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K-Riesz bases and K-g-Riesz bases in Hilbert C^*-module

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Abstract

This paper is devoted to studying the K-Riesz bases and the K-g-Riesz bases in Hilbert C^*-modules; we characterize the concept of K-Riesz bases by a bounded below operator and the standard orthonormal basis for Hilbert C^*-modules $\mathcal{H}$. Also We give some properties and characterization of K-g-Riesz bases by a bounded surjective operator and g-orthonormal basis for $\mathcal{H}$. Finally we consider the relationships between K-Riesz bases and K-g-Riesz bases.

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1. Introduction

Frames for Hilbert spaces were first introduced in 1952 by Duffin and Schaefer [1] for study of non-harmonic Fourier series. Frame theory is an exciting, dynamic and fast paced subject with applications to a wide variety of areas in mathematics and engineering, operator theory, harmonic analysis, wavelet theory, wireless communication, data transmission with erasures, processing signal, processing image, and many other fields.

The notion of frames in Hilbert C^*-module was introduced and investigated in [4]. K-frames and K-g-frames in Hilbert C^*-module are introduced in ([6],[2]).

Sun in [12] introduced g-frames and g-Riesz bases in Hilbert spaces and investigated some of their properties. For more details on Riesz bases in Hilbert C^*-module and for a discussion of basic properties of g-frames and g-Riesz bases in Hilbert C^*-module we refer to ([3],[8] and [7]).

In this paper, we introduce the notion of K-Riesz basis and K-g-Riesz basis in Hilbert C^*-module which are a generalization of K-Riesz bases and K-g-Riesz bases in Hilbert space introduced by Sheraki and Abdollahpour [11] and we establish some results.
The paper is organized as follows, In section 2 we briefly recall the definitions and basic properties of Hilbert C^*-modules, frame, K-frame, g-frame, K-g-frame, Riesz basis, and g-Riesz bases in Hilbert C^*-modules. In section 3, we construct and characterize the concept of K-Riesz basis in Hilbert C^*-module. In section 4, we investigate the notion of K-g-Riesz bases in Hilbert C^*-module.

2. Preliminaries

Definition 2.1. *[5] Let $\mathcal{A}$ be a unitary C^*-algebra and $\mathcal{H}$ be a left $\mathcal{A}$-module such that the linear structures of $\mathcal{A}$ and $\mathcal{H}$ are compatible. $\mathcal{H}$ is a pre-Hilbert $\mathcal{A}$-module if $\mathcal{H}$ is equipped with an $\mathcal{A}$-valued inner product $\langle .,. \rangle_{\mathcal{A}} : \mathcal{H} \times \mathcal{H} \to \mathcal{A}$ such that*

(i) For all $x \in \mathcal{H}$, $\langle x, x\rangle_{\mathcal{A}} \geq 0$, and $\langle x, x\rangle_{\mathcal{A}} = 0$ if and only if $x = 0$.

(ii) $\langle ax + y, z\rangle_{\mathcal{A}} = a\langle x, z\rangle_{\mathcal{A}} + \langle y, z\rangle_{\mathcal{A}}$, for all $a \in \mathcal{A}$ and $x, y, z \in \mathcal{H}$.

(iii) $\langle x, y\rangle_{\mathcal{A}} = \langle y, x\rangle^*_{\mathcal{A}}$, for all $x, y \in \mathcal{H}$.

For $x \in \mathcal{H}$, we define $\|x\| = \|\langle x, x\rangle_{\mathcal{A}}\|_{\mathcal{A}}^{\frac{1}{2}}$. If $\mathcal{H}$ is complete with $\|.\|$, it is called a Hilbert $\mathcal{A}$-module or a Hilbert C^*-module over $\mathcal{A}$.

Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert C^*-modules over $\mathcal{A}$, a map $T : \mathcal{H} \to \mathcal{K}$ is said to be adjointable if there exists a map $T^* : \mathcal{K} \to \mathcal{H}$ such that $\langle Tx, y\rangle_{\mathcal{A}} = \langle x, T^*y\rangle_{\mathcal{A}}$ for all $x \in \mathcal{H}$ and $y \in \mathcal{K}$.

We reserve the notation $End^*_{\mathcal{A}}(\mathcal{H}, \mathcal{K})$ for the set of all adjointable operators from $\mathcal{H}$ to $\mathcal{K}$ and $End^*_{\mathcal{A}}(\mathcal{H}, \mathcal{H})$ is abbreviated to $End^*_{\mathcal{A}}(\mathcal{H})$.

Throughout this paper, $\mathcal{H}$ is considered to be a Hilbert $\mathcal{A}-$module, and let $\{\mathcal{H}_i\}_{i\in I}$ be a collection of Hilbert $\mathcal{A}$-modules, where I is a finite or countable index set.
Let $l^2(\{\mathcal{H}_i\}_{i\in I})$ be the Hilbert $\mathcal{A}$-module defined by

$$l^2(\{\mathcal{H}_i\}_{i\in I}) = \left\{ \{x_i\}_{i\in I} : \forall i \in I,\ \ x_i \in \mathcal{H}_i \ \text{ and } \sum_{i\in I}\langle x_i, x_i\rangle_{\mathcal{A}} \ \text{ converge in } \ \|.\|_{\mathcal{A}} \right\},$$

with $\mathcal{A}$-valued inner product $\langle \{x_i\}_{i\in I}, \{y_i\}_{i\in I}\rangle = \sum_{i\in I}\langle x_i, y_i\rangle_{\mathcal{A}}$, for all $\{x_i\}_{i\in I}, \{y_i\}_{i\in I} \in l^2(\{\mathcal{H}_i\}_{i\in I})$.

And let $l^2(\mathcal{A})$ be the Hilbert $\mathcal{A}$-module defined by

$$l^2(\mathcal{A}) = \left\{ \{a_i\}_{i\in I} \subseteq \mathcal{A} : \sum_{i\in I} a_i a_i^* \ \text{ converge in } \ \|.\|_{\mathcal{A}} \right\},$$

with $\mathcal{A}$-valued inner product $\langle \{a_i\}_{i\in I}, \{b_i\}_{i\in I}\rangle = \sum_{i\in I} a_i b_i^*$, for all $\{a_i\}_{i\in I}, \{b_i\}_{i\in I} \in l^2(\mathcal{A})$.

Proposition 2.2. *[9] Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert C^*-modules over a C^*-algebra $\mathcal{A}$ and $T \in End^*_{\mathcal{A}}(\mathcal{H}, \mathcal{K})$. The following statements are equivalent:*

(1) *T is surjective.*

(2) *T^* is bounded below with respect to the norm, i.e., there is $m > 0$ such that*
$$m\,\|x\| \leq \|T^*x\|\,, \ \text{ for all } x \in \mathcal{K}.$$

(3) *T^* is bounded below with respect to the inner product, i.e., there is $m' > 0$ such that*
$$m'\langle x, x\rangle \leq \langle T^*x, T^*x\rangle, \ \text{ for all } x \in \mathcal{K}.$$

Proposition 2.3. *[9] For $T \in End^*_{\mathcal{A}}(\mathcal{H})$, we have $\langle Tx, Tx\rangle \leq \|T\|^2 \langle x, x\rangle$, for all $x \in \mathcal{H}$.*

Definition 2.4. *[4] A sequence $\{f_i\}_{i\in I}$ in $\mathcal{H}$ is called a Frame if there exist a constants $A, B > 0$ such that for each $f \in \mathcal{H}$,*

$$A\langle f, f\rangle \leq \sum_{i\in I}\langle f, f_i\rangle\langle f_i, f\rangle \leq B\langle f, f\rangle. \tag{2.1}$$

The numbers A and B are called lower and upper bound of the frame, respectively. If $A = B = \lambda$, the frame is called λ-tight. If $A = B = 1$, it is called a normalized tight frame or a Parseval frame. If only upper inequality of 2.1 hold, then $\{f_i\}_{i\in I}$ is called a Bessel sequence for $\mathcal{H}$.

