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On the total edge irregularity strength of certain classes of cycle related graphs

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Abstract

For a graph $G = (V, E)$, an edge irregular total $k-$labeling is a labeling of the vertices and edges of G with labels from the set $\{1, 2, ..., k\}$ such that any two different edges have distinct weights. The sum of the label of edge uv and the labels of vertices u and v determine the weight of the edge uv. The smallest possible k for which the graph G has an edge irregular total $k-$labeling is called the total edge irregularity strength of G. We determine the exact value of the total edge irregularity strength for some cycle related graphs.

Keywords: *edge irregularity strength; total edge irregularity strength; corona product of graphs; biwheel; triangular snake graph; graphs obtained by duplication of edges and vertices of cycles.*

Subject Classification Code: *05C78*

1. Introduction

Consider a simple undirected graph $G = (V, E)$. An edge irregular total labeling $f : V \cup E \rightarrow \{1, 2, ..., k\}$ of G was defined by Bača et al.[2] as the labeling of vertices and edges of G in such a way that for any different edges e and f, weights of e and f are distinct. The weight of an edge $e = xy$ is $wt(xy) = f(x) + f(xy) + f(y)$. The minimum k for which the graph G has an edge irregular total k labeling is called the total edge irregularity strength of the graph G, $tes(G)$.
The lower bound and upper bound of the total edge irregularity strength of any graph is given by Bača et al. in [2] as:

Theorem 1.1. *[2] Let $G = (V, E)$ be a graph with vertex set V and a non-empty edge set E. Then $\lceil \frac{|E|+2}{3} \rceil \leq tes(G) \leq |E|$.*

Theorem 1.2. *[2] For any graph G with maximum degree $\Delta = \Delta(G)$, $tes(G) \geq max\{\lceil \frac{|E|+2}{3} \rceil, \lceil \frac{\Delta(G)+1}{2} \rceil\}$.*

The total edge irregularity strength of path P_n, cycle C_n, star S_n, wheel W_n and friendship graph F_n were determined in [2] as: $tes(P_n) = tes(C_n) = \lceil \frac{n+2}{3} \rceil$; $tes(S_n) = \lceil \frac{n+1}{2} \rceil$; $tes(W_n) = \lceil \frac{2n+2}{3} \rceil$; and $tes(F_n) = \lceil \frac{3n+2}{3} \rceil$.
The following conjecture that gives the exact value of total edge irregularity strength of an arbitrary graph G was posed by Ivancŏ and Jendroĺ in [4]. They also proved the conjecture to be true for all trees.

Conjecture 1.3. *[4] Let G be an arbitrary graph different from K_5. Then $tes(G) = max\{\lceil \frac{|E|+2}{3} \rceil, \lceil \frac{\Delta+1}{2} \rceil\}$.*

Jendroĺ, Miškuf, and Soták in [5] found the exact value of total edge irregularity strength of complete graphs and complete bipartite graphs as: $tes(K_5) = 5$; for $n \geq 6$, $tes(K_n) = \lceil \frac{n^2-n-4}{6} \rceil$; and $tes(K_{m,n}) = \lceil \frac{mn+2}{3} \rceil$, for $n, m \geq 2$. Many other results on total edge irregularity strength can be found in [1], [6], [7], [8]. In this paper, we determine the total edge irregularity strength of the corona graph $C_n \odot mK_1$, double triangular snake graph $D(T_n)$, biwheel B_{2n} and the graphs obtained by duplicating edges and vertices of a cycle.
For the basic definitions of graph theory, we refer [3],[9].

Definition 1.4. *The corona product of two graphs G and H, denoted by, $G \odot H$, is a graph obtained by taking one copy of G (which has n vertices) and n copies $H_1, H_2, ..., H_n$ of H and then joining the i^{th} vertex of G to every vertex in H_i.*

Definition 1.5. *A triangular snake T_n is obtained from a path $w_1w_2...w_{n+1}$ by joining w_i and w_{i+1} to a new vertex v_i for $1 \leq i \leq n$. A double triangular snake $D(T_n)$ is obained from T_n by adding a new vertex u_i for $1 \leq i \leq n$ and edges u_iw_i and u_iw_{i+1} for $1 \leq i \leq n$.*

Definition 1.6. *Biwheel is a graph obtained from an even cycle of length $2n$ such that odd numbered vertices are joined to a new vertex and even numbered vertices are joined to another new vertex. It is denoted by B_{2n}.*

Definition 1.7. *Duplication of an edge $e = uv$ by a new vertex w in a graph G produces a new graph G' such that $N(w) = \{u, v\}$.*

Definition 1.8. *Duplication of a vertex v_k by a new edge $e = {v_k}'{v_k}"$ in a graph G produces a new graph G^* such that $N({v_k}') \cap N({v_k}") = v_k$.*

2. Total Edge Irregularity Strength of some Cycle Related Graphs.

Corona product $C_n \odot mK_1$ is the graph obtained by taking one copy of C_n and n copies of mK_1 and then joining each vertex of C_n to the vertices of mK_1. In the following theorem we determine the exact value of the total edge irregularity strength of corona product $C_n \odot mK_1$, $m \geq 1$.

