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k-Zumkeller graphs through splitting of graphs

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Abstract

Let $G = (V, E)$ be a simple graph with vertex set V and edges set E. A $1-1$ function $f : V \to N$ is said to induce a k-Zumkeller graph G if the induced edge function $f^* : E \to N$ defined by $f^*(xy) = f(x)f(y)$ satisfies the following conditions:

1. *$f^*(xy)$ is a Zumkeller number for every $xy \in E$.*
2. *The total distinct Zumkeller numbers on the edges of G is k.*

In this article, we compute k-Zumkeller graphs through the graph splitting operation on path, cycle and star graphs.

AMS Subject Classification: *05C78.*

Keywords: *Zumkeller number, k-Zumkeller graph, splitting graph.*

1. Introduction

Let $G = (V, E)$ be a finite, simple, connected graph where V is the vertex set and E is the edge set of G. A labeled graph G is a graph obtained by assigning labels, traditionally by integers, to the vertices of V and/or edges of E. Alexander Rosa [11] first discussed the labeled graphs in the year 1967. Thousands of research articles have been published in the last 60 years using various labeling techniques. The terminologies and recent progress of labeled graphs are refered from Gallian [8].

Graph labeling is one of the most fascinating and vibrant area of graph theory. Labeled graphs are becoming an increasingly useful family of mathematical models for broad and wide range of applications. The applications of labeled graphs are found in [5]. The cryptography concepts depend on graph labeling techniques. One can find in [6, 10] the application of labeled graphs in cryptography. Graph labeling research is becoming popular and interesting nowadays. It has grown into an important branch of interdisciplinary mathematics-science study. Investigating the existence of one particular type of graph labeling to complex graph families is a potential area of research.

Reinhard Zumkeller introduced the Zumkeller numbers in 2003 by generalising the concept of practical numbers [13]. These numbers can be seen in the A083207 edition of the encyclopedia of integer sequences. The Zumkeller numbers and their partitions can be seen in [7]. The Zumkeller numbers play a vital role in theory of numbers. The properties of the Zumkeller numbers are referred to [15].

In the year 2015, Balamurugan et al.,[1] investigated a new variant of Zumkeller labeling known as k-Zumkeller labeling by minimizing the number of distinct Zumkeller numbers. It is the process of using appropriate functions to assign k different Zumkeller numbers to the graphs edges. In 2018, Balamurugan et al.,[2] extended the existence of k-Zumkeller graph to few more graphs such as path, cycle, comb graphs, ladder graphs and square grid. Recently in the year 2021, Basher in [3, 4] showed the existence of k-Zumkeller graph of super subdivision of some graphs, cartesian products of cycles and paths. They have also proved that the tensor products of cycles and paths admit k-Zumkeller labeling with $k = 2, 4, 5, 6, 7, 8, 9$.

Splitting graph of a graph was introduced by Sampathkumar and Walikar in 1980s [12]. If G is a graph with p vertices and q edges, then the splitting graph of G has $2p$ vertices and $3q$ edges. If a given graph G is tree or it contains an even cycle then its splitting graph is a planar graph.

If G contains an odd cycle of length n, $n \geq 5$ or non-outer planar or cycle with a chord then its splitting graph is a non-planar graph. This important property of a splitting graph provides a variety of interesting planar graphs. Because of this novelty property we have been motivated to find their k-Zumkeller labeling. In addition to that the splitting graphs have interesting applications. For example, here we consider the transmission of signals through a cell phone tower. Apparently all cell phone towers of a particular company would be interlinked to each other in a serial way and all towers in a row are connected. Additionally there could be link towers which would act as normal stand-alone towers as well as a back-up link for a main tower. When there is a maintenance activity going on or when there is an outage in one of the main towers, the link/back-up one of the failed tower will connect to its adjacent main towers so that it by-passes the failed main tower in between. This set up resembles the splitting graph and avoids break in linkage even any one of the main tower fails.

Harary [9] and West [14] have been cited for the graph concepts and terminologies of this paper. The splitting graphs of path, cycle and star and their k-Zumkeller graphs have been computed in this research article with appropriate illustrations.

2. Preliminaries

The basic concepts pertaining to this paper are presented in this section. The concept of Zumkeller numbers and splitting graphs are dicussed with appropriate example. The definition of k-Zumkeller labeling with suitable example is also provided in this section.

