_5 \cdot \partial_{z_5}, \\
&\theta_7 = 3x_3 \cdot \partial_{x_3} + 3y_3 \cdot \partial_{y_3} - 2z_1 \cdot \partial_{z_1} - 2z_2 \cdot \partial_{z_2} + z_3 \cdot \partial_{z_3} - 2z_4 \cdot \partial_{z_4} - 2z_5 \cdot \partial_{z_5}, \\
&\theta_8 = 6x_2 \cdot \partial_{x_2} + 6y_2 \cdot \partial_{y_2} - 5z_1 \cdot \partial_{z_1} + z_2 \cdot \partial_{z_2} - 5z_3 \cdot \partial_{z_3} - 5z_4 \cdot \partial_{z_4} - 5z_5 \cdot \partial_{z_5}, \\
&\theta_9 = x_2 \cdot \partial_{x_5} + y_2 \cdot \partial_{y_5} + z_2 \cdot \partial_{z_5}, \\
&\theta_{10} = 3x_1 \cdot \partial_{x_1} + 3y_1 \cdot \partial_{y_1} + z_1 \cdot \partial_{z_1} - 2z_2 \cdot \partial_{z_2} - 2z_3 \cdot \partial_{z_3} - 2z_4 \cdot \partial_{z_4} - 2z_5 \cdot \partial_{z_5}, \\
&\theta_{11} = x_1 \cdot \partial_{y_1} + x_2 \cdot \partial_{y_2} + x_3 \cdot \partial_{y_3} + x_4 \cdot \partial_{y_4} + x_5 \cdot \partial_{y_5}, \\
&\theta_{12} = x_1 \cdot \partial_{z_1} + x_2 \cdot \partial_{z_2} + x_3 \cdot \partial_{z_3} + x_4 \cdot \partial_{z_4} + x_5 \cdot \partial_{z_5},
\end{aligned}
$$

$$
\begin{aligned}
\theta_{13} = y_1 \cdot \partial_{y_1} + y_2 \cdot \partial_{y_2} + y_3 \cdot \partial_{y_3} + y_4 \cdot \partial_{y_4} + y_5 \cdot \partial_{y_5} - z_1 \cdot \partial_{z_1} - z_2 \cdot \partial_{z_2} \\
\text{-z}_3 \cdot \partial_{z_3} - z_4 \cdot \partial_{z_4} - z_5 \cdot \partial_{z_5},
\end{aligned}
$$

and

$$\begin{aligned}
\theta_{14} \quad &= 3x_1x_3y_2z_2 \cdot \partial_{x_2} + 180x_1x_2y_3z_3 \cdot \partial_{x_3} + (192x_1x_2y_4z_3 - 9x_1x_3y_4z_2 \\
&+12x_1x_3y_2z_4 - 12x_1x_2y_3z_4) \cdot \partial_{x_4} + (15x_1x_3y_5z_2 - 12x_1x_3y_2z_5) \cdot \partial_{x_5} \\
&+(3x_3y_1y_2z_2 + 60x_2y_1y_2z_3 - 60x_1y_2^2z_3) \cdot \partial_{y_2} + (3x_3y_1y_3z_2 - 3x_1y_3^2z_2 \\
&-120x_3y_1y_2z_3 + 180x_2y_1y_3z_3 + 120x_1y_2y_3z_3) \cdot \partial_{y_3} + (12x_4y_1y_3z_2 \\
&-9x_3y_1y_4z_2 - 12x_1y_3y_4z_2 - 132x_4y_1y_2z_3 + 192x_2y_1y_4z_3 + 132x_1y_2y_4z_3 \\
&+12x_3y_1y_2z_4 - 12x_2y_1y_3z_4) \cdot \partial_{y_4} + (15x_3y_1y_5z_2 - 12x_5y_1y_3z_2 + 12x_1y_3y_5z_2 \\
&+60x_2y_1y_5z_3 - 60x_1y_2y_5z_3 - 12x_3y_1y_2z_5 + 12x_2y_1y_3z_5 - 12x_1y_2y_3z_5) \\
&\cdot \partial_{y_5} + (4x_1y_3z_2^2 - x_3y_1z_2^2 + 4x_3y_2z_1z_2 - 4x_2y_3z_1z_2 + 176x_2y_2z_1z_3 \\
&-204x_2y_1z_2z_3 + 28x_1y_2z_2z_3) \cdot \partial_{z_2} + (204x_2y_3z_1z_3 - 28x_3y_2z_1z_3 - 24x_2y_1z_3^2 \\
&+28x_1y_2z_3^2 + 181x_1y_3z_2z_3 - 181x_3y_1z_2z_3) \cdot \partial_{z_3} + (8x_4y_3z_1z_2 - 8x_3y_4z_1z_2 \\
&-40x_4y_2z_1z_3 + 216x_2y_4z_1z_3 - 180x_4y_1z_2z_3 + 180x_1y_4z_2z_3 + 12x_3y_2z_1z_4 \\
&-12x_2y_3z_1z_4 - x_3y_1z_2z_4 - 8x_1y_3z_2z_4 - 24x_2y_1z_3z_4 + 40x_1y_2z_3z_4) \cdot \partial_{z_4} \\
&+(16x_3y_5z_1z_2 - 16x_5y_3z_1z_2 - 16x_5y_2z_1z_3 + 192x_2y_5z_1z_3 - 72x_5y_1z_2z_3 \\
&+12x_1y_5z_2z_3 - 12x_3y_2z_1z_5 + 12x_2y_3z_1z_5 - x_3y_1z_2z_5 + 4x_1y_3z_2z_5 \\
&-132x_2y_1z_3z_5 + 16x_1y_2z_3z_5) \cdot \partial_{z_5}.
\end{aligned}$$

We claim that the set $\{E, \theta_1, \theta_2, ..., \theta_{14}\}$ gives us a basis for $D(\mathcal{H})$. It is enough to observe that the determinant of Saito's matrix A is equal to

$$\mathrm{Det}(A) = 23328 \cdot Q_7,$$

which completes the proof. □

3. Further numerical experiments

In order to understand better the geometry of determinantal hyperplane arrangements, we decided to investigate all possible arrangements $\mathcal{C}$ given by triplets $F_{ijk} = f_if_jf_k$ and given by 4-tuples $F_{ijkl} = f_if_jf_kf_l$ provided that the indices are pairwise distinct. Our first observation is the following.