Theorem 2.1. *Let $C_n \odot mK_1$ be a corona graph with $n \geq 3$ and $m \geq 1$. Then $tes(C_n \odot mK_1) = \lceil \frac{(m+1)n+2}{3} \rceil$.*

Proof. *The corona product $C_n \odot mK_1$ is a graph with the vertex set $V(C_n \odot mK_1) = \{u_i, {v_i}^j; 1 \leq i \leq n, 1 \leq j \leq m\}$ and edge set $E(C_n \odot mK_1) = \{u_iu_{i+1}; 1 \leq i \leq n-1\} \cup \{u_i{v_i}^j; 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{u_nu_1\}$. The graph $C_n \odot mK_1$ has $(m+1)n$ vertices and $(m+1)n$ edges. The maximum degree of $C_n \odot mK_1$, $\Delta(C_n \odot mK_1)$ is $m+2$. Then, by Theorem 1.2, $tes(C_n \odot mK_1) \geq$ max $\{\lceil \frac{(m+1)n+2}{3} \rceil, \lceil \frac{m+3}{2} \rceil\}$. Therefore, $tes(C_n \odot mK_1) \geq \lceil \frac{(m+1)n+2}{3} \rceil$. To prove the equality, it is sufficient to prove the existence of a total edge irregular $\lceil \frac{(m+1)n+2}{3} \rceil$ -labeling. Let $k = \lceil \frac{(m+1)n+2}{3} \rceil$.*
Define total labeling $f : V(C_n \odot mK_1) \cup E(C_n \odot mK_1) \rightarrow \{1, 2, ..., k\}$ as follows:

$$f(u_i) = \begin{cases} 1 + (i-1)\lceil \frac{m}{2} \rceil; & 1 \le i \le \lceil \frac{n}{2} \rceil \\ k - (n-i)\lceil \frac{m}{2} \rceil; & \lceil \frac{n}{2} \rceil < i \le n. \end{cases}$$

For $1 \le j \le m$;

$$f({v_i}^j) = \begin{cases} 1 + (i-1)\lceil \frac{m}{2} \rceil; & 1 \le i \le \lceil \frac{n}{2} \rceil \\ k - (n-i)\lceil \frac{m}{2} \rceil; & \lceil \frac{n}{2} \rceil < i \le n. \end{cases}$$

Case 1*: When* m *is even.*

$$f(u_i u_{i+1}) = \begin{cases} \frac{m}{2} + i; & 1 \le i < \lceil \frac{n}{2} \rceil \\ k - (\frac{m}{2} + n - i); & \lceil \frac{n}{2} \rceil < i < n, \ |E| \equiv 0 \ (mod\ 3) \\ k - (\frac{m}{2} + n - i - 1); & \lceil \frac{n}{2} \rceil < i < n, \ |E| \equiv 1 \ (mod\ 3) \\ k - (\frac{m}{2} + n - i + 1); & \lceil \frac{n}{2} \rceil < i < n, \ |E| \equiv 2 \ (mod\ 3). \end{cases}$$

$$f(u_{\lceil \frac{n}{2} \rceil} u_{\lceil \frac{n}{2} \rceil + 1}) = \begin{cases} \lfloor \frac{k}{2} \rfloor; & |E| \equiv 0 \ (mod\ 3) \\ \lceil \frac{k}{2} \rceil; & |E| \equiv 1 \ (mod\ 3) \\ \lceil \frac{k}{2} \rceil - 1; & |E| \equiv 2 \ (mod\ 3). \end{cases}$$

When n *is even,*

$$f(u_n u_1) = \begin{cases} \lfloor \frac{k}{2} \rfloor + 1; & |E| \equiv 0 \ (mod\ 3), \ |E| \equiv 1 \ (mod\ 3) \\ \frac{k}{2}; & |E| \equiv 2 \ (mod\ 3). \end{cases}$$

When n *is odd,*

$$f(u_n u_1) = \begin{cases} \lceil \frac{k}{2} \rceil + \frac{m}{2} + 1; & |E| \equiv 0 \ (mod\ 3), \ |E| \equiv 1 \ (mod\ 3) \\ \lceil \frac{k}{2} \rceil + \frac{m}{2}; & |E| \equiv 2 \ (mod\ 3). \end{cases}$$

For $1 \le j \le m$;

$$f(u_i {v_i}^j) = \begin{cases} i + j - 1; & 1 \le i \le \lceil \frac{n}{2} \rceil \\ k - (m-j) - (n-i) - 1; & \lceil \frac{n}{2} \rceil < i \le n, \ |E| \equiv 0 \ (mod\ 3) \\ k - (m-j) - (n-i); & \lceil \frac{n}{2} \rceil < i \le n, \ |E| \equiv 1 \ (mod\ 3) \\ k - (m-j) - (n-i) - 2; & \lceil \frac{n}{2} \rceil < i \le n, \ |E| \equiv 2 \ (mod\ 3). \end{cases}$$