Definition 2.5. *[3] A frame $\{f_i\}_{i\in I}$ for $\mathcal{H}$ is said to be a Riesz basis for $\mathcal{H}$ if it satisfies:*

- (i) *$f_i \neq 0$ for all $i \in I$.*
- (ii) *if an $\mathcal{A}$-linear combination $\sum_{j\in S} a_j f_j$ with coefficients $\{a_j : j \in S\} \subseteq \mathcal{A}$ and $S \subseteq I$ is equal to zero, then every summand $a_j f_j$ is equal to zero.*

Corollary 2.6. *[3] Suppose that $\{f_i\}_{i\in I}$ is a frame of $\mathcal{H}$, then $\{f_i\}_{i\in I}$ is a Riesz basis if and only if*

- (i) *$f_i \neq 0$ for all $i \in I$.*
- (ii) *if $\sum_{i\in I} a_i f_i = 0$ for some sequence $\{a_i; i \in I\} \in l^2(\mathcal{A})$, then $a_i f_i = 0$ for each $i \in I$.*

Definition 2.7. *[13] A sequence $\{\Lambda_i \in End^*_{\mathcal{A}}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is called a g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$ if there exist two positive constants A, B, such that*

$$A\langle f, f\rangle \leq \sum_{i\in I}\langle \Lambda_i f, \Lambda_i f\rangle \leq B\langle f, f\rangle, \text{ for all } f \in \mathcal{H}. \tag{2.2}$$

The constants A and B are called the lower and upper bounds of g-frame, respectively. If only the right hand inequality of 2.2 holds, $\{\Lambda_i\}_{i\in I}$ is called a g-bessel sequence for $\mathcal{H}$.

Definition 2.8. *[7] A g-frame $\{\Lambda_i \in End^*_{\mathcal{A}}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$ is said to be a g-Riesz basis if it satisfies:*

(i) *$\Lambda_i \neq 0$ for all $i \in I$.*

(ii) *if an $\mathcal{A}$-linear combination $\sum_{j\in S} \Lambda_j^* g_j$ is equal to zero, then every summand $\Lambda_j^* g_j$ equal to zero, where $g_j \in \mathcal{H}_i$ and $S \subseteq I$.*

Corollary 2.9. *[7] Let $\{\Lambda_i \in End_{\mathcal{A}}^*(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$, then $\{\Lambda_i \in End_{\mathcal{A}}^*(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g-Riesz basis if and only if,*

(i) *$\Lambda_i \neq 0$ for all $i \in I$.*

(ii) *if $\sum_{i\in I} \Lambda_i^* g_i = 0$ for some sequence $\{g_i\}_{i\in I} \in l^2(\{\mathcal{H}_i\}_{i\in I})$, then $\Lambda_i^* g_i = 0$ for each $i \in I$.*

Definition 2.10. *[10] A sequence $\{\Lambda_i \in End_{\mathcal{A}}^*(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is called a g-orthonormal basis for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$ if*

$$\left\langle \Lambda_i^* g_i, \Lambda_j^* g_j \right\rangle = \delta_{i,j} \left\langle g_i, g_j \right\rangle, \text{ for all } i, j \in I,\ g_i \in \mathcal{H}_i, \text{ and } g_j \in \mathcal{H}_j, \tag{2.3}$$

and

$$\sum_{i\in I} \left\langle \Lambda_i f, \Lambda_i f \right\rangle = \left\langle f, f \right\rangle, \text{ for all } f \in \mathcal{H}. \tag{2.4}$$

Definition 2.11. *[6] For $K \in End_{\mathcal{A}}^*(\mathcal{H})$, a sequence $\{f_i\}_{i\in I}$ in $\mathcal{H}$ is called a K-Frame if there exist constants $A, B > 0$ such that for each $f \in \mathcal{H}$,*

$$A\langle K^* f, K^* f\rangle \leq \sum_{i\in I} \langle f, f_i\rangle \langle f_i, f\rangle \leq B\langle f, f\rangle.$$

We define the analysis operator for a K-Frame $\{f_i\}_{i\in I}$ as:

$$U : \mathcal{H} \to l^2(\mathcal{A}), \quad U(f) = \{\langle f, f_i\rangle\}_{i\in I}.$$

It is easy to show that the adjoint operator of U is

$$U^* : l^2(\mathcal{A}) \to \mathcal{H}, \quad U^*(\{a_i\}_{i\in I}) = \sum_{i\in I} a_i f_i. \tag{2.5}$$

U^* is called the synthesis operator for $\{f_i\}_{i\in I}$.

By composing U^* and U, the frame operator S for the K-frames is given by,

$$S : \mathcal{H} \to \mathcal{H}, \quad S(f) = U^* U(f) = \sum_{i\in I} \langle f, f_i\rangle f_i.$$

Proposition 2.12. *[4] $\{f_i\}_{i\in I}$ is a Bessel sequence for $\mathcal{H}$ with bound B, if and only if the operator U^* defined in (2.5) is a well defined bounded operator with $\|U^*\| \leq \sqrt{B}$.*

Definition 2.13. *[2] For $K \in End_{\mathcal{A}}^*(\mathcal{H})$, a sequence $\{\Lambda_i \in End_{\mathcal{A}}^*(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is called a K-g-Frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$ if there exist constants $A, B > 0$ such that for each $f \in \mathcal{H}$,*

$$A\langle K^*f, K^*f\rangle \leq \sum_{i\in I}\langle \Lambda_i f, \Lambda_i f\rangle \leq B\langle f, f\rangle. \tag{2.6}$$

We define the analysis operator for a K-g-Frame
$\{\Lambda_i \in End_{\mathcal{A}}^*(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ as:

$$T_\Lambda : \mathcal{H} \to l^2(\{\mathcal{H}_i\}_{i\in I}), \quad T_\Lambda(f) = \{\Lambda_i f\}_{i\in I}.$$

It is easy to show that the adjoint operator of T_Λ is

$$T_\Lambda^* : l^2(\{\mathcal{H}_i\}_{i\in I}) \to \mathcal{H}, \quad T_\Lambda^*(\{g_i\}_{i\in I}) = \sum_{i\in I} \Lambda_i^* g_i. \tag{2.7}$$

T_Λ^* is called the synthesis operator for $\{\Lambda_i\}_{i\in I}$.
By composing T_Λ^* and T_Λ, the frame operator S_Λ for the K-g-frames is given by,

$$S_\Lambda : \mathcal{H} \to \mathcal{H}, \quad S_\Lambda(f) = T_\Lambda^* T_\Lambda(f) = \sum_{i\in I} \Lambda_i^* \Lambda_i f.$$

Proposition 2.14. *[13] $\{\Lambda_i \in End_{\mathcal{A}}^*(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a g-Bessel sequence for $\mathcal{H}$ with bound B, if and only if the operator T_Λ^* defined in (2.7) is a well defined bounded operator with $\|T_\Lambda^*\| \leq \sqrt{B}$.*

Lemma 2.15. *[6] Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert C^*-modules and $T \in End_{\mathcal{A}}^*(\mathcal{H}_1, \mathcal{H}_2)$. Then*

(i) $R(T)^\perp = N(T^*), \quad \overline{R(T)} \subseteq R(T)^{\perp\perp} = N(T^*)^\perp.$

(ii) *$R(T)$ is closed if and only if $R(T^*)$ is closed, and in this case* $R(T) = N(T^*)^\perp$ and $R(T^*) = N(T)^\perp$.

