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Graph folding and chromatic number

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Abstract

Given a connected graph G, identify two vertices if they have a common neighbor and then reduce the resulting multiple edges to simple edges. Repeat the process until the result is a complete graph. This process is called folding a graph.

We show here that any connected graph G which is not complete folds onto the connected graph K_p where $p = \chi(G)$, the chromatic number of G. Furthermore, the set of all integers p such that G folds onto K_p consist of consecutive integers, the smallest of which is $\chi(G)$.

One particular result of this study is that a sharp upper bound was obtained on the largest complete graph which a graph can be folded onto.

Keywords*: Chromatic number, Folding, Bipartite graph, Complete graph*

AMS 2020 Subject Classification: *05C50, 05C76.*

1. Introduction

Consider the path P_3 of order 3. We may imagine the two edges of the path to be pieces of thin rectangular plates 1 unit long hinged at the common vertex of the two edges of P_3. An illustration is given in Figure 1.

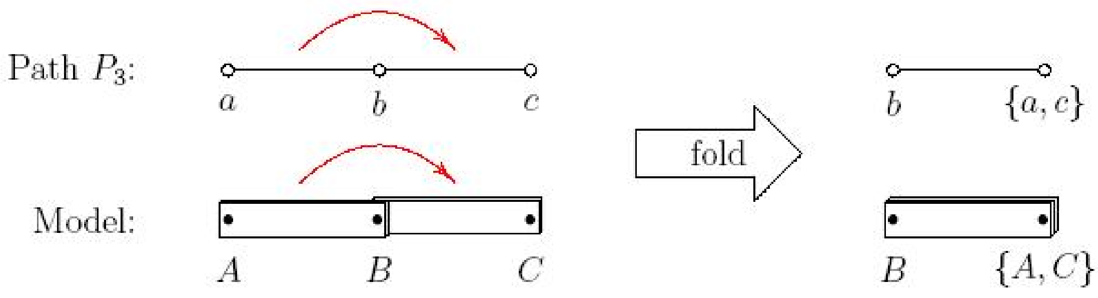

Figure 1: Folding the path P_3

As is usually done, the vertex-set of a graph G is denoted by $V(G)$, and its edge-set is denoted by $E(G)$. In a graph G, the neighbor-set of a vertex x, denoted by $N_G(x)$ or simply $N(x)$, is defined as the set of all vertices y adjacent to x.

Definition 1. *To fold a graph G means to identify two non-adjacent vertices x and y satisfying $N(x) \cap N(y) \neq \emptyset$ and then reducing multiple edges to simple edges. We say that G folds onto the graph H if H can be obtained from G by applying a sequence (possibly empty) of foldings.*

It follows from the definition that if G folds onto G_1 and G_1 folds onto G_2 then G folds onto G_2. Also, every graph folds onto itself (by the empty sequence of foldings).

Since any two distinct vertices in a complete graph are adjacent, then the complete graph K_n folds onto K_n only and to no other graph.

By a sequence of foldings any connected graph folds onto some complete graph because each time we fold, we decrease the number of pairs of non-adjacent vertices.

The above definition of folding can be found in [5]. It was first defined by the author in 2001. However, this concept was defined even earlier by

Cook and Evans in [3] in 1979 from the notion of graph homomorphisms. In their paper, they defined the following. An *elementary homomorphism* is an identification of two non-adjacent vertices. By the *identification* of non-adjacent vertices u and v in a graph G, we mean constructing another graph G' from G by removing the vertices u and v and all edges incident with u and v from G and adding a vertex w and edges from w to all vertices adjacent to either u or v. A *homomorphism* for us means a sequence of elementary homomorphisms, and a *simple fold* is an elementary homomorphism in which the identified vertices have a common neighbor. Kholy and Esawy [4] defined a similar concept in 2005 and they also used the term folding.

As an example, let us fold the Petersen graph shown in Figure 2 onto the complete graph K_5. Shown in the same figure is the result of identifying 1 and b, and reducing multiple edges to simple edges.