Definition 1. *Let n be a positive integer. Let A_1 and A_2 be two sets of disjoint positive factors of n. If the sum of the numbers of A_1 is equal to the sum of the numbers of A_2 then n is said to be a Zumkeller number. For example, the positive factors of* 56 *can be partitioned as* $A_1 = \{1, 2, 7, 8, 14, 28\}$ *and* $A_2 = \{4, 56\}$ *such that sum of each set is* 60*. Hence* 56 *is a Zumkeller number.*

Definition 2. *Let G be a graph with vertex set V and edge set E. The splitting graph of G is obtained as follows: For each vertex $v \in V$, define a new vertex v' such that v' is joined to all vertices of G adjacent to v. The graph $S(G)$ thus obtained is called the splitting graph of G. If G has n vertices and m edges, then $S(G)$ has $2n$ vertices and $3m$ edges.*

Example 1. *The graph G and splitting graph of G are shown in Figure 1.*

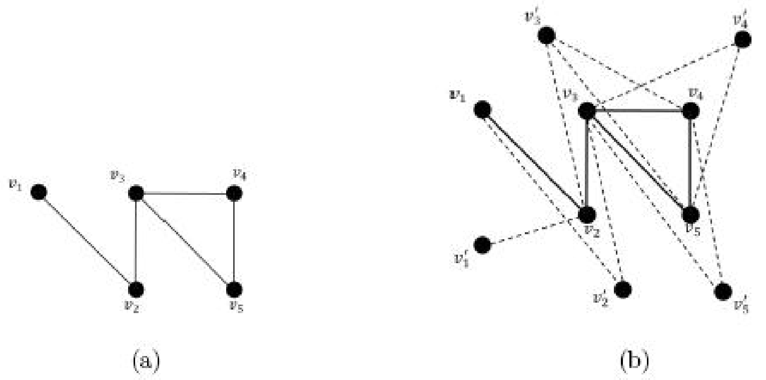

Figure 1: (a) Graph G (b) Splitting graph of G

Definition 3. *Let $G = (V, E)$ be a simple graph with vertex set V and edges set E. A $1-1$ function $f : V \to N$ is said to induce a k-Zumkeller graph G if the induced edge function $f^* : E \to N$ defined by $f^*(xy) = f(x)f(y)$ satisfies the following conditions:*

1. *$f^*(xy)$ is a Zumkeller number for every $xy \in E$.*
2. *The total distinct Zumkeller numbers on the edges of G is k.*

Definition 4. *Let $G(V, E)$ be a simple graph. If G admits a k-Zumkeller labeling then it is said to be a k-Zumkeller graph.*

Example 2. *A 3-Zumkeller graph is given in Figure 2.*

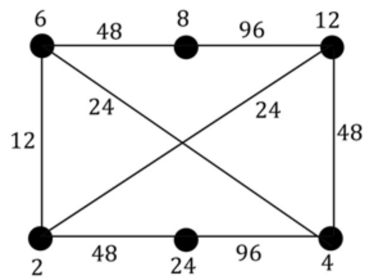

Figure 2: A 4-Zumkeller graph

3. Splitting operation on graphs and their k-Zumkeller graphs

The splitting of paths, cycles, star graphs and their k-Zumkeller labeling are investigated in this section. Futher, results pertaining to the degree and its connection with k-Zumkeller labeling have been discussed.

Theorem 1. *The splitting graph of the path P_n, $n > 2$, $n \equiv 0(mod\ 2)$ is a 4-Zumkeller graph.*

Proof. Let $V = U \cup V$ where $U = \{u_i : 1 \leq i \leq n\}$ and $V = \{v_i : 1 \leq i \leq n\}$ be the vertex set and $E = \{v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{u_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{v_i u_{i+1} : 1 \leq i \leq n-1\}$ be the edge set of the splitting graph of path P_n.

Define a $1-1$ function $f : V \to N$ such that

$$f(v_i) = 2^{\frac{i+1}{2}}, \qquad i = 1, 3, 5, ..., n-1$$

$$f(v_{i+1}) = p2^{\frac{n-i+1}{2}}, \qquad i = 1, 3, 5, ..., n-1$$

$$f(u_i) = 2^{\frac{n+i+1}{2}}, \qquad i = 1, 3, 5, ..., n-1$$

$$f(u_{i+1}) = p2^{\frac{2n-i+1}{2}}, \qquad i = 1, 3, 5, ..., n-1$$

where p is a prime number such that $2 < p < 10$.