3. Main result

3.1. K-Riesz bases

Definition 3.1. *For $K \in End^*_{\mathcal{A}}(\mathcal{H})$, a sequence $\{f_i\}_{i\in I}$ in $\mathcal{H}$ is called a K-Riesz basis if the following holds:*

(*i*) $\{f \in \mathcal{H} : \langle f, f_i\rangle = 0, \text{ for all } i \in I\} \subset N(K^*)$.

(*ii*) *There exist constants $A, B > 0$ such that for all $\{a_i\}_{i\in I} \in l^2(\mathcal{A})$,*

$$A\sum_{i\in I} a_i a_i^* \leq \left\langle \sum_{i\in I} a_i f_i, \sum_{i\in I} a_i f_i \right\rangle \leq B\sum_{i\in I} a_i a_i^*.$$

Theorem 3.2. *Let $\{f_i\}_{i\in I}$ be a K-frame for $\mathcal{H}$, then $\{f_i\}_{i\in I}$ is a K-Riesz basis for $\mathcal{H}$ if and only if, the synthesis operator U^* for K-frame $\{f_i\}_{i\in I}$ is bounded below.*

Proof. ($\Rightarrow$), Assume that $\{f_i\}_{i\in I}$ is a K-Riesz basis for $\mathcal{H}$.
i.e.

$$\{f \in \mathcal{H} : \langle f, f_i\rangle = 0, \text{ for all } i \in I\} \subset N(K^*),$$

and there exist constants $A, B > 0$ such that for all $\{a_i\}_{i\in I} \in l^2(\mathcal{A})$,

$$A\sum_{i\in I} a_i a_i^* \leq \left\langle \sum_{i\in I} a_i f_i, \sum_{i\in I} a_i f_i \right\rangle \leq B\sum_{i\in I} a_i a_i^*.$$

Then

$$A\left\langle \{a_i\}_{i\in I}, \{a_i\}_{i\in I}\right\rangle \leq \left\langle U^*(\{a_i\}_{i\in I}), U^*(\{a_i\}_{i\in I})\right\rangle \leq B\left\langle \{a_i\}_{i\in I}, \{a_i\}_{i\in I}\right\rangle.$$

Thus U^* is a bounded below.
($\Leftarrow$), Since $\{f_i\}_{i\in I}$ is a K-frame, then there exist constants $A, B > 0$ such that for each $f \in \mathcal{H}$,

$$A\langle K^*f, K^*f\rangle \leq \sum_{i\in I}\langle f, f_i\rangle\langle f_i, f\rangle \leq B\langle f, f\rangle.$$

Thus if $\langle f, f_i\rangle = 0$ for all $i \in I$, then $A\langle K^*f, K^*f\rangle = 0$, so $f \in N(K^*)$. Therefore, $\{f \in \mathcal{H} : \langle f, f_i\rangle = 0, \text{ for all } i \in I\} \subset N(K^*)$. Since $\{f_i\}_{i\in I}$ is Bessel sequence, by Proposition 2.12, U^* is bounded and there exists $B > 0$ such that $\|U^*\| \leq \sqrt{B}$. It means that, for all $\{a_i\}_{i\in I} \in l^2(\mathcal{A})$

$$\begin{aligned}\left\langle \sum_{i\in I} a_i f_i, \sum_{i\in I} a_i f_i \right\rangle &= \langle U^*(\{a_i\}_{i\in I}), U^*(\{a_i\}_{i\in I})\rangle \\ &\leq B\,\langle \{a_i\}_{i\in I}, \{a_i\}_{i\in I}\rangle \\ &= B\sum_{i\in I} a_i a_i^*.\end{aligned}$$

Since U^* is bounded below, then there is $m > 0$ such that for all $\{a_i\}_{i\in I} \in l^2(\mathcal{A})$,

$$\begin{aligned}m\sum_{i\in I} a_i a_i^* &= m\,\langle \{a_i\}_{i\in I}, \{a_i\}_{i\in I}\rangle \\ &\leq \langle U^*(\{a_i\}_{i\in I}), U^*(\{a_i\}_{i\in I})\rangle \\ &= \left\langle \sum_{i\in I} a_i f_i, \sum_{i\in I} a_i f_i \right\rangle.\end{aligned}$$

Therefore,

$$m\sum_{i\in I} a_i a_i^* \leq \left\langle \sum_{i\in I} a_i f_i, \sum_{i\in I} a_i f_i \right\rangle \leq B\sum_{i\in I} a_i a_i^*.$$

So $\{f_i\}_{i\in I}$ is a K-Riesz basis for $\mathcal{H}$. □

Theorem 3.3. *A sequence $\{f_i\}_{i\in I}$ is a K-Riesz basis for $\mathcal{H}$ if and only if there exists a bounded below operator $\Theta \in End_{\mathcal{A}}^*(\mathcal{H})$ such that $R(K) \subset R(\Theta)$ and $\Theta e_i = f_i$ for all $i \in I$, where $\{e_i\}_{i\in I}$ is a standard orthonormal basis of $\mathcal{H}$.*

Proof. Suppose that $\{f_i\}_{i\in I}$ is a K-Riesz basis for $\mathcal{H}$. Then

$$\{f \in \mathcal{H} : \langle f, f_i\rangle = 0, \text{ for all } i \in I\} \subset N(K^*). \tag{3.1}$$

and there exist $A, B > 0$ such that for all $\{a_i\}_{i\in I} \in l^2(\mathcal{A})$,

$$A\sum_{i\in I} a_i a_i^* \leq \left\langle \sum_{i\in I} a_i f_i, \sum_{i\in I} a_i f_i \right\rangle \leq B\sum_{i\in I} a_i a_i^*. \tag{3.2}$$

Define $\Theta : \mathcal{H} \to \mathcal{H}$ by

$$\Theta(x) = \sum_{i\in I} \langle x, e_i\rangle f_i.$$

We have $\{\langle x, e_i\rangle\}_{i\in I} \in l^2(\mathcal{A})$, for all $x \in \mathcal{H}$. Then by (3.2),

$$\begin{aligned}\langle\Theta(x),\Theta(x)\rangle &= \left\langle \sum_{i\in I}\langle x,e_i\rangle f_i, \sum_{i\in I}\langle x,e_i\rangle f_i\right\rangle\\ &\leq B\sum_{i\in I}\langle x,e_i\rangle\langle e_i,x\rangle\\ &= B\langle x,x\rangle.\end{aligned}$$

Then, Θ is well defined bounded operator and $\Theta e_i = f_i$ for all $i\in I$. Also, (3.2) implies that

$$A\langle x,x\rangle \leq \langle\Theta(x),\Theta(x)\rangle \leq B\langle x,x\rangle,\ \forall x\in\mathcal{H}. \tag{3.3}$$

By (3.3) we conclude that Θ is bounded below.