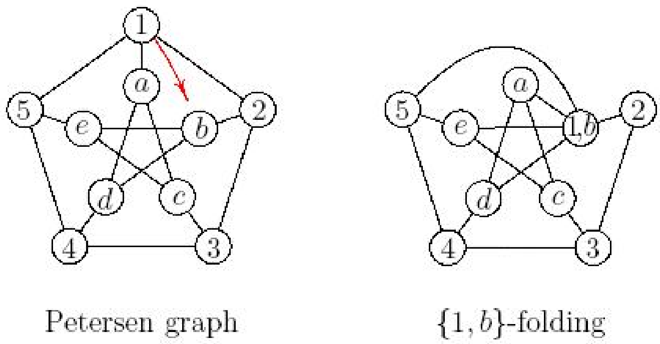

Figure 2: Folding the Petersen graph

For convenience let $\{x, y\}$-folding mean identifying the non-adjacent vertices x and y with $N(x) \cap N(y) \neq \emptyset$ and then reducing multiple edges to simple edges.

One way of folding the Petersen graph onto the complete graph K_5 is by applying the following sequence of foldings:

$$\begin{array}{ll} 1 & : \{1, b\}\text{-folding} \\ 2 & : \{2, c\}\text{-folding} \\ 3 & : \{3, d\}\text{-folding} \\ 4 & : \{4, e\}\text{-folding} \\ 5 & : \{5, a\}\text{-folding} \end{array}$$

We show here that if a connected graph has chromatic number p then it folds onto the complete graph K_p. Furthermore, orders of the complete graphs onto which connected graph folds form a set of consecutive integers, the smallest of which is the chromatic number of the graph.

2. Folding graphs onto complete graphs

An assignment of colors to the vertices of a graph from a set of k colors such that adjacent vertices get different colors is called a proper k-vertex coloring of the graph, or simply, a k-coloring.

More formally, a k-coloring of a graph G is a mapping $\lambda\colon V(G) \to \{1, 2, \ldots, k\}$ such that $\lambda(a) \neq \lambda(b)$ if $ab \in E(G)$. The smallest integer k for which a graph G has a k-coloring is called the chromatic number of G, denoted by $\chi(G)$. Note that if a graph G has a c-coloring, then $\chi(G) \leq c$.

If G is a graph with connected components $G_1, G_2, \ldots, G_c$ then a sequence of foldings will transform G to a disjoint union of c complete graphs because each G_i folds onto some complete graph by applying a sequence of foldings. For this reason, we confine ourselves to folding connected graphs.

Our first theorem gives the effect of folding on the chromatic number of a graph. The result in Theorem 1 was stated without proof as a corollary to an analogous Theorem from [6] in terms of elementary homomorphism. We provide an explicit proof of this result for completeness.

Theorem 1. *Let H be obtained from G by an $\{a, b\}$-folding. Then $\chi(G) \leq \chi(H) \leq 1 + \chi(G)$.*

Proof. Let H be obtained from G by an $\{a, b\}$-folding. Let $\chi(G) = p$ and $\chi(H) = q$. Consider a p-coloring α of G using p. Define a $(p+1)$-coloring β of H as follows: If $x \in V(H) \setminus \{a, b\}$, let $\beta(x) = \alpha(x)$. Color the vertex $v = \{a, b\}$ in H arising from the identification of a and b using the color $\beta(v) = p + 1$. Then $\chi(H) \leq p + 1$. Now consider a coloring γ of H using q colors. Define a q-coloring λ of G as follows: If $x \notin \{a, b\}$, let $\lambda(x) = \gamma(x)$. Let $\lambda(a) = \lambda(b) = \gamma(v)$, where v is the vertex obtained by identifying a and b. Then $\chi(G) \leq q$. Hence, $\chi(G) \leq \chi(H) \leq \chi(G) + 1$. □

The next theorem states that we can properly choose a folding so that the chromatic number is preserved.

Theorem 2. *If G is a connected graph which is not complete, then there exists vertices $a, b \in V(G)$ with $N(a) \cap N(b) \neq \emptyset$ such that the graph G_1 obtained from G by the $\{a, b\}$-folding satisfies $\chi(G_1) = \chi(G)$.*

Proof. Let G be a connected graph which is not complete. If G is a cycle C_n, then $n \geq 4$. Let $1, 2, \ldots, n$ be the consecutive vertices of C_n. By the $\{1, n-1\}$-folding followed by the $\{n, 2\}$-folding we get the cycle C_{n-2} in case $n > 4$ and K_2 in case $n = 4$. In both cases, the chromatic number is preserved.