The induced function $f^* : E \to N$ provides the following Zumkeller numbers on the edges of splitting graph of P_n:

(3.1) $f^*(v_i v_{i+1}) = f(v_i)f(v_{i+1}) = 2^{\frac{i+1}{2}} p2^{\frac{n-i+1}{2}} = p2^{\frac{i+1+n-i+1}{2}} = p2^{\frac{n+2}{2}}$

$f^*(v_{i+1}v_{i+2}) = f(v_{i+1})f(v_{i+2}) = p2^{\frac{n-i+1}{2}} 2^{\frac{i+3}{2}} = p2^{\frac{n-i+1+i+3}{2}} = p2^{\frac{n+4}{2}}$
(3.2)

$f^*(u_i v_{i+1}) = f(u_i)f(v_{i+1}) = 2^{\frac{n+i+1}{2}} p2^{\frac{n-i+1}{2}} = p2^{\frac{n+i+1+n-i+1}{2}} = p2^{n+1}$
(3.3)

$f^*(u_{i+1}v_{i+2}) = f(u_{i+1})f(v_{i+2}) = p2^{\frac{2n-i+1}{2}} 2^{\frac{i+3}{2}} = p2^{\frac{2n-i+1+i+3}{2}} = p2^{n+2}$
(3.4)

$f^*(v_i u_{i+1}) = f(v_i)f(u_{i+1}) = 2^{\frac{i+1}{2}} p2^{\frac{2n-i+1}{2}} = p2^{\frac{i+1+2n-i+1}{2}} = p2^{n+1}$
(3.5)

$f^*(v_{i+1}u_{i+2}) = f(v_{i+1})f(u_{i+2}) = p2^{\frac{n-i+1}{2}} 2^{\frac{n+i+3}{2}} = p2^{\frac{2n-i+1+i+3}{2}} = p2^{n+2}$
(3.6)
where $i = 1, 3, 5, ..., n-1$

It is observed from the equations (3.1) to (3.6) that the edges of the splitting graph of P_n receive only the four distinct Zumkeller numbers viz., $p2^{\frac{n+2}{2}}$, $p2^{\frac{n+4}{2}}$, $p2^{n+1}$, $p2^{n+2}$. Hence the graph is a 4-Zumkeller graph. $\square$

Example 3. *A 4-Zumkeller labeling of the splitting graph of path P_6 is shown in Figure 3.*

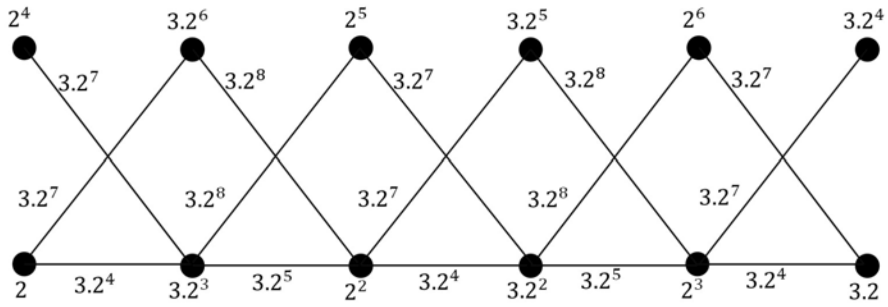

Figure 3: 4-*Zumkeller graph of* $S(P_6)$

Corollary 1. *The splitting graph of the path P_2 is a 3-Zumkeller graph.*

Theorem 2. *The splitting graph of the path P_n, $n \geq 3$, $n \equiv 1(mod\ 2)$ is a 5-Zumkeller graph.*

Proof. *Let $V = U \cup V$ where $U = \{u_i : 1 \leq i \leq n\}$ and $V = \{v_i : 1 \leq i \leq n\}$ be the vertex set. Let $E = \{v_iv_{i+1} : 1 \leq i \leq n-1\} \cup \{u_iv_{i+1} : 1 \leq i \leq n-1\} \cup \{v_iu_{i+1} : 1 \leq i \leq n-1\}$ be the edge set of the splitting graph of path P_n.*

Define a $1-1$ function $f : V \to N$ such that

$$
\begin{aligned}
f(v_i) &= 2^{\frac{i+1}{2}}, & i &= 1, 3, 5, ..., n \\
f(v_{i+1}) &= p2^{\frac{n-i}{2}}, & i &= 1, 3, 5, ..., n-2 \\
f(u_i) &= 2^{\frac{n+i+2}{2}}, & i &= 1, 3, 5, ..., n \\
f(u_{i+1}) &= p2^{\frac{2n-i-1}{2}}, & i &= 1, 3, 5, ..., n-2
\end{aligned}
$$

where p is a prime number such that $2 < p < 10$.