Case 2*: When* m *is odd.*

$$f(u_i u_{i+1}) = \begin{cases} \lceil \frac{m}{2} \rceil; & 1 \le i < \lceil \frac{n}{2} \rceil \\ k - \lceil \frac{m}{2} \rceil; & \lceil \frac{n}{2} \rceil < i < n, \ |E| \equiv 0 \ (mod\ 3) \\ k - \lfloor \frac{m}{2} \rfloor; & \lceil \frac{n}{2} \rceil < i < n, \ |E| \equiv 1 \ (mod\ 3) \\ k - \lceil \frac{m}{2} \rceil - 1; & \lceil \frac{n}{2} \rceil < i < n, \ |E| \equiv 2 \ (mod\ 3). \end{cases}$$

$$f(u_{\lceil\frac{n}{2}\rceil}u_{\lceil\frac{n}{2}\rceil+1}) = \begin{cases} \lfloor\frac{k}{2}\rfloor; & |E| \equiv 0 \ (mod\ 3) \\ \frac{k}{2}; & |E| \equiv 1 \ (mod\ 3) \\ \frac{k}{2} - 1; & |E| \equiv 2 \ (mod\ 3). \end{cases}$$

When n is even,

$$f(u_n u_1) = \begin{cases} \lfloor\frac{k}{2}\rfloor + 1; & |E| \equiv 0 \ (mod\ 3),\ \ |E| \equiv 1 \ (mod\ 3) \\ \frac{k}{2}; & |E| \equiv 2 \ (mod\ 3). \end{cases}$$

When n is odd,

$$f(u_n u_1) = \begin{cases} \lfloor\frac{k}{2}\rfloor + \lceil\frac{m}{2}\rceil + 1; & |E| \equiv 0 \ (mod\ 3),\ \ |E| \equiv 1 \ (mod\ 3) \\ \frac{k}{2} + \lceil\frac{m}{2}\rceil; & |E| \equiv 2 \ (mod\ 3). \end{cases}$$

For $1 \leq j \leq m$;

$$f(u_i v_i^{\,j}) = \begin{cases} j; & 1 \leq i \leq \lceil\frac{n}{2}\rceil \\ k - m + j - 1; & \lceil\frac{n}{2}\rceil < i \leq n,\ \ |E| \equiv 0 \ (mod\ 3) \\ k - m + j; & \lceil\frac{n}{2}\rceil < i \leq n,\ \ |E| \equiv 1 \ (mod\ 3) \\ k - m + j - 2; & \lceil\frac{n}{2}\rceil < i \leq n,\ \ |E| \equiv 2 \ (mod\ 3). \end{cases}$$

Then the edge weight function $wt_f : E(C_n \odot mK_1) \rightarrow \{3, 4, ..., |E| + 2\}$ is as follows:

$$wt_f(u_i v_i^{\,j}) = \begin{cases} (i-1)(m+1) + j + 2; & 1 \leq i \leq \lceil\frac{n}{2}\rceil,\ \ 1 \leq j \leq m \\ i(m+1) - (m-j) + 2; & \lceil\frac{n}{2}\rceil < i \leq n,\ \ 1 \leq j \leq m. \end{cases}$$

$$wt_f(u_i u_{i+1}) = \begin{cases} (m+1)i + 2; & 1 \leq i < \lceil\frac{n}{2}\rceil \\ (m+1)i + 3; & \lceil\frac{n}{2}\rceil < i < n. \end{cases}$$

$$wt_f(u_{\lceil\frac{n}{2}\rceil}u_{\lceil\frac{n}{2}+1\rceil}) = \lceil\frac{n}{2}\rceil(m+1) + 2.$$

$$wt_f(u_n u_1) = \lceil\frac{n}{2}\rceil(m+1) + 3.$$

The edge weight function wt_f is bijective and hence f is an edge irregular total $k-$ labeling. Therefore, $tes(C_n \odot mK_1) \leq k$.

□

Theorem 2.2. *Let $D(T_n)$ be a double triangular snake graph. Then for $n \geq 1$, $tes(D(T_n)) = \lceil\frac{5n+2}{3}\rceil$.*

Proof. *Double triangular snake graph $D(T_n)$ is a graph with vertex set $V(D(T_n)) = \{w_i; 1 \leq i \leq n+1\} \cup \{u_i; 1 \leq i \leq n\} \cup \{v_i; 1 \leq i \leq n\}$ and edge set $E(D(T_n)) = \{w_iw_{i+1}; 1 \leq i \leq n\} \cup \{u_iw_i; 1 \leq i \leq n\} \cup \{v_iw_i; 1 \leq i \leq n\} \cup \{u_iw_{i+1}; 1 \leq i \leq n\} \cup \{v_iw_{i+1}; 1 \leq i \leq n\}$. The graph $D(T_n)$ has $3n+1$ vertices and $5n$ edges. Then by Theorem (1.2), $tes(D(T_n)) \geq \lceil \frac{5n+2}{3} \rceil$. To prove the equality, we find the existence of a total edge irregular $\lceil \frac{5n+2}{3} \rceil$- labeling. Let $k = \lceil \frac{5n+2}{3} \rceil$.*
Define total labeling $f : V(D(T_n)) \cup E(D(T_n)) \to \{1, 2, ..., k\}$ as follows:

Case 1: *When $n = 1, 2$.*

$$f(w_i) = \begin{cases} i; & 1 \leq i \leq n \\ i+1; & i = n+1. \end{cases}$$

$$f(u_i) = 2i - 1; \quad 1 \leq i \leq n.$$

$$f(v_i) = 2i; \quad 1 \leq i \leq n.$$

$$f(v_iw_i) = f(u_iw_i) = 2i - 1; \quad 1 \leq i \leq n.$$

For $1 \leq i \leq n$;

$$f(w_iw_{i+1}) = \begin{cases} i; & \text{if } \ n = 1 \\ 2i; & \text{if } \ n = 2. \end{cases}$$

$$f(u_iw_{i+1}) = f(v_iw_{i+1}) = \begin{cases} i+1; & \text{if } \ n = 1 \\ i+2; & \text{if } \ n = 2. \end{cases}$$

Case 2: *When $n \geq 3$.*

$$f(w_i) = \begin{cases} i; & 1 \leq i \leq \lfloor \frac{n}{2} \rfloor + 1 \\ k - n + i - 1; & \lfloor \frac{n}{2} \rfloor + 1 < i \leq n+1. \end{cases}$$

$$f(u_i) = \begin{cases} 2i - 1; & 1 \leq i \leq \lfloor \frac{n}{2} \rfloor \\ k - 2(n-i) - 1; & \lfloor \frac{n}{2} \rfloor < i \leq n. \end{cases}$$

$$f(v_i) = \begin{cases} 2i; & 1 \leq i \leq \lfloor \frac{n}{2} \rfloor \\ k - 2(n-i); & \lfloor \frac{n}{2} \rfloor < i \leq n. \end{cases}$$

$$f(w_iw_{i+1}) = \begin{cases} 3i - 1; & 1 \leq i \leq \lfloor \frac{n}{2} \rfloor \\ 3i + 2(n-k) + 1; & \lfloor \frac{n}{2} \rfloor + 1 < i \leq n. \end{cases}$$

$$f(v_iw_i) = f(u_iw_i) = \begin{cases} 2i-1; & 1 \le i \le \lfloor \frac{n}{2} \rfloor \\ 3n-2k+2i; & \lfloor \frac{n}{2} \rfloor + 1 < i \le n. \end{cases}$$

$$f(u_iw_{i+1}) = f(v_iw_{i+1}) = \begin{cases} 2i+1; & 1 \le i \le \lfloor \frac{n}{2} \rfloor \\ 3n-2k+2i+2; & \lfloor \frac{n}{2} \rfloor + 1 \le i \le n. \end{cases}$$

For $i = \lfloor \frac{n}{2} \rfloor + 1$ *and* n *is even;*

$$f(w_iw_{i+1}) = \begin{cases} \lceil \frac{k}{2} \rceil + 1; & n \equiv 0 \ (mod\ 3), n \equiv 1 \ (mod\ 3) \\ \frac{k}{2} + 2; & n \equiv 2 \ (mod\ 3). \end{cases}$$

For $i = \lfloor \frac{n}{2} \rfloor + 1$ *and* n *is odd;*

$$f(w_iw_{i+1}) = \begin{cases} \lfloor \frac{k}{2} \rfloor; & n \equiv 0 \ (mod\ 3), n \equiv 1 \ (mod\ 3) \\ \lceil \frac{k}{2} \rceil; & n \equiv 2 \ (mod\ 3). \end{cases}$$

For $i = \lfloor \frac{n}{2} \rfloor + 1$ *and* n *is even;*

$$f(v_iw_i) = f(u_iw_i) = \begin{cases} \lceil \frac{k}{2} \rceil + \frac{n}{2} - 1; & n \equiv 0 \ (mod\ 3), n \equiv 1 \ (mod\ 3) \\ \frac{k}{2} + \frac{n}{2}; & n \equiv 2 \ (mod\ 3). \end{cases}$$

For $i = \lfloor \frac{n}{2} \rfloor + 1$ *and* n *is odd;*

$$f(v_iw_i) = f(u_iw_i) = \begin{cases} \lfloor \frac{k}{2} \rfloor + \lfloor \frac{n}{2} \rfloor - 1; & n \equiv 0 \ (mod\ 3), n \equiv 1 \ (mod\ 3) \\ \lfloor \frac{k}{2} \rfloor + \lfloor \frac{n}{2} \rfloor; & n \equiv 2 \ (mod\ 3). \end{cases}$$

Then the edge weight function $wt_f : E(D(T_n)) \rightarrow \{3, 4, ..., |E| + 2\}$ *is as follows:*

$$wt_f(u_iw_i) = 5i - 2; \quad 1 \le i \le n.$$

$$wt_f(v_iw_i) = 5i - 1; \quad 1 \le i \le n.$$

$$wt_f(w_iw_{i+1}) = 5i; \quad 1 \le i \le n.$$

$$wt_f(u_iw_{i+1}) = 5i + 1; \quad 1 \le i \le n.$$

$$wt_f(v_iw_{i+1}) = 5i + 2; \quad 1 \le i \le n.$$

The edge weight function wt_f *is bijective and hence* f *is a edge irregular total* $k-$ *labeling. Therefore,* $tes(D(T_n)) \le k$. □

Theorem 2.3. *Let* B_{2n} *be a biwheel. Then for* $n \ge 2$, $tes(B_{2n}) = \lceil \frac{4n+2}{3} \rceil$.