If G is not a cycle, then $\chi(G) \leq \Delta$, the maximum degree of a vertex in G according to Brooks' theorem [1]. Let $\chi(G) = c$ and let $\lambda\colon V(G) \to \{1, 2, \ldots, c\}$ be a coloring of G. Denote by S_i the set of vertices in G with color i, $i = 1, 2, \ldots, c$. Let x be a vertex in G with $\deg(x) = \Delta$. Since $\Delta \geq c$, then x must have two neighbor a and b belong to a common set S_i. Let G_1 be the graph resulting from the $\{a, b\}$-folding. Clearly, G_1 is c-colorable and so $\chi(G_1) \leq c$. By Theorem 1, $\chi(G_1) \geq \chi(G)$. Therefore, $\chi(G_1) = \chi(G)$. □

Theorem 3. *A connected graph folds onto the complete graph K_2 if and only if it is bipartite. Furthermore, a connected bipartite graph folds only onto K_2 and to no other complete graph.*

Proof. Let G be a connected bipartite graph. Then $V(G)$ can be partitioned into two sets A and B such that every edge of G is of the form xy where $x \in A$ and $y \in B$. By Theorem 4, G folds onto K_2 since $\chi(G) = 2$. Let $a, b \in V(G)$ and $N(a) \cap N(b) \neq \emptyset$. Without loss of generality we may assume that both a and b belong to A. If G_1 is the result of the $\{a, b\}$-folding of G then clearly G_1 is bipartite. Thus, if $G = G_0 \to G_1 \to G_2 \to \cdots \to G_k = K_p$, then $p = 2$ since each G_i is a bipartite graph. Therefore, G folds onto K_2 and to no other complete graph.

Conversely, let G be a connected graph that folds onto K_2. By Theorem 1, $\chi(G) \leq \chi(K_2) = 2$. Therefore, $\chi(G) = 1$ or 2. If $\chi(G) = 1$, G is the trivial graph and it cannot fold onto K_2. It follows that $\chi(G) = 2$ and so G is bipartite. □

Figure 3 illustrates a way of folding the planar grid $P_3 \times P_3$ onto the complete graph K_2 using a sequence of 7 foldings.

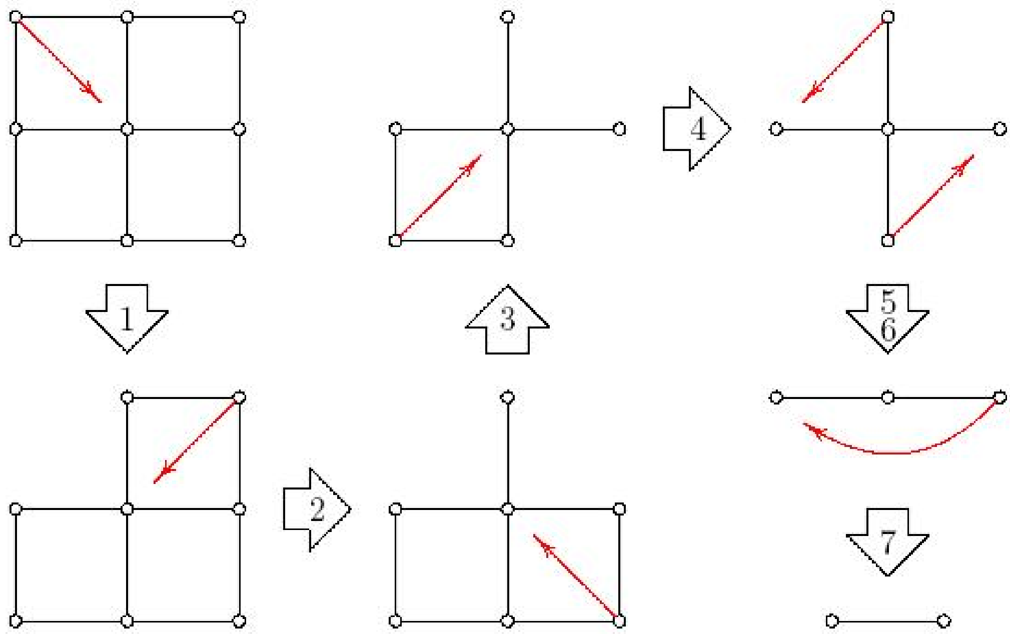

Figure 3: Folding $P_3 \times P_3$ onto K_2

It is known, and easy to verify, that the chromatic number of the Petersen graph is 3. The Petersen graph folds onto K_3, as our next result will show.