The induced function $f^* : E \to N$ provides the following Zumkeller numbers on the edges of $S(P_n)$:

$$f^*(v_iv_{i+1}) = f(v_i)f(v_{i+1}) = 2^{\frac{i+1}{2}}p2^{\frac{n-i}{2}} = p2^{\frac{i+1+n-i}{2}} = p2^{\frac{n+1}{2}} \tag{3.7}$$

(3.8) $f^*(v_{i+1}v_{i+2}) = f(v_{i+1})f(v_{i+2}) = p2^{\frac{n-i}{2}}2^{\frac{i+3}{2}} = p2^{\frac{n-i+i+3}{2}} = p2^{\frac{n+3}{2}}$

(3.9) $f^*(u_iv_{i+1}) = f(u_i)f(v_{i+1}) = 2^{\frac{n+i+2}{2}}p2^{\frac{n-i}{2}} = p2^{\frac{n+i+2+n-i}{2}} = p2^{n+1}$

$f^*(u_{i+1}v_{i+2}) = f(u_{i+1})f(v_{i+2}) = p2^{\frac{2n-i-1}{2}}2^{\frac{i+3}{2}} = p2^{\frac{2n-i-1+i+3}{2}} = p2^{n+1}$
(3.10)

(3.11) $f^*(v_iu_{i+1}) = f(v_i)f(u_{i+1}) = 2^{\frac{i+1}{2}}p2^{\frac{2n-i-1}{2}} = p2^{\frac{i+1+2n-i-1}{2}} = p2^n$

$f^*(v_{i+1}u_{i+2}) = f(v_{i+1})f(u_{i+2}) = p2^{\frac{n-i}{2}}2^{\frac{n+i+4}{2}} = p2^{\frac{n-i+n+i+4}{2}} = p2^{n+2}$
(3.12)
where $i = 1, 3, 5, ..., n-2$

It is noted from the equations (3.7) to (3.12) that the edges of the splitting graph of P_n receive only the five distinct Zumkeller numbers viz., $p2^{\frac{n+1}{2}}$, $p2^{\frac{n+3}{2}}$, $p2^{n+1}$, $p2^{n+2}$, $p2^n$. Hence the graph is a 5-Zumkeller graph. □

Example 4. *A 5-Zumkeller labeling for the splitting graph of path P_5 is shown in Figure 4.*

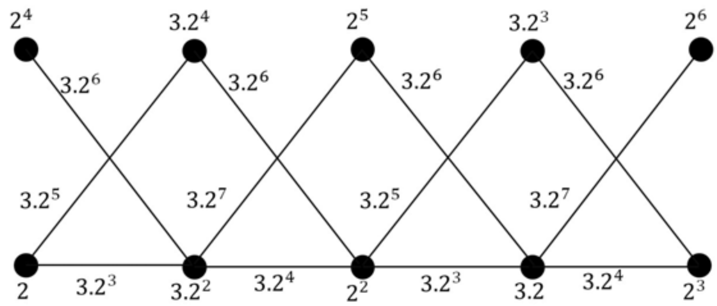

Figure 4: 5-*Zumkeller graph of* $S(P_5)$

Corollary 2. *The total number of distinct Zumkeller numbers in the k-Zumkeller labeling of the splitting graph of P_n is*

1. Δ *when* $n \equiv 0(mod\ 2)$
2. $\Delta + 1$ *when* $n \equiv 1(mod\ 2)$

where Δ is the maximum degree of the splitting graph of P_n.

Theorem 3. *The splitting graph of the cycle C_n ,$n \geq 4$, $n \equiv 0(mod\ 2)$ is a 5-Zumkeller graph.*

Proof. *Let $V = U \cup V$ where $U = \{u_i : 1 \leq i \leq n\}$ and $V = \{v_i : 1 \leq i \leq n\}$ be the vertex set. Let $E = \{v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{v_1 v_n\} \cup \{u_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{u_n v_1\} \cup \{v_i u_{i+1} : 1 \leq i \leq n-1\} \cup \{u_1 v_n\}$ be the edge set of the splitting graph of cycle C_n.*

Define a $1-1$ function $f : V \to N$ such that

$$\begin{aligned} f(v_i) &= 2^{\frac{i+1}{2}}, & i &= 1, 3, 5, ..., n-1 \\ f(v_{i+1}) &= p2^{\frac{n-i+1}{2}}, & i &= 1, 3, 5, ..., n-1 \\ f(u_i) &= 2^{\frac{n+i+1}{2}}, & i &= 1, 3, 5, ..., n-1 \\ f(u_{i+1}) &= p2^{\frac{2n-i+1}{2}}, & i &= 1, 3, 5, ..., n-1 \end{aligned}$$

where p is a prime number such that $2 < p < 10$.