Proof. *Biwheel B_{2n} is a graph with vertex set $V(B_{2n}) = \{u_i; 1 \leq i \leq 2n\} \cup \{v_1, v_2\}$ and edge set $E(B_{2n}) = \{u_iu_{i+1}; 1 \leq i \leq 2n\} \cup \{u_{2n}u_1\} \cup \{v_1u_i; i \text{ is odd}\} \cup \{v_2u_i; i \text{ is even}\}$. The graph B_{2n} has $2n+2$ vertices and $4n$ edges. Then by Theorem (1.2), $tes(B_{2n}) \geq \lceil \frac{4n+2}{3} \rceil$. Let $k = \lceil \frac{4n+2}{3} \rceil$. Then $tes(B_{2n}) \geq k$. To prove the equality, we find the existence of a total edge irregular $k-$labeling.*
Define total labeling $f : V(B_{2n}) \cup E(B_{2n}) \to \{1, 2, ..., k\}$ as follows:

$$f(u_i) = \begin{cases} \lfloor \frac{i}{2} \rfloor + 1; & 1 \leq i \leq n \\ k - n + \lfloor \frac{i}{2} \rfloor; & n < i \leq 2n. \end{cases}$$

$$f(v_1) = 1, f(v_2) = k.$$

$$f(u_iu_{i+1}) = \begin{cases} \lceil \frac{n}{2} \rceil; & 1 \leq i < n \\ k - \lceil \frac{n}{2} \rceil; & n < i < 2n \text{ and } n \equiv 0 \ (mod\ 3) \\ k - \lceil \frac{n}{2} \rceil + 1; & n < i < 2n \text{ and } n \equiv 1 \ (mod\ 3) \\ k - \lceil \frac{n}{2} \rceil - 1; & n < i < 2n \text{ and } n \equiv 2 \ (mod\ 3). \end{cases}$$

$$f(u_nu_{n+1}) = \begin{cases} \lfloor \frac{k}{2} \rfloor; & n \equiv 0 \ (mod\ 3) \text{ and } n \equiv 1 \ (mod\ 3) \\ \frac{k}{2} - 1; & n \equiv 2 \ (mod\ 3). \end{cases}$$

$$f(u_{2n}u_1) = \begin{cases} \lfloor \frac{k}{2} \rfloor + 1; & n \equiv 0 \ (mod\ 3) \text{ and } n \equiv 1 \ (mod\ 3) \\ \frac{k}{2}; & n \equiv 2 \ (mod\ 3). \end{cases}$$

For $1 \leq i \leq n$ and i is odd, $f(v_1u_i) = 1$.
For $n < i \leq 2n$ and i is odd,

$$f(v_1u_i) = \begin{cases} \lfloor \frac{k}{2} \rfloor; & n \equiv 0 \ (mod\ 3) \text{ and } n \equiv 1 \ (mod\ 3) \\ \frac{k}{2} - 1; & n \equiv 2 \ (mod\ 3). \end{cases}$$

For $1 \leq i \leq n$ and i is even,

$$f(v_2u_i) = \begin{cases} \lfloor \frac{k}{2} \rfloor + 1; & n \equiv 0 \ (mod\ 3) \text{ and } n \equiv 1 \ (mod\ 3) \\ \frac{k}{2}; & n \equiv 2 \ (mod\ 3). \end{cases}$$

For $n < i \leq 2n$ and i is even,

$$f(v_2u_i) = \begin{cases} k - 1; & n \equiv 0 \ (mod\ 3) \\ k; & n \equiv 1 \ (mod\ 3) \\ k - 2; & n \equiv 2 \ (mod\ 3). \end{cases}$$

Then the edge weight function $wt_f : E(B_{2n}) \to \{3, 4, ..., |E| + 2\}$ is as follows:

For $1 \leq i \leq n$ *and* i *is odd;* $wt_f(v_1u_i) = \lfloor \frac{i}{2} \rfloor + 3$.
For $1 \leq i < n$; $wt_f(u_iu_{i+1}) = i + \lceil \frac{n}{2} \rceil + 2$.
For $n < i \leq 2n$ *and* i *is odd;* $wt_f(v_1u_i) = \lfloor \frac{i}{2} \rfloor + n + 2$.
$wt_f(u_nu_{n+1}) = 2n + 2$.
$wt_f(u_{2n}u_1) = 2n + 3$.
For $1 \leq i \leq n$ *and* i *is even;* $wt_f(v_2u_i) = \lfloor \frac{i}{2} \rfloor + 2n + 3$.
For $n < i < 2n$; $wt_f(u_iu_{i+1}) = i + \lfloor \frac{3n}{2} \rfloor + 3$.
For $n < i \leq 2n$ *and* i *is even;* $wt_f(v_1u_i) = \lfloor \frac{i}{2} \rfloor + 3n + 2$.