We have

$$\Theta^*:\mathcal{H}\to\mathcal{H},\quad \Theta^*(f)=\sum_{i\in I}\langle f,f_i\rangle e_i. \tag{3.4}$$

Now, if $f\in\mathcal{H}$ and $\Theta^*(f)=0$ then for all $i\in I$, $\langle f,f_i\rangle = 0$. Thus by (3.1), $f\in N(K^*)$. It means that $N(\Theta^*)\subset N(K^*)$, and $N(K^*)^\perp\subset N(\Theta^*)^\perp$. By Lemma 2.15, $R(K)\subset\overline{R(K)}\subset\overline{R(\Theta)}=R(\Theta)$.

Conversely, let $\Theta\in End^*_{\mathcal{A}}(\mathcal{H})$ be a bounded below operator such that for all $i\in I$, $\Theta e_i = f_i$ and $R(K)\subset R(\Theta)$. Then for all $\{a_i\}_{i\in I}\in l^2(\mathcal{A})$, we have

$$\begin{aligned}\left\langle \sum_{i\in I}a_i f_i, \sum_{i\in I}a_i f_i\right\rangle &= \left\langle \sum_{i\in I}a_i\Theta e_i, \sum_{i\in I}a_i\Theta e_i\right\rangle\\ &\leq \|\Theta\|^2\cdot\sum_{i\in I}a_i a_i^*,\end{aligned}$$

and there is $m>0$ such that

$$\begin{aligned}m\sum_{i\in I}a_i a_i^* &= m\left\langle \sum_{i\in I}a_i e_i, \sum_{i\in I}a_i e_i\right\rangle\\ &\leq \left\langle \Theta\left(\sum_{i\in I}a_i e_i\right), \Theta\left(\sum_{i\in I}a_i e_i\right)\right\rangle\\ &= \left\langle \sum_{i\in I}a_i f_i, \sum_{i\in I}a_i f_i\right\rangle.\end{aligned}$$

This shows that (3.2) holds for $\{f_i\}_{i\in I}$ with m and $\|\Theta\|^2$.

Let $f\in\mathcal{H}$ such that for all $i\in I$, $\langle f,f_i\rangle=0$, then

$$0=\langle f,\Theta e_i\rangle=\langle\Theta^* f,e_i\rangle,$$

thus $\Theta^* f = 0$. Since $R(K) \subset R(\Theta)$, by Lemma 2.15, $N(\Theta^*) \subset N(K^*)$. Therefore,

$$\{f \in \mathcal{H} : \langle f, f_i \rangle = 0, \text{ for all } i \in I\} \subset N(K^*).$$

$\square$

3.2. K-g-Riesz bases

Definition 3.4. *For $K \in End_{\mathcal{A}}^*(\mathcal{H})$, a sequence $\{\Lambda_i \in End_{\mathcal{A}}^*(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ in $\mathcal{H}$ is called a K-g-Riesz basis if the following holds:*

(i) $\{f \in \mathcal{H} : \Lambda_i f = 0, \text{ for all } i \in I\} \subset N(K^*)$.

(ii) *There exist constants $A, B > 0$ such that for all $\{g_i\}_{i\in I} \in l^2(\{\mathcal{H}_i\}_{i\in I})$,*

$$A \sum_{i\in I} \langle g_i, g_i \rangle \leq \left\langle \sum_{i\in I} \Lambda_i^* g_i, \sum_{i\in I} \Lambda_i^* g_i \right\rangle \leq B \sum_{i\in I} \langle g_i, g_i \rangle .$$

Example 3.5. *Let $\{e_i\}_{i=0}^{\infty}$ be the standard orthonormal basis of $\mathcal{H}$. For all $i \in \mathbf{N}$, define $\Lambda_i : \mathcal{H} \to \mathcal{A}$ by $\Lambda_i f = \langle f, e_i \rangle$ and $K : \mathcal{H} \to \mathcal{H}$ by $Kf = \sum_{i=1}^{\infty} \langle f, e_i \rangle e_i$. We see that $K^* f = \sum_{i=1}^{\infty} \langle f, e_i \rangle e_i$, and for $a \in \mathcal{A}$, $\Lambda_i^* a = a e_i$. For all $f \in \mathcal{H}$, we have*

$$\langle K^* f, K^* f \rangle = \left\langle \sum_{i=1}^{\infty} \langle f, e_i \rangle e_i, \sum_{i=1}^{\infty} \langle f, e_i \rangle e_i \right\rangle = \sum_{i=1}^{\infty} \langle f, e_i \rangle \langle e_i, f \rangle = \sum_{i=1}^{\infty} \langle \Lambda_i f, \Lambda_i f \rangle .$$

*If $\Lambda_i f = 0$, for all $i \in \mathbf{N}$, then $K^* f = 0$,
i.e., $\{f \in \mathcal{H} : \Lambda_i f = 0, \text{ for all } i \in \mathbf{N}\} \subset N(K^*)$. Also, for every $\{a_i\}_{i\in\mathbf{N}} \in l^2(\mathcal{A})$,*

$$\left\langle \sum_{i\in\mathbf{N}} \Lambda_i^* a_i, \sum_{i\in\mathbf{N}} \Lambda_i^* a_i \right\rangle = \left\langle \sum_{i\in\mathbf{N}} a_i e_i, \sum_{i\in\mathbf{N}} a_i e_i \right\rangle = \sum_{i\in\mathbf{N}} a_i a_i^* = \sum_{i\in\mathbf{N}} \langle a_i, a_i \rangle .$$

Therefore, $\{\Lambda_i\}_{i\in\mathbf{N}}$ is a K-g-Riesz basis for $\mathcal{H}$ with respect to $\mathcal{A}$.