The induced function $f^* : E \to N$ provides the following Zumkeller numbers on the edges of $S(C_n)$:

$$f^*(v_i v_{i+1}) = f(v_i) f(v_{i+1}) = 2^{\frac{i+1}{2}} p2^{\frac{n-i+1}{2}} = p2^{\frac{i+1+n-i+1}{2}} = p2^{\frac{n+2}{2}} \tag{3.13}$$

$$f^*(v_{i+1} v_{i+2}) = f(v_{i+1}) f(v_{i+2}) = p2^{\frac{n-i+1}{2}} 2^{\frac{i+3}{2}} = p2^{\frac{n-i+1+i+3}{2}} = p2^{\frac{n+4}{2}} \tag{3.14}$$

$$f^*(v_1 v_n) = f(v_1) f(v_n) = 2^1 p2^1 = p2^2 \tag{3.15}$$

$$f^*(u_iv_{i+1}) = f(u_i)f(v_{i+1}) = 2^{\frac{n+i+1}{2}}p2^{\frac{n-i+1}{2}} = p2^{\frac{n+i+1+n-i+1}{2}} = p2^{n+1} \tag{3.16}$$

$$f^*(u_{i+1}v_{i+2}) = f(u_{i+1})f(v_{i+2}) = p2^{\frac{2n-i+1}{2}}2^{\frac{i+3}{2}} = p2^{\frac{2n-i+1+i+3}{2}} = p2^{n+2} \tag{3.17}$$

$$f^*(v_iu_{i+1}) = f(v_i)f(u_{i+1}) = 2^{\frac{i+1}{2}}p2^{\frac{2n-i+1}{2}} = p2^{\frac{i+1+2n-i+1}{2}} = p2^{n+1} \tag{3.18}$$

$$f^*(v_{i+1}u_{i+2}) = f(v_{i+1})f(u_{i+2}) = p2^{\frac{n-i+1}{2}}2^{\frac{n+i+3}{2}} = p2^{\frac{2n-i+1+i+3}{2}} = p2^{n+2} \tag{3.19}$$

$$(3.20) f^*(u_nv_1) = f(u_n)f(v_1) = p2^{\frac{2n-n+1+1}{2}}2^1 = p2^{\frac{2n-n+1+1+2}{2}} = p2^{\frac{n+4}{2}}$$

$$f^*(u_1v_n) = f(u_1)f(v_n) = 2^{\frac{n+2}{2}}p2^1 = p2^{\frac{n+2+2}{2}} = p2^{\frac{n+4}{2}} \tag{3.21}$$

where $i = 1, 3, 5, ..., n-1$

It is noted from the equations (3.13) to (3.21) that the edges of the splitting graph of C_n *receive only the five distinct Zumkeller numbers viz.,* $p2^{\frac{n+2}{2}}$, $p2^{\frac{n+4}{2}}$, $p2^{n+1}$, $p2^{n+2}$, $p2^2$. *Hence the graph is a* 5*-Zumkeller graph.* □

Example 5. *A* 5*-Zumkeller graph of the splitting graph of cycle* C_8 *is shown in Figure 5.*

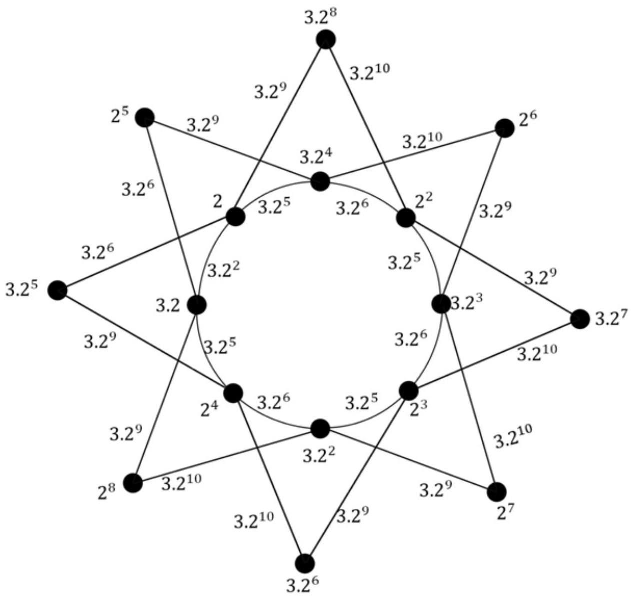

Figure 5: 5-*Zumkeller graph of* $S(C_8)$

Proposition 4. *The splitting graph of the cycle* C_3 *is a* 6-*Zumkeller graph.*

Proof. *Let* $V = U \cup V$ *where* $U = \{u_1, u_2, u_3\}$ *and* $V = \{v_1, v_2, v_3\}$ *be the vertex set. Let* $E = \{v_1v_2, v_2v_3, v_1v_3, u_1v_2, u_1v_3, u_2v_1, u_2v_3, u_3v_1, u_3v_2\}$ *be the edge set of the splitting graph of cycle* C_3.

Define a $1-1$ *function* $f : V \to N$ *such that*

$f(v_1) = 2,$
$f(v_2) = 2p_1,$
$f(v_3) = 2p_2,$
$f(u_1) = 2^2,$
$f(u_2) = 2^2p_1,$
$f(u_3) = 2^2p_2.$

where p_1, p_2 *is a prime number such that* $2 < p_1, p_2 < 10$.