The edge weight function wt_f *is bijective and hence* f *is a edge irregular total* $k-$ *labeling. Therefore,* $tes(B_{2n}) \leq k$. $\square$

Theorem 2.4. *Let* ${C_n}'$ *be the graph obtained by duplicating each edge by a vertex in a cycle* C_n*. Then* $tes({C_n}') = n + 1$.

Proof. *Let* $\{u_1, u_2, ..., u_n\}$ *be the vertices of the cycle* C_n *and* $\{v_1, v_2, ..., v_n\}$ *be the added vertices to obtain* ${C_n}'$*. Then* $E({C_n}') = \{u_iu_i + 1; 1 \leq i < n\} \cup \{u_nu_1\} \cup \{u_iv_i; 1 \leq i \leq n\} \cup \{u_{i+1}v_i; 1 \leq i < n\} \cup \{u_1v_n\}$.
The graph ${C_n}'$ *has* $3n$ *edges and* $2n$ *vertices. By Theorem 1.2,* $tes({C_n}') \geq \lceil \frac{3n+2}{3} \rceil = n+1$*. To prove the equality, we define a total edge irregular* $n+1-$ *labeling. Define total labeling* $f : V({C_n}') \cup E({C_n}') \rightarrow \{1, 2, ..., n + 1\}$ *as follows:*

$$f(u_i) = i; \qquad 1 \leq i \leq n.$$

$$f(v_i) = \begin{cases} 1; & i = 1 \\ i + 1; & 1 < i \leq n. \end{cases}$$

$$f(u_iu_{i+1}) = \begin{cases} i; & 1 \leq i < \lceil \frac{n}{2} \rceil \\ i + 2; & \lceil \frac{n}{2} \rceil \leq i < n. \end{cases}$$

$$f(u_nu_1) = f(u_1v_n) = \begin{cases} \frac{n}{2} - 1; & n \text{ is even} \\ \lceil \frac{n}{2} \rceil; & n \text{ is odd.} \end{cases}$$

$$f(u_iv_i) = \begin{cases} 1; & i = 1 \\ i - 1; & 1 < i < \lceil \frac{n}{2} \rceil \\ i + 1; & \lceil \frac{n}{2} \rceil \leq i \leq n. \end{cases}$$

$$f(u_{i+1}v_i) = \begin{cases} 2; & i = 1 \\ i; & 1 < i < \lceil \frac{n}{2} \rceil \\ i+2; & \lceil \frac{n}{2} \rceil \leq i \leq n. \end{cases}$$

Then the edge weight function $wt_f : E(C_n{}') \to \{3, 4, ..., |E|+2\}$ *is as follows:*
$wt(u_1v_1) = 3$.
For $1 < i < \lceil \frac{n}{2} \rceil$; $wt_f(u_iv_i) = 3i$.
For $\lceil \frac{n}{2} \rceil \leq i \leq n$; $wt_f(u_iv_i) = 3i + 2$.
For $1 \leq i < \lceil \frac{n}{2} \rceil$; $wt_f(u_iu_{i+1}) = 3i + 1$.
For $\lceil \frac{n}{2} \rceil \leq i < n$; $wt_f(u_iu_{i+1}) = 3i + 3$.
$wt_f(u_2v_1) = 5$.
$wt_f(u_nu_1) = 3\lceil \frac{n}{2} \rceil$.
$wt_f(u_1v_n) = 3\lceil \frac{n}{2} \rceil + 1$.
For $1 < i < \lceil \frac{n}{2} \rceil$; $wt_f(u_{i+1}v_i) = 3i + 2$.
For $\lceil \frac{n}{2} \rceil \leq i < n$; $wt_f(u_{i+1}v_i) = 3i + 4$.

The edge weight function wt_f *is bijective and hence* f *is a edge irregular total* $k-$ *labeling. Therefore,* $tes(C_n{}') \leq k$. □

Theorem 2.5. *Let* $C_n{}^*$ *be the graph obtained by duplicating each vertex by an edge in a cycle* C_n. *Then* $tes(C_n{}^*) = \lceil \frac{4n+2}{3} \rceil$.