Theorem 3.6. *Let $\{\Lambda_i \in End_{\mathcal{A}}^*(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$, then $\{\Lambda_i\}_{i\in I}$ is a K-g-Riesz basis for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$ if and only if, the synthesis operator T_Λ^* for K-g-frame $\{\Lambda_i\}_{i\in I}$ is bounded below.*

Proof. ($\Rightarrow$), We assume that $\{\Lambda_i\}_{i\in I}$ is a K-g-Riesz basis for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$. Then

$$\{f \in \mathcal{H} : \Lambda_i f = 0, \text{ for all } i \in I\} \subset N(K^*),$$

and there exist constants $A, B > 0$ such that for all $\{g_i\}_{i\in I} \in l^2(\{\mathcal{H}_i\}_{i\in I})$,

$$A \sum_{i\in I} \langle g_i, g_i\rangle \leq \left\langle \sum_{i\in I} \Lambda_i^* g_i, \sum_{i\in I} \Lambda_i^* g_i \right\rangle \leq B \sum_{i\in I} \langle g_i, g_i\rangle .$$

Then

$$A \langle \{g_i\}_{i\in I}, \{g_i\}_{i\in I}\rangle \leq \langle T_\Lambda^*(\{g_i\}_{i\in I}), T_\Lambda^*(\{g_i\}_{i\in I})\rangle \leq B \langle \{g_i\}_{i\in I}, \{g_i\}_{i\in I}\rangle .$$

Thus T_Λ^* is bounded below.
($\Leftarrow$), Since $\{\Lambda_i\}_{i\in I}$ is a K-g-Frame, then there exist constants $A, B > 0$ such that for all $f \in \mathcal{H}$,

$$A\langle K^* f, K^* f\rangle \leq \sum_{i\in I} \langle \Lambda_i f, \Lambda_i f\rangle \leq B\langle f, f\rangle.$$

Thus if $\Lambda_i f = 0$ for all $i \in I$, then $A\langle K^* f, K^* f\rangle = 0$, so $f \in N(K^*)$. Therefore, $\{f \in \mathcal{H} : \Lambda_i f = 0, \text{ for all } i \in I\} \subset N(K^*)$. Since $\{\Lambda_i\}_{i\in I}$ is g-Bessel sequence, by Proposition 2.14, T_Λ^* is bounded and there exists $B > 0$ such that $\|T_\Lambda^*\| \leq \sqrt{B}$. It means that, for all $\{g_i\}_{i\in I} \in l^2(\{\mathcal{H}_i\}_{i\in I})$,

$$\begin{aligned}\left\langle \sum_{i\in I} \Lambda_i^* g_i, \sum_{i\in I} \Lambda_i^* g_i \right\rangle &= \langle T_\Lambda^*(\{g_i\}_{i\in I}), T_\Lambda^*(\{g_i\}_{i\in I})\rangle \\ &\leq B \langle \{g_i\}_{i\in I}, \{g_i\}_{i\in I}\rangle \\ &= B \sum_{i\in I} \langle g_i, g_i\rangle .\end{aligned}$$

Since T_Λ^* is bounded below, then there is $m > 0$ such that

$$\begin{aligned} m \sum_{i\in I} \langle g_i, g_i\rangle &= m \langle \{g_i\}_{i\in I}, \{g_i\}_{i\in I}\rangle \\ &\leq \langle T_\Lambda^*(\{g_i\}_{i\in I}), T_\Lambda^*(\{g_i\}_{i\in I})\rangle \\ &= \left\langle \sum_{i\in I} \Lambda_i^* g_i, \sum_{i\in I} \Lambda_i^* g_i \right\rangle, \quad \forall \{g_i\}_{i\in I} \in l^2(\{\mathcal{H}_i\}_{i\in I}).\end{aligned}$$

Therefore, for all $\{g_i\}_{i\in I} \in l^2(\{\mathcal{H}_i\}_{i\in I})$

$$m \sum_{i\in I} \langle g_i, g_i\rangle \leq \left\langle \sum_{i\in I} \Lambda_i^* g_i, \sum_{i\in I} \Lambda_i^* g_i \right\rangle \leq B \sum_{i\in I} \langle g_i, g_i\rangle .$$

So $\{\Lambda_i\}_{i\in I}$ is a K-g-Riesz basis for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$. $\square$

Example 3.7. *Let $\{e_i\}_{i=0}^{\infty}$ be a standard orthonormal basis of $\mathcal{H}$, and for all $i \in \mathbf{N}$, let $\mathcal{H}_i = \mathcal{A}^2$, for all $i \in \mathbf{N}$. We define bounded operators $\Lambda_i : \mathcal{H} \to \mathcal{A}^2$ by $\Lambda_i f = (\langle f, e_i\rangle, \langle f, e_{i+1}\rangle)$, and $K : \mathcal{H} \to \mathcal{H}$ by $Kf = \sum_{i=0}^{\infty} \langle f, e_i\rangle e_{i+1} + \sum_{i=0}^{\infty} \langle f, e_{i+1}\rangle e_i$. We see that*

$$K^* f = \sum_{i=0}^{\infty} \langle f, e_i\rangle e_{i+1} + \sum_{i=0}^{\infty} \langle f, e_{i+1}\rangle e_i, \text{ for all } f \in \mathcal{H}.$$

and

$$\begin{aligned}\langle \Lambda_i f, \Lambda_i f\rangle &= \langle (\langle f, e_i\rangle, \langle f, e_{i+1}\rangle), (\langle f, e_i\rangle, \langle f, e_{i+1}\rangle)\rangle \\ &= \langle f, e_i\rangle \langle e_i, f\rangle + \langle f, e_{i+1}\rangle \langle e_{i+1}, f\rangle, \text{ for all } f \in \mathcal{H} \text{ and } i \in \mathbf{N}.\end{aligned}$$

For all $f \in \mathcal{H}$, we have

$$\begin{aligned}\langle K^* f, K^* f\rangle &= \Big\langle \sum_{i=0}^{\infty} \langle f, e_i\rangle e_{i+1} + \sum_{i=0}^{\infty} \langle f, e_{i+1}\rangle e_i, \sum_{i=0}^{\infty} \langle f, e_i\rangle e_{i+1} \\ &+ \sum_{i=0}^{\infty} \langle f, e_{i+1}\rangle e_i \\ &\leq \sum_{i=0}^{\infty} \langle f, e_i\rangle \langle e_i, f\rangle + \sum_{i=0}^{\infty} \langle f, e_{i+1}\rangle \langle e_{i+1}, f\rangle \\ &\leq \sum_{i=0}^{\infty} \langle \Lambda_i f, \Lambda_i f\rangle .\end{aligned}$$

Therefore,

$$\begin{aligned}\langle K^* f, K^* f\rangle &\leq \sum_{i=0}^{\infty} \langle \Lambda_i f, \Lambda_i f\rangle \\ &= \sum_{i=0}^{\infty} \langle f, e_i\rangle \langle e_i, f\rangle + \sum_{i=0}^{\infty} \langle f, e_{i+1}\rangle \langle e_{i+1}, f\rangle \\ &\leq 2 \langle f, f\rangle .\end{aligned}$$

We conclude that $\{\Lambda_i\}_{i\in\mathbf{N}}$ is a K-g-frame for $\mathcal{H}$ with respect to $\mathcal{A}^2$. Also, if $\Lambda_i f = 0$, for all $i \in \mathbf{N}$ and $f \in \mathcal{H}$ then, $\langle K^* f, K^* f\rangle = 0$, i.e. $\{f \in \mathcal{H} : \Lambda_i f = 0, \text{ for all } i \in \mathbf{N}\} \subset N(K^*)$.