The induced function $f^* : E \to N$ *provides the following Zumkeller numbers on the edges of* $S(C_3)$:

$$f^*(v_1v_2) = f(v_1)f(v_2) = 2p_12 = p_12^2 \tag{3.22}$$

$$f^*(v_2v_3) = f(v_2)f(v_3) = p_12p_22 = p_2p_12^2 \tag{3.23}$$

$$f^*(v_1v_3) = f(v_1)f(v_3) = 2p_22 = p_22^2 \tag{3.24}$$

$$f^*(u_1v_2) = f(u_1)f(v_2) = 2^2p_12 = p_12^3 \tag{3.25}$$

$$f^*(u_1v_3) = f(u_1)f(v_3) = 2^2p_22 = p_22^3 \tag{3.26}$$

$$f^*(u_2v_1) = f(u_2)f(v_1) = p_12^22 = p_12^3 \tag{3.27}$$

$$f^*(u_2v_3) = f(u_2)f(v_3) = p_12^2p_22 = p_2p_12^3 \tag{3.28}$$

$$f^*(u_3v_1) = f(u_3)f(v_1) = p_22^22 = p_22^3 \tag{3.29}$$

$$f^*(u_3v_2) = f(u_3)f(v_2) = p_2p_12^22 = p_2p_12^3 \tag{3.30}$$

From the equations (3.22) to (3.30) it is observed that the edges of the splitting graph of C_3 *receive only the six distinct Zumkeller numbers viz.,* $p_12^2, p_22^2, p_2p_12^2, p_12^3, p_22^3, p_2p_12^3$*. Hence the graph is a* 6*-Zumkeller graph.* □

Example 6. *A* 6*-Zumkeller graph of the splitting graph of cycle* C_3 *is shown in Figure 6.*

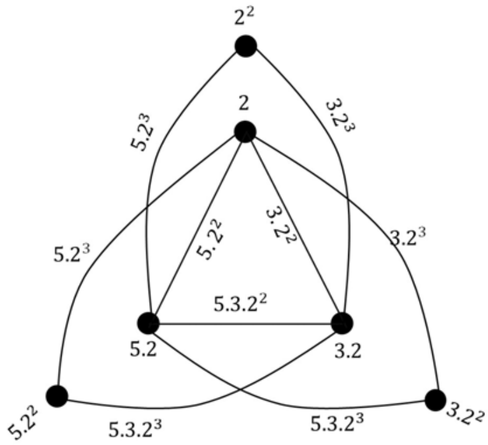

Figure 6: 6-*Zumkeller graph of* $S(C_3)$

Theorem 5. *The splitting graph of the cycle C_n, $n \geq 5$, $n \equiv 1(mod\ 2)$ is a 10-Zumkeller graph.*

Proof. *Let $V = U \cup V$ where $U = \{u_i : 1 \leq i \leq n\}$ and $V = \{v_i : 1 \leq i \leq n\}$ be the vertex set. Let $E = \{v_iv_{i+1} : 1 \leq i \leq n-1\} \cup \{v_1v_n\} \cup \{u_iv_{i+1} : 1 \leq i \leq n-1\} \cup \{u_nv_1\} \cup \{v_iu_{i+1} : 1 \leq i \leq n-1\} \cup \{u_1v_n\}$ be the edge set of the splitting graph of cycle C_n.*

Define a $1-1$ function $f : V \to N$ such that

$$
\begin{aligned}
f(v_i) &= 2^{\frac{i+1}{2}}, && i = 1,3,5,...,n-2\\
f(v_{i+1}) &= p_1 2^{\frac{n-i}{2}}, && i = 1,3,5,...,n-2\\
& && f(v_n) = p_2 2\\
f(u_i) &= 2^{\frac{n+i}{2}}, && i = 1,3,5,...,n-2\\
f(u_{i+1}) &= p_1 2^{\frac{2n-i-1}{2}}, && i = 1,3,5,...,n-2\\
& && f(u_n) = p_2 2^2
\end{aligned}
$$

where p_1 and p_2 are prime numbers such that $2 < p_1, p_2 < 10$.