Proof. *Let* $\{u_1, u_2, ..., u_n\}$ *be the vertices of the cycle* C_n *and* $\{v_1, v_2, ..., v_{2n}\}$ *be the added vertices to obtain* $C_n{}^*$. *Then* $E(C_n{}^*) = \{u_iu_{i+1}; 1 \leq i < n\} \cup \{u_nu_1\} \cup \{u_iv_{2i-1}; 1 \leq i \leq n\} \cup \{u_iv_{2i}; 1 \leq i \leq n\} \cup \{v_{2i-1}v_{2i}; 1 \leq i \leq n\}$. *The graph* $C_n{}^*$ *has* $4n$ *edges and* $3n$ *vertices. By Theorem 1.2,* $tes(C_n{}^*) \geq \lceil \frac{4n+2}{3} \rceil$. *Let* $k = \lceil \frac{4n+2}{3} \rceil$. *To prove the equality, we prove the existence of a total edge irregular* $k-$ *labeling. Define total labeling* $f : V(C_n{}^*) \cup E(C_n{}^*) \to \{1, 2, ..., k\}$ *as follows:*

Case 1: *When* n *is even.*

$$f(u_i) = \begin{cases} i; & 1 \leq i \leq \frac{n}{2} \\ k-n+i, & \frac{n}{2} < i \leq n. \end{cases}$$

For $1 \leq i \leq n$; $f(v_{2i-1}) = i$ *and* $f(v_{2i}) = k - n + i$.

(i) $n \equiv 0 \pmod 3$ *and* $n \equiv 1 \pmod 3$.

$$f(u_iu_{i+1}) = \begin{cases} 1; & 1 \le i < \frac{n}{2} \\ \lfloor \frac{k}{2} \rfloor + 1; & i = \frac{n}{2} \\ k-1; & \frac{n}{2} < i < n \text{ and } n \equiv 0 \pmod 3 \\ k; & \frac{n}{2} < i < n \text{ and } n \equiv 1 \pmod 3. \end{cases}$$

$$f(u_nu_1) = \lfloor \frac{k}{2} \rfloor.$$

$$f(u_iv_{2i-1}) = \begin{cases} 1; & 1 \le i \le \frac{n}{2} \\ \lfloor \frac{k}{2} \rfloor + 1; & \frac{n}{2} < i \le n. \end{cases}$$

$$f(u_iv_{2i}) = \begin{cases} \lfloor \frac{k}{2} \rfloor - 1; & 1 \le i \le \frac{n}{2} \\ k-1; & \frac{n}{2} < i \le n \text{ and } n \equiv 0 \pmod 3 \\ k; & \frac{n}{2} < i \le n \text{ and } n \equiv 1 \pmod 3. \end{cases}$$

$$f(v_{2i-1}v_{2i}) = \begin{cases} \lfloor \frac{k}{2} \rfloor; & 1 \le i \le \frac{n}{2} \\ \lfloor \frac{k}{2} \rfloor + 2; & \frac{n}{2} < i \le n. \end{cases}$$

(ii) $n \equiv 2 \pmod 3$.

$$f(u_iu_{i+1}) = \begin{cases} 1; & 1 \le i < \frac{n}{2} \\ \frac{k}{2}; & i = \frac{n}{2} \\ k-2; & \frac{n}{2} < i < n. \end{cases}$$

$$f(u_nu_1) = \frac{k}{2} - 1$$

$$f(u_iv_{2i-1}) = \begin{cases} 1; & 1 \le i \le \frac{n}{2} \\ \frac{k}{2}; & \frac{n}{2} < i \le n. \end{cases}$$

$$f(u_iv_{2i}) = \begin{cases} \frac{k}{2} - 2; & 1 \le i \le \frac{n}{2} \\ k-2; & \frac{n}{2} < i \le n. \end{cases}$$

$$f(v_{2i-1}v_{2i}) = \begin{cases} \frac{k}{2} - 1; & 1 \le i \le \frac{n}{2} \\ \frac{k}{2} + 1; & \frac{n}{2} < i \le n. \end{cases}$$

Then the edge weight function $wt_f : E({C_n}^*) \to \{3, 4, ..., |E| + 2\}$ *is as follows:*
For $1 \le i \le \frac{n}{2}$; $wt_f(u_iv_{2i-1}) = 2i + 1$.
For $1 \le i < \frac{n}{2}$; $wt_f(u_iu_{i+1}) = 2i + 2$.

For $1 \le i \le \frac{n}{2}$; $wt_f(u_i v_{2i}) = 2i + n$.
For $1 \le i \le \frac{n}{2}$; $wt_f(v_{2i-1} v_{2i}) = 2i + n + 1$.
$wt_f(u_n u_1) = 2n + 2$.
$wt_f(u_{\frac{n}{2}} u_{\frac{n}{2}+1}) = 2n + 3$.
For $\frac{n}{2} < i \le n$; $wt_f(u_i v_{2i-1}) = 2i + n + 2$.
For $\frac{n}{2} < i \le n$; $wt_f(v_{2i-1} v_{2i}) = 2i + n + 3$.
For $\frac{n}{2} < i \le n$; $wt_f(u_i v_{2i}) = 2i + 2n + 2$.
For $\frac{n}{2} < i < n$; $wt_f(u_i u_{i+1}) = 2i + 2n + 3$.