Moreover, for all $i \in \mathbf{N}$,

$$\Lambda_i^*(a, b) = a e_i + b e_{i+1}, \text{ for all } (a, b) \in \mathcal{A}^2. \tag{3.5}$$

Let $\{(a_i, b_i)\}_{i=0}^{\infty}$ be a sequence with the property that

$$(a_0, b_0) = (0, -1), \quad (a_1, b_1) = (1, 0), \quad (a_i, b_i) = (0, 0), \text{ for all } i \geq 2.$$

Then, $\{(a_i, b_i)\}_{i=0}^{\infty} \in l^2(\{\mathcal{H}_i\}_{i\in\mathbf{N}})$ *and by (3.5),*

$$\sum_{i=0}^{\infty} \Lambda_i^*(a_i, b_i) = \Lambda_0^*(a_0, b_0) + \Lambda_1^*(a_1, b_1) = \Lambda_0^*(0, -1) + \Lambda_1^*(1, 0) = -e_2 + e_2 = 0,$$

but $(a_0, b_0) \neq (0, 0)$*. Therefore* T_Λ^* *is not injective, then* T_Λ^* *is not bounded below, and by Theorem 3.6,* $\{\Lambda_i \in End_{\mathcal{A}}^*(\mathcal{H}, \mathcal{H}_i) : i \in \mathbf{N}\}$ *is not a K-g-Riesz basis for* $\mathcal{H}$ *with respect to* $\{\mathcal{H}_i\}_{i\in\mathbf{N}}$.

Lemma 3.8. *Let* $\{\Lambda_i \in End_{\mathcal{A}}^*(\mathcal{H}, \mathcal{H}_i) : i \in I\}$*. Suppose that for each* $i \in I$*,* $\{e_{i,j} : j \in J_i\}$ *is a standard orthonormal basis for* $\mathcal{H}_i$*, where* J_i *is a subset of* $\mathbf{Z}$*. Then consider*

$$u_{i,j} = \Lambda_i^* e_{i,j};\ \text{ for all } i \in I \text{ and } j \in J_i. \tag{3.6}$$

We call $\{u_{i,j} : i \in I,\ j \in J_i\}$*, the sequence induced by* $\{\Lambda_i\}_{i\in I}$ *with respect to* $\{e_{i,j} : i \in I,\ j \in J_i\}$*. Also for all* $i \in I$*, we have the following relations:*

$$\Lambda_i f = \sum_{j\in J_i} \langle f, u_{i,j}\rangle\, e_{i,j} \ \text{ for all } f \in \mathcal{H}, \tag{3.7}$$

$$\Lambda_i^* g_i = \sum_{j\in J_i} \langle g_i, e_{i,j}\rangle\, u_{i,j} \ \text{ for all } g_i \in \mathcal{H}_i. \tag{3.8}$$

Proof. Since for all $i \in I$, $\{e_{i,j} : j \in J_i\}$ is an standard orthonormal basis for $\mathcal{H}_i$, then for all $f \in \mathcal{H}$ and all $i \in I$ we have

$$\Lambda_i f = \sum_{j\in J_i} \langle \Lambda_i f, e_{i,j}\rangle\, e_{i,j} = \sum_{j\in J_i} \langle f, \Lambda_i^* e_{i,j}\rangle\, e_{i,j} = \sum_{j\in J_i} \langle f, u_{i,j}\rangle\, e_{i,j},$$

and for all $g_i \in \mathcal{H}_i$ we have

$$\Lambda_i^* g_i = \Lambda_i^* \left(\sum_{j\in J_i} \langle g_i, e_{i,j}\rangle\, e_{i,j} \right) = \sum_{j\in J_i} \langle g_i, e_{i,j}\rangle\, \Lambda_i^* e_{i,j} = \sum_{j\in J_i} \langle g_i, e_{i,j}\rangle\, u_{i,j}.$$

□

Theorem 3.9. *Let* $\{\Lambda_i \in End_{\mathcal{A}}^*(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ *and* $(u_{i,j})$ *be defined as in (3.6). Then* $\{\Lambda_i\}_{i\in I}$ *is a K-g-Riesz basis for* $\mathcal{H}$ *with respect to* $\{\mathcal{H}_i\}_{i\in I}$ *if and only if* $\{u_{i,j} : i \in I,\ j \in J_i\}$ *is a K-Riesz basis for* $\mathcal{H}$.

Proof. First, assume that $\{\Lambda_i\}_{i\in I}$ is a K-g-Riesz basis for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$. Then

$$\{f \in \mathcal{H} : \Lambda_i f = 0, \text{ for all } i \in I\} \subset N(K^*),$$

and there exist constants $A, B > 0$ such that for all $\{g_i\}_{i\in I} \in l^2(\{\mathcal{H}_i\}_{i\in I})$,

$$A \sum_{i\in I} \langle g_i, g_i\rangle \leq \left\langle \sum_{i\in I} \Lambda_i^* g_i, \sum_{i\in I} \Lambda_i^* g_i \right\rangle \leq B \sum_{i\in I} \langle g_i, g_i\rangle . \tag{3.9}$$

Since for all $i \in I$, $\{e_{i,j} : j \in J_i\}$ is an standard orthonormal basis for $\mathcal{H}_i$, every $g_i \in \mathcal{H}_i$ has an expansion of the form $g_i = \sum_{j\in J_i} a_{i,j} e_{i,j}$, where $\{a_{i,j} : j \in J_i\} \in l^2(J_i)$. It follows that (3.9) is equivalent to

$$A \sum_{i\in I} \sum_{j\in J_i} a_{i,j} a_{i,j}^* \leq \left\langle \sum_{i\in I} \sum_{j\in J_i} a_{i,j} u_{i,j}, \sum_{i\in I} \sum_{j\in J_i} a_{i,j} u_{i,j} \right\rangle \leq B \sum_{i\in I} \sum_{j\in J_i} a_{i,j} a_{i,j}^*.$$

On the other hand, we see from $\Lambda_i f = \sum_{j\in J_i} \langle f, u_{i,j}\rangle e_{i,j}$ for all $i \in I$, that

$$\{f \in \mathcal{H} : \Lambda_i f = 0, \text{ for all } i \in I\} = \{f \in \mathcal{H} : \langle f, u_{i,j}\rangle = 0, \text{ for all }$$

$$i \in I \text{ and } j \in J_i\}.$$

Hence

$$\{f \in \mathcal{H} : \Lambda_i f = 0, \text{ for all } i \in I\} \subset N(K^*),$$

if and only if

$$\{f \in \mathcal{H} : \langle f, u_{i,j}\rangle = 0, \text{ for all } i \in I \text{ and } j \in J_i\} \subset N(K^*).$$

Therefore, $\{\Lambda_i\}_{i\in I}$ is a K-g-Riesz basis for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$ if and only if $\{u_{i,j} : i \in I, j \in J_i\}$ is a K-Riesz basis for $\mathcal{H}$. $\square$

Theorem 3.10. *Let $\{\Lambda_i \in End_{\mathcal{A}}^*(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ and $\{u_{i,j} : i \in I, j \in J_i\}$ be defined as in* (3.6). *Then $\{\Lambda_i\}_{i\in I}$ is a g-orthonormal basis for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$ if and only if $\{u_{i,j} : i \in I, j \in J_i\}$ is a standard orthonormal basis for $\mathcal{H}$.*

Proof. ($\Rightarrow$) Assume that $\{\Lambda_i\}_{i\in I}$ is g-orthonormal basis for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$. It follows from (3.6) and (2.3) that

$$\begin{aligned}\langle u_{i_1,j_1}, u_{i_2,j_2}\rangle &= \left\langle \Lambda^*_{i_1} e_{i_1,j_1}, \Lambda^*_{i_2} e_{i_2,j_2}\right\rangle \\ &= \delta_{i_1,i_2}\langle e_{i_1,j_1}, e_{i_2,j_2}\rangle \\ &= \delta_{i_1,i_2}\delta_{j_1,j_2},\end{aligned}$$

for all $i_1, i_2 \in I, \quad j_1 \in J_{i_1}, \quad j_2 \in J_{i_2}$.