The induced function $f^* : E \to N$ provides the following Zumkeller numbers on the edges of $S(C_n)$:

$$(3.31)\ f^*(v_iv_{i+1}) = f(v_i)f(v_{i+1}) = 2^{\frac{i+1}{2}} p_1 2^{\frac{n-i}{2}} = p_1 2^{\frac{i+1+n-i}{2}} = p_1 2^{\frac{n+1}{2}}$$

$$f^*(v_{i+1}v_{i+2}) = f(v_{i+1})f(v_{i+2}) = p_1 2^{\frac{n-i}{2}} 2^{\frac{i+3}{2}} = p_1 2^{\frac{n-i+i+3}{2}} = p_1 2^{\frac{n+3}{2}}$$
(3.32)

$$(3.33) \qquad f^*(v_{n-1}v_n) = f(v_{n-1})f(v_n) = p_1 2 p_2 2 = p_2 p_1 2^2$$

$$(3.34) \qquad f^*(v_1v_n) = f(v_1)f(v_n) = 2^1 p_2 2^1 = p_2 2^2$$

$$(3.35)\ f^*(u_iv_{i+1}) = f(u_i)f(v_{i+1}) = 2^{\frac{n+i}{2}} p_1 2^{\frac{n-i}{2}} = p_1 2^{\frac{n+i+n-i}{2}} = p_1 2^n$$

$$f^*(u_{i+1}v_{i+2}) = f(u_{i+1})f(v_{i+2}) = p_1 2^{\frac{2n-i-1}{2}} 2^{\frac{i+3}{2}} = p_1 2^{\frac{2n-i-1+i+3}{2}} = p_1 2^{n+1} \tag{3.36}$$

$$f^*(u_{n-1}v_n) = f(u_{n-1})f(v_n) = p_1 2^{\frac{2n-(n-2)-1}{2}} p_2 2 = p_2 p_1 2^{\frac{n+3}{2}} \tag{3.37}$$

$$f^*(u_n v_1) = f(u_n)f(v_1) = p_2 2^2 2^1 = p_2 2^3 \tag{3.38}$$

$$f^*(v_i u_{i+1}) = f(v_i)f(u_{i+1}) = 2^{\frac{i+1}{2}} p_1 2^{\frac{2n-i-1}{2}} = p_1 2^{\frac{i+1+2n-i-1}{2}} = p_1 2^n \tag{3.39}$$

$$f^*(v_{i+1}u_{i+2}) = f(v_{i+1})f(u_{i+2}) = p_1 2^{\frac{n-i}{2}} 2^{\frac{n+i+2}{2}} = p_1 2^{\frac{2n-i+i+2}{2}} = p_1 2^{n+1} \tag{3.40}$$

$$f^*(v_{n-1}u_n) = f(v_{n-1})f(u_n) = p_1 2 p_2 2^2 = p_2 p_1 2^3 \tag{3.41}$$

$$f^*(v_n u_1) = f(v_n)f(u_1) = p_2 2 2^{\frac{n+1}{2}} = p_2 2^{\frac{n+3}{2}} \tag{3.42}$$

where $i = 1, 3, 5, ..., n-2$

It is noted from the equations (3.31) to (3.42) that the edges of the splitting graph of C_n *receives the ten distinct Zumkeller numbers viz.,* $p_1 2^{\frac{n+1}{2}}$, $p_1 2^{\frac{n+3}{2}}$, $p_2 p_1 2^2$, $p_2 2^2$, $p_1 2^{n+1}$, $p_1 2^n$, $p_2 p_1 2^{\frac{n+3}{2}}$, $p_2 2^3$, $p_2 2^{\frac{n+3}{2}}$, $p_2 p_1 2^3$. *Hence the graph is a* 10*-Zumkeller graph.*

□

Example 7. *A* 10*-Zumkeller graph of the splitting graph of cycle* C_5 *is shown in Figure 7.*

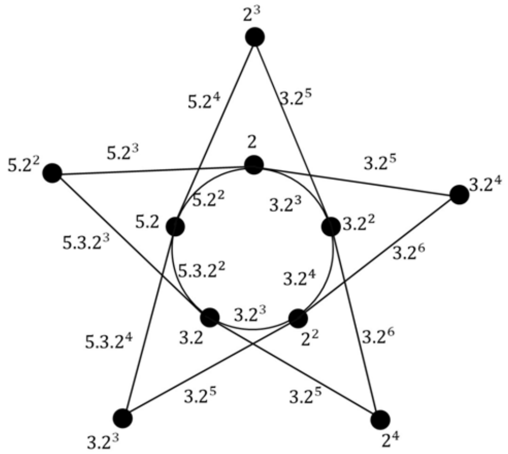

Figure 7: 10-*Zumkeller graph of* $S(C_5)$

Corollary 3. *The total number of distinct Zumkeller numbers on k-Zumkeller splitting graph of* C_n *is*

1. $\Delta + 1$ *when* $n \equiv 0(mod\ 2)$
2. $\Delta + 2$ *when* $n = 3$
3. $\Delta + 6$ *when* $n \equiv 1(mod\ 2)$

where Δ *is the maximum degree of the splitting graph of* C_n.