Case 2: *When n is odd.*

$$f(u_i) = \begin{cases} i; & 1 \le i \le \lceil \frac{n}{2} \rceil \\ k - n + i, & \lceil \frac{n}{2} \rceil < i \le n. \end{cases}$$

For $1 \le i \le n$; $\quad f(v_{2i-1}) = i$ and $f(v_{2i}) = k - n + i$.

(i) When $n \equiv 0 \pmod 3$ and $n \equiv 1 \pmod 3$.

$$f(u_i u_{i+1}) = \begin{cases} 1; & 1 \le i < \lceil \frac{n}{2} \rceil \\ \lfloor \frac{k}{2} \rfloor + 2; & i = \lceil \frac{n}{2} \rceil \\ k - 1; & \lceil \frac{n}{2} \rceil < i < n \text{ and } n \equiv 0 \pmod 3 \\ k; & \lceil \frac{n}{2} \rceil < i < n \text{ and } n \equiv 1 \pmod 3. \end{cases}$$

$$f(u_n u_1) = \lfloor \frac{k}{2} \rfloor + 2.$$

$$f(u_i v_{2i-1}) = \begin{cases} 1; & 1 \le i \le \lceil \frac{n}{2} \rceil \\ \lfloor \frac{k}{2} \rfloor + 2; & \lceil \frac{n}{2} \rceil < i \le n. \end{cases}$$

$$f(u_i v_{2i}) = \begin{cases} \lfloor \frac{k}{2} \rfloor; & 1 \le i \le \lceil \frac{n}{2} \rceil \\ k - 1; & \lceil \frac{n}{2} \rceil < i \le n \text{ and } n \equiv 0 \pmod 3 \\ k; & \lceil \frac{n}{2} \rceil < i \le n \text{ and } n \equiv 1 \pmod 3. \end{cases}$$

$$f(v_{2i-1} v_{2i}) = \begin{cases} \lfloor \frac{k}{2} \rfloor + 1; & 1 \le i \le \lceil \frac{n}{2} \rceil \\ \lfloor \frac{k}{2} \rfloor + 3; & \lceil \frac{n}{2} \rceil < i \le n. \end{cases}$$

(ii) When $n \equiv 2 \pmod 3$.

$$f(u_i u_{i+1}) = \begin{cases} 1; & 1 \le i < \lceil \frac{n}{2} \rceil \\ \frac{k}{2} + 1; & i = \lceil \frac{n}{2} \rceil \\ k - 2; & \lceil \frac{n}{2} \rceil < i < n. \end{cases}$$

$$f(u_n u_1) = \frac{k}{2} + 1.$$

$$f(u_i v_{2i-1}) = \begin{cases} 1; & 1 \le i \le \lceil \frac{n}{2} \rceil \\ \frac{k}{2} + 1; & \lceil \frac{n}{2} \rceil < i \le n. \end{cases}$$

$$f(u_i v_{2i}) = \begin{cases} \frac{k}{2} - 1; & 1 \le i \le \lceil \frac{n}{2} \rceil \\ k - 2; & \lceil \frac{n}{2} \rceil < i \le n. \end{cases}$$

$$f(v_{2i-1} v_{2i}) = \begin{cases} \frac{k}{2}; & 1 \le i \le \lceil \frac{n}{2} \rceil \\ \frac{k}{2} + 2; & \lceil \frac{n}{2} \rceil < i \le n. \end{cases}$$

Then the edge weight function $wt_f : E(C_n{}^*) \to \{3, 4, ..., |E| + 2\}$ *is as follows:*
For $1 \le i \le \lceil \frac{n}{2} \rceil$; $wt_f(u_i v_{2i-1}) = 2i + 1.$
For $1 \le i < \lceil \frac{n}{2} \rceil$; $wt_f(u_i u_{i+1}) = 2i + 2.$
For $1 \le i \le \lceil \frac{n}{2} \rceil$; $wt_f(u_i v_{2i}) = 2i + n + 1.$
For $1 \le i \le \lceil \frac{n}{2} \rceil$; $wt_f(v_{2i-1} v_{2i}) = 2i + n + 2.$
$wt_f(u_n u_1) = 2n + 4.$
$wt_f(u_{\lceil \frac{n}{2} \rceil} u_{\lceil \frac{n}{2} \rceil + 1}) = 2n + 5.$
For $\lceil \frac{n}{2} \rceil < i \le n$; $wt_f(u_i v_{2i-1}) = 2i + n + 3.$
For $\lceil \frac{n}{2} \rceil < i \le n$; $wt_f(v_{2i-1} v_{2i}) = 2i + n + 4.$
For $\lceil \frac{n}{2} \rceil < i \le n$; $wt_f(u_i v_{2i}) = 2i + 2n + 2.$
For $\lceil \frac{n}{2} \rceil < i < n$; $wt_f(u_i u_{i+1}) = 2i + 2n + 3.$

In both the cases, the edge weight function $wt_f : E(C_n{}^*) \to \{3, 4, ..., |E| + 2\}$ *is bijective and hence* f *is a edge irregular total* $k-$ *labeling. Therefore,* $tes(C_n{}^*) \le k.$ □

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