Hence $\{u_{i,j} : \ i \in I, \ j \in J_i\}$ is a standard orthonormal sequence. Moreover, observe that

$$\text{for all } f \in \mathcal{H}, \quad \langle f, f\rangle = \sum_{i\in I}\langle \Lambda_i f, \Lambda_i f\rangle = \sum_{i\in I}\left\langle \sum_{j\in J_i}\langle f, u_{i,j}\rangle e_{i,j}, \sum_{j\in J_i}\langle f, u_{i,j}\rangle e_{i,j}\right\rangle = \sum_{i\in I}\sum_{j\in J_i}\langle f, u_{i,j}\rangle\langle u_{i,j}, f\rangle.$$

We have $\{u_{i,j} : \ i \in I, \ j \in J_i\}$ is a standard orthonormal basis for $\mathcal{H}$. ($\Leftarrow$) we need only to show that (2.3) holds. In fact, we see from (3.8) that for any $i \neq j \in I, \quad g_i \in \mathcal{H}_i, \quad g_j \in \mathcal{H}_j$,

$$\left\langle \Lambda^*_i g_i, \Lambda^*_j g_j\right\rangle = \left\langle \sum_{k\in J_i}\langle g_i, e_{i,k}\rangle u_{i,k}, \sum_{l\in J_j}\langle g_j, e_{j,l}\rangle u_{j,l}\right\rangle = 0,$$

and for all $g_i \in \mathcal{H}_i$,

$$\langle \Lambda^*_i g_i, \Lambda^*_i g_i\rangle = \left\langle \sum_{k\in J_i}\langle g_i, e_{i,k}\rangle u_{i,k}, \sum_{l\in J_i}\langle g_i, e_{i,l}\rangle u_{i,l}\right\rangle = \langle g_i, g_i\rangle.$$

Thus $\{\Lambda_i\}_{i\in I}$ is g-orthonormal basis for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$. □

Theorem 3.11. *A sequence $\{\Lambda_i \in End^*_{\mathcal{A}}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a K-g-Riesz basis for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$ if and only if there exists a g-orthonormal basis $\{\Gamma_i \in End^*_{\mathcal{A}}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ for $\mathcal{H}$ and a bounded surjective operator $U \in End^*_{\mathcal{A}}(\mathcal{H})$ such that $\Lambda_i = \Gamma_i U$ for all $i \in I$, and $R(K) \subset R(U^*)$.*

Proof. Let $\{e_{i,j} : j \in J_i\}$ be the standard orthonormal basis of $\mathcal{H}_i$, for all $i \in I$. We assume that $\{\Lambda_i\}_{i\in I}$ is a K-g-Riesz basis for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$. By Theorem 3.9, we can find a K-Riesz basis $\{u_{i,j} : \ i \in I, \ j \in J_i\}$ for $\mathcal{H}$ such that

$$\Lambda_i f = \sum_{j\in J_i}\langle f, u_{i,j}\rangle e_{i,j}, \ \text{ for all } i \in I, \ \text{ and } f \in \mathcal{H}.$$

Take a standard orthonormal basis $\{v_{i,j}: \ i \in I, \ j \in J_i\}$ for $\mathcal{H}$. Since $\{u_{i,j}: i \in I, \ j \in J_i\}$ is a K-Riesz basis for $\mathcal{H}$, by Theorem 3.3, there exists a bounded below operator $\Theta \in End^*_{\mathcal{A}}(\mathcal{H})$ such that

$$\Theta v_{i,j} = u_{i,j}, \text{ for all } i \in I, \text{ and } j \in J_i$$

and $R(K) \subset R(\Theta)$. Put $U = \Theta^*$, then by Proposition 2.2, $U: \mathcal{H} \to \mathcal{H}$ is a bounded surjective operator and $R(K) \subset R(U^*)$. For all $i \in I$, let $\Gamma_i \in End^*_{\mathcal{A}}(\mathcal{H}, \mathcal{H}_i)$ be such that

$$\Gamma_i g = \sum_{j \in J_i} \langle g, v_{i,j} \rangle e_{i,j}, \text{ for all } g \in \mathcal{H}.$$

By Theorem 3.10, $\{\Gamma_i\}_{i\in I}$ is a g-orthonormal basis for $\mathcal{H}$. Moreover, for any $f \in \mathcal{H}$ and $i \in I$,

$$\Gamma_i U f = \sum_{j \in J_i} \langle Uf, v_{i,j} \rangle e_{i,j} = \sum_{j \in J_i} \langle f, \Theta v_{i,j} \rangle e_{i,j} = \sum_{j \in J_i} \langle f, u_{i,j} \rangle e_{i,j} = \Lambda_i f.$$

Hence for all $i \in I$, $\Lambda_i = \Gamma_i U$.

Conversely, let $\{\Gamma_i\}_{i\in I}$ be a g-orthonormal basis for $\mathcal{H}$ and U be a bounded surjective operator on $\mathcal{H}$ such that $\Lambda_i = \Gamma_i U$ for all $i \in I$, and $R(K) \subset R(U^*)$. Then

$$\left\langle \Gamma_i^* g_i, \Gamma_j^* g_j \right\rangle = \delta_{i,j} \left\langle g_i, g_j \right\rangle, \text{ for all } i, j \in I, \ g_i \in \mathcal{H}_i, \text{and } g_j \in \mathcal{H}_j,$$

and

$$\sum_{i \in I} \left\langle \Gamma_i f, \Gamma_i f \right\rangle = \left\langle f, f \right\rangle, \text{ for all } f \in \mathcal{H}.$$

If $\Lambda_i f = 0$, for all $i \in I$ and $f \in \mathcal{H}$, then

$$0 = \sum_{i \in I} \left\langle \Lambda_i f, \Lambda_i f \right\rangle = \sum_{i \in I} \left\langle \Gamma_i U f, \Gamma_i U f \right\rangle = \left\langle Uf, Uf \right\rangle,$$

i.e., $f \in N(U)$. From $R(K) \subset R(U^*)$ and by Lemma 2.15, $f \in N(K^*)$. Thus, $\{f \in \mathcal{H} : \Lambda_i f = 0, \text{ for all } i \in I\} \subset N(K^*)$. By Theorem 3.10, we can find a standard orthonormal basis $\{v_{i,j}: \ i \in I, \ j \in J_i\}$ for $\mathcal{H}$ such that $\Gamma_i g = \sum_{j \in J_i} \langle g, v_{i,j} \rangle e_{i,j}$, for all $i \in I$ and $g \in \mathcal{H}$. Hence,

$$\Lambda_i f = \Gamma_i U f = \sum_{j \in J_i} \langle Uf, v_{i,j} \rangle e_{i,j} = \sum_{j \in J_i} \langle f, U^* v_{i,j} \rangle e_{i,j}, \text{ for all } i \in I \text{ and } f \in \mathcal{H}.$$