Theorem 6. *The splitting graph of the star* $K_{1,n}$, $n \geq 2$ *is a* $2n$*-Zumkeller graph.*

Proof. *Let* $V = U \cup V$ *where* $U = \{u\} \cup \{u_i : 1 \leq i \leq n\}$ *and* $V = \{v\} \cup \{v_i : 1 \leq i \leq n\}$ *be the vertex set. Let* $E = \{vv_i : 1 \leq i \leq n\} \cup \{uv_i : 1 \leq i \leq n\} \cup \{vu_i : 1 \leq i \leq n\}$ *be the edge set of the splitting graph of star* $K_{1,n}$.

Define a $1 - 1$ *function* $f : V \to N$ *such that*
f(v)=2 p
f(u)=p 2^2
$f(v_i) = 2^{2i-1}, \qquad i = 1, 2, 3, ..., n$
$f(u_i) = 2^{2i}, \qquad i = 1, 2, 3, ..., n$
where p *is a prime number such that* $2 < p < 10$.

The induced function $f^* : E \to N$ provides the following Zumkeller numbers on the edges of $S(K_{1,n})$:

$$f^*(vv_i) = f(v)f(v_i) = p22^{2i-1} = p2^{2i} \tag{3.43}$$

$$f^*(uv_i) = f(u)f(v_i) = p2^2 2^{2i-1} = p2^{2i+1} \tag{3.44}$$

$$f^*(vu_i) = f(v)f(u_i) = p22^{2i} = p2^{2i+1} \tag{3.45}$$

where $i = 1, 2, 3, ..., n$

It is clear from the equations (3.43) to (3.45) that the edges of the splitting graph of $K_{1,n}$ receive only $2n$ distinct Zumkeller numbers viz., $p2^{i+1}$, $p2^{2i}$ for $i = 1, 2, 3, ..., n$. Hence the graph is a $2n$-Zumkeller graph. □

Example 8. *A 10-Zumkeller graph of the splitting graph of star $K_{1,5}$ is shown in Figure 8.*

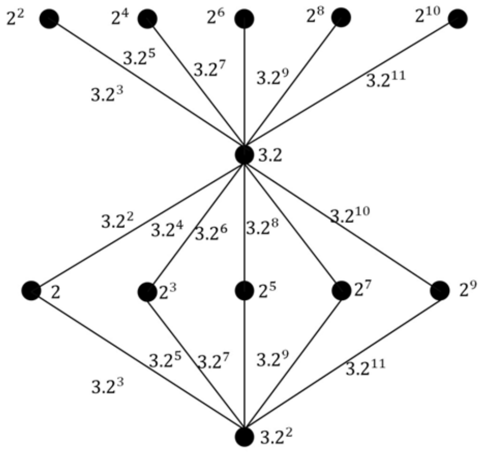

Figure 8: 10-*Zumkeller graph of* $S(K_{1,5})$

Corollary 4. *The total number of distinct Zumkeller numbers in k-Zumkeller labeling of the splitting graph of star $K_{1,n}$, is the maximum degree of the splitting graph of star $K_{1,n}$.*

The following table gives a comparison study of the k-Zumkeller labeling of graphs and k-Zumkeller labeling of their splitting graphs. The k-values of these graphs and maximum degree (Δ) are tabulated.

Table 1: *Comparison of k-Zumkeller labeling of graphs and k-Zumkeller labeling of their splitting graphs.*

S.No	Graph G	k-Zumkeller labeling of G (k-value)	Max. degree of $G(\Delta)$	Splitting graph of $S(G)$	k-Zumkeller labeling of $S(G)$ (k-value)	Max. degree of $S(G)(\Delta)$
1	Path graphP_n $n \equiv 0(mod\ 2)$	2	2	$S(P_n)$ $n \equiv 0(mod\ 2)$	4	4
2	Path graphP_n $n \equiv 1(mod\ 2)$	2	2	$S(P_n)$ $n \equiv 1(mod\ 2)$	5	4
3	Cycle graphC_n $n \equiv 0(mod\ 2)$	3	2	$S(C_n)$ $n \equiv 0(mod\ 2)$	5	4
4	Cycle graphC_n $n = 3$	3	2	$S(C_n)$ $n = 3$	6	4
5	Cycle graphC_n $n \equiv 1(mod\ 2)$	4	2	$S(C_n)$ $n \equiv 1(mod\ 2)$	10	4
6	Star graph$K_{1,n}$	n	n	$S(K_{1,n})$	2n	2n

4. Conclusion

In this article, the splitting graphs of path, cycle and star and their k-Zumkeller graphs have been computed with appropriate examples. The value of k in k- Zumkeller graphs have been compared with the maximum degree of the graph and tabulated the results accordingly. The optimal way of finding the values of k in k- Zumkeller labeling of graphs has many applications in science and technology. Investigating the existence of k-Zumkeller graphs with minimum value of k is a potential and challenging area of research and interesting too. Extending this k- Zumkeller labeling to other classes of graphs is a future scope of research.

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