Theorem 4. *If G is a connected graph with chromatic number $\chi(G) = p$, then G folds onto the complete graph K_p and to no other smaller complete graph.*

Proof. The theorem is trivially true if G is complete so assume that G is not complete. By Lemma 2, there exist vertices a and b in the graph G with $N(a) \cap N(b) \neq \emptyset$ such that the graph G_1 obtained from G by the $\{a, b\}$-folding has the same chromatic number as G. We repeatedly apply the Lemma until we obtain a complete graph with chromatic number $\chi(G)$. Thus the order of the complete graph is $\chi(G)$. Theorem 1 states that folding a graph never decreases the chromatic number. Therefore, the smallest order of a complete graph onto which G folds is $\chi(G)$. $\square$

We have seen that the Petersen graph folds onto the complete graph K_5 and onto K_3 as well. Thus, in general, a connected graph folds onto some complete graph that is not unique.

Theorem 5. *Let G be a connected graph and let q be the largest order of a complete graph onto which G folds. Then for every integer r with $\chi(G) \le r \le q$, G folds onto K_r.*

Proof. Let $G = G_0 \to G_1 \to G_2 \to \cdots \to G_k = K_p$ be a sequence of foldings that folds G onto K_q. By Theorem 1, $\chi(G_i) \le \chi(G_{i+1}) \le \chi(G_i)+1$ for $i = 0, 1, \ldots, k-1$ Therefore for each r satisfying $\chi(G) \le r \le q$, there exists a graph G_i in the sequence $G_0, G_1, \ldots, G_k$ with $\chi(G_i) = r$. By Theorem 4, G_i folds onto K_r. Since G folds onto G_i and G_i folds onto K_r, then G folds onto K_r. $\square$

The sum of two graphs G and H, denoted by $G+H$, is formed by taking the disjoint union of G and H and then adding all edges of the form ab where $a \in V(G)$ and $b \in V(H)$.

Theorem 6. *Let G and H be connected graphs. Then $G + H$ folds onto K_n if and only if for some integers p and q with $p+q = n$, G folds onto K_p and H folds onto K_q.*

Proof. In the graph $G+H$, every vertex in G is adjacent to each vertex in H. Therefore folding $G + H$ is accomplished by identifying vertices in G or vertices in H only. $\square$

3. Largest complete folding of a graph

We have seen that every connected graph folds onto a set of complete graphs whose orders form a set of consecutive integers. Thus, there is a largest complete folding of a connected graph.

Theorem 7. *Let G be a connected graph of order n and size m. If G folds onto K_r then*

$$\chi(G) \le r \le \left\lfloor \frac{1}{2}\left(3 + \sqrt{8(m-n)+9}\right) \right\rfloor$$

Proof. We already know that $r \geq \chi(G)$ because of Theorem 4. Let $G = G_0 \rightarrow G_1 \rightarrow \cdots \rightarrow G_k = K_r$ be a folding of G onto some complete graph K_r. In each step, the order decreases by 1 and the size (number of edges) decreases by at least 1. Therefore we must have $m - k \geq \binom{n-k}{2}$. Setting $r = n - k$, we have $m - n + r \geq \binom{r}{2}$, which gives

$$\begin{aligned}
\binom{r}{2} &\leq m - n + r \\
r(r-1) &\leq 2(m-n) + 2r \\
r^2 - 3r - 2(m-n) &\leq 0 \\
r &\leq \frac{1}{2}\left(3 + \sqrt{8(m-n)+9}\right) \\
r &\leq \left\lfloor \frac{1}{2}\left(3 + \sqrt{8(m-n)+9}\right) \right\rfloor
\end{aligned}$$

This completes the proof of the theorem. □

The bound given in the theorem is sharp because its value is 5 for the Petersen graph and the Petersen graph folds onto K_5.

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