By Proposition 2.2, U^* is a bounded below operator and $R(K) \subset R(U^*)$, then by Theorem 3.3, $\{U^* v_{i,j}: \ i \in I, \ j \in J_i\}$ is a K-Riesz basis for $\mathcal{H}$. Thus, by Theorem 3.9, we conclude that $\{\Lambda_i\}_{i\in I}$ is a K-g-Riesz basis for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$. $\square$

Theorem 3.12. *Let $\{\Lambda_i \in End^*_{\mathcal{A}}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a K-g-Riesz basis for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$ and let $\{w_{i,j}\}_{j\in M_i}$ be a Riesz basis for $\mathcal{H}_i$, for all $i \in I$ with bounded C_i and D_i such that $0 < \inf\limits_i C_i$ and $\sup\limits_i D_i < \infty$, where M_i is a subset of $\mathbf{Z}$. Then $\{\Lambda_i^* w_{i,j}\}_{i\in I, j\in M_i}$ is a K-Riesz basis for $\mathcal{H}$.*

Proof. Since $\{\Lambda_i \in End^*_{\mathcal{A}}(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a K-g-Riesz basis for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$, then

$$\{f \in \mathcal{H} : \Lambda_i f = 0, \text{ for all } i \in I\} \subset N(K^*),$$

and there exist constants $A, B > 0$ such that for all $\{g_i\}_{i\in I} \in l^2(\{\mathcal{H}_i\}_{i\in I})$,

$$A \sum_{i\in I} \langle g_i, g_i\rangle \leq \left\langle \sum_{i\in I} \Lambda_i^* g_i, \sum_{i\in I} \Lambda_i^* g_i \right\rangle \leq B \sum_{i\in I} \langle g_i, g_i\rangle . \tag{3.10}$$

Moreover, since for all $i \in I$, $\{w_{i,j}\}_{j\in M_i}$ is a Riesz basis for $\mathcal{H}_i$,

$$\{g_i \in \mathcal{H}_i : \langle g_i, w_{i,j}\rangle = 0, \text{ for all } j \in M_i\} = \{0\},$$

and for each $\{a_{i,j}\}_{j\in M_i} \in l^2(M_i)$,

$$C_i \sum_{j\in M_i} a_{i,j}a^*_{i,j} \leq \left\langle \sum_{j\in M_i} a_{i,j}w_{i,j}, \sum_{j\in M_i} a_{i,j}w_{i,j} \right\rangle \leq D_i \sum_{j\in M_i} a_{i,j}a^*_{i,j}, \text{for all} i \in I. \tag{3.11}$$

We have

$$\begin{aligned} \{f \in \mathcal{H} : \langle \Lambda_i f, w_{i,j}\rangle &= 0, \text{ for all } i \in I \text{ and } j \in M_i\} \\ &= \{f \in \mathcal{H} : \Lambda_i f = 0, \text{ for all } i \in I\} \\ &\subset N(K^*). \end{aligned}$$

Thus $\{f \in \mathcal{H} : \langle f, \Lambda_i^* w_{i,j}\rangle = 0, \text{ for all } i \in I \text{ and } j \in M_i\} \subset N(K^*)$. If $\inf\limits_i C_i = C$ and $\sup\limits_i D_i = D$ then by (3.10) and (3.11), for all $\{\beta_{i,j}\}_{i\in I, j\in M_i} \in l^2(I)$, we have

$$\begin{aligned} AC \sum_{i\in I}\sum_{j\in M_i} \beta_{i,j}\beta^*_{i,j} &\leq A \sum_{i\in I} \left\langle \sum_{j\in M_i} \beta_{i,j}w_{i,j}, \sum_{j\in M_i} \beta_{i,j}w_{i,j} \right\rangle \\ &\leq \left\langle \sum_{i\in I} \Lambda_i^* \left(\sum_{j\in M_i} \beta_{i,j}w_{i,j} \right), \sum_{i\in I} \Lambda_i^* \left(\sum_{j\in M_i} \beta_{i,j}w_{i,j} \right) \right\rangle \\ &\leq B \sum_{i\in I} \left\langle \sum_{j\in M_i} \beta_{i,j}w_{i,j}, \sum_{j\in M_i} \beta_{i,j}w_{i,j} \right\rangle \\ &\leq BD \sum_{i\in I}\sum_{j\in M_i} \beta_{i,j}\beta^*_{i,j}. \end{aligned}$$

So

$$AC \sum_{i\in I}\sum_{j\in M_i} \beta_{i,j}\beta_{i,j}^* \leq \left\langle \sum_{i\in I}\sum_{j\in M_i} \beta_{i,j}\Lambda_i^* w_{i,j}, \sum_{i\in I}\sum_{j\in M_i} \beta_{i,j}\Lambda_i^* w_{i,j} \right\rangle$$
$$\leq BD \sum_{i\in I}\sum_{j\in M_i} \beta_{i,j}\beta_{i,j}^*.$$

Thus $\{\Lambda_i^* w_{i,j}\}_{i\in I, j\in M_i}$ is a K-Riesz basis for $\mathcal{H}$. □

Theorem 3.13. *Let $\{\Lambda_i\}_{i\in I}$ is a K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$. Suppose there exists a finite subset σ of I for which $\{\Lambda_i\}_{i\in I\backslash\sigma}$ is a K-g-Riesz basis for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I\backslash\sigma}$. If $\sum_{i\in I}\Lambda_i^* g_i$ is converges, then $\{g_i\}_{i\in I} \in l^2(\{\mathcal{H}_i\}_{i\in I})$.*

Proof. Suppose that $\sum_{i\in I}\Lambda_i^* g_i$ converges, where $g_i \in \mathcal{H}_i$ for all $i \in I$. So $\sum_{i\in I\backslash\sigma}\Lambda_i^* g_i$ converges. Since $\{\Lambda_i\}_{i\in I\backslash\sigma}$ is a K-g-Riesz basis for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I\backslash\sigma}$, by Theorem 3.11, there exist a bounded surjective operator $U : \mathcal{H} \to \mathcal{H}$ and a g-orthonormal basis $\{\Gamma_i\}_{i\in I\backslash\sigma}$ for $\mathcal{H}$ such that $\Lambda_i = \Gamma_i U$ for all $i \in I\backslash\sigma$ and $R(K) \subset R(U^*)$.
So

$$\sum_{i\in I\backslash\sigma}\Lambda_i^* g_i = \sum_{i\in I\backslash\sigma}(\Gamma_i U)^* g_i = U^*\left(\sum_{i\in I\backslash\sigma}\Gamma_i^* g_i\right).$$

Since $\{\Gamma_i\}_{i\in I\backslash\sigma}$ is a g-orthonormal basis, we have

$$\sum_{i\in I\backslash\sigma}\langle g_i, g_i\rangle = \left\langle \sum_{i\in I\backslash\sigma}\Gamma_i^* g_i, \sum_{i\in I\backslash\sigma}\Gamma_i^* g_i \right\rangle < \infty.$$

Then $\{g_i\}_{i\in I\backslash\sigma} \in l^2(\{\mathcal{H}_i\}_{i\in I\backslash\sigma})$ and this implies that $\{g_i\}_{i\in I} \in l^2(\{\mathcal{H}_i\}_{i\in I})$.
□

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