Proposition 3.1. *Let* $\mathcal{C} \subset \mathbf{P}^{14}_{\mathbf{C}}$ *be a determinantal arrangement defined by the equation* $F_{ijk} = f_if_jf_k$*, where* $i, j, k \in \{1, ..., 10\}$ *and the indices are pairwise distinct. Then* $\mathcal{C}$ *is never free.*

Proof. Using a simple **Singular** routine, we examined all choices of indices, obtaining 120 determinantal arrangements, and in each case $\mathrm{pdim}(S/J_{F_{ijk}}) > 2$, which completes the proof. □

After that, we focused on determinantal arrangements $\mathcal{C}$ given by $F_{ijkl} = f_i f_j f_k f_l$. We have exactly 210 such arrangements, and among them, we have exactly 5 special arrangements, namely

a) $\mathcal{C}_1 \subset \mathbf{P}^{14}_{\mathbf{C}}$ given by F_{1234},

b) $\mathcal{C}_2 \subset \mathbf{P}^{14}_{\mathbf{C}}$ given by F_{1567},

c) $\mathcal{C}_3 \subset \mathbf{P}^{14}_{\mathbf{C}}$ given by F_{2589},

d) $\mathcal{C}_4 \subset \mathbf{P}^{14}_{\mathbf{C}}$ given by F_{36810},

e) $\mathcal{C}_5 \subset \mathbf{P}^{14}_{\mathbf{C}}$ given by F_{47910}.

These arrangements can be viewed as determinantal arrangements constructed as products of the maximal minors of appropriate generic 3×4 matrix of indeterminates. Thus, due to Buchweitz and Mond [1] arrangements $\mathcal{C}_i$ with $i \in \{1, 2, 3, 4, 5\}$ are free.

Another important class of hypersurface arrangements was introduced by Buśe, Dimca, Schenck, and Sticlaru, and such arrangements are called nearly-free.

Definition 3.2. *([2, Definition 2.6]) A reduced hypersurface $\mathcal{C} \subset \mathbf{P}^n_{\mathbf{C}}$ given by $F = 0$ is nearly-free if its Milnor algebra $M(F)$ admits a graded free resolution of the form*

$$0 \to S(-d_n - d) \to S(-d_n - d + 1) \oplus \left(\oplus_{i=0}^{n-1} S(-d_i - d + 1) \right) \to S^{n+1}(d+1) \to S$$

for some integers $d_0 \leq d_1 \leq d_2 \leq \ldots \leq d_n$, where $d = deg(F)$.

Next, we checked whether some of the remaining 205 determinantal arrangements $\mathcal{C}$ given by $F_{ijkl} = 0$ are nearly-free. It turns out that among 205 arrangements we found 58 having this peculiar property that their Milnor algebras $M(F_{ijkl})$ have the following minimal resolution:

$$0 \to S(-15) \to S^{15}(-12) \to S^{15}(-11) \to S,$$

so these are not nearly-free arrangements, but to some extend are close to them. Having a complete picture of the minimal resolution we can also calculate the regularity of $S/J_{F_{ijkl}}$ which is equal to

$$\mathrm{reg}(S/J_{F_{ijkl}}) = 12.$$

Acknowledgments

We would like to warmly thank Piotr Pokora for his useful comments and suggestions.

Paulina Wiśniewska was partially supported by the National Science Center (Poland) Sonata Grant Nr **2018/31/D/ST1/00177**.

References

[1] R.-O. Buchweitz and D. Mond, *Singularities and Computer Algebra.* London Mathematical Society Lecture Note Series, vol. 324, pp. 41-77. Cambridge: Cambridge University Press, 2006.

[2] L. Busé, A. Dimca, H. Schenck, and G. Sticlaru, "The Hessian polyno- mial and the Jacobian ideal of a reduced surface in P^{n}", *Advances in Mathematics,* vol. 392, Article ID 108035, 2021. doi: 10.1016/j.aim.2021.108035

[3] W. Decker, G.-M. Greuel, G. Pfister, and H. Schönemann, *Singular 4-1-1 – A computer algebra system for polynomial computations,* 2018. [On line]. Available: http://www.singular.uni-kl.de

[4] A. Dimca, *Hyperplane arrangements. An introduction. Universitext.* Cham: Springer, 2017.

[5] H. Schenck and Ş O. Tohǎneanu, "Freeness of Conic-Line Arrangements in P^{2}", *Commentarii Mathematici Helvetici,* vol. 84, pp. 235-258, 2009.

[6] A. Yim, "Homological properties of determinantal arrangements", J*ournal of Algebra,* vol. 471, pp. 220-239, 2017. doi: 10.1016/j.jalgebra.2016.09.019

Marek Janasz
Department of Mathematics,
Pedagogical University of Krakow,
ul. Podchorażych 2,
PL-30-084 Kraków,
Poland
e-mail: marek.janasz@up.krakow.pl
Corresponding author

and

Paulina Wiśniewska
Department of Mathematics and Doctoral School,
Pedagogical University of Krakow,
ul. Podchorażych 2,
PL-30-084 Kraków,
Poland
e-mail: wisniewska.paulina.m@gmail.com