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Maximal graphical realization of a topology

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Abstract

Given a topological space, the graphical realizations of it with as many edges as possible, called maximal graphical realizations, are studied here. Every finite topological space admits a maximal graphical realization. However, there are graphs which are not maximal graphical realizations of any topology. A tree of odd order is never a maximal graphical realization of a topological space. Maximal graphical realization of a topology is a cycle if and only if it is C_3. It is shown that chain topologies admit unique maximal graphical realizations. A lower bound for the size of a maximal graphical realization is also obtained.

Keywords: *Set-indexer; topology; t-set-graceful; optimal space.*

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1. Introduction

Acharya [1] initiated the study of set-valuations by introducing the notions of set-indexer and topological set-indexer for a given graph. Subsequently, several authors [4], [5], [7], [8], [11], [15] studied set-valuations of graphs and obtained many significant results. In [12, 16], the authors investigated topological set-indexers and derived the topological number of certain graphs. The additional study [13] established topological set-gracefulness of certain stars, paths and related graphs. Further, the topological set-gracefulness of subgraphs, especially spanning subgraphs, of a topologically set-graceful graph has been examined in [6]. Later in [14], the authors concentrated on topologies formed by the vertex labels of a topological set-indexer of a given graph. The graph then is called a graphical realization of the topology and the topological space is said to be a graphically realizable space.

This paper sheds more light on graphical realizations by exploring graphical realizations with maximum size for a given topological space. Such a graphical realization is called maximal graphical realization and every topology with finitely many open sets has a maximal graphical realization. It is proved that star graph is a maximal graphical realization of a finite topological space if and only if the space is quasi-discrete. Further, no cycle of order greater than three is a maximal graphical realization of a topology. It is shown that chain topologies admit unique maximal graphical realizations and conjectured that the converse is also true. Investigations are also carried out on the lower bound for the size of a maximal graphical realization.

2. Preliminaries

In this section we include certain definitions and known results needed for the subsequent development of the study. For a nonempty set X, the set of all subsets of X is denoted by 2^X. By A^c, we mean, the complement of a set A. We always denote a graph under consideration by G and its vertex and edge sets by V and E respectively. By $G' \subseteq G$ we mean G' is a subgraph of G while $G' \subset G$ means G' is a proper subgraph of G. The empty graph of order n is denoted by N_n. By $G[v_1, \ldots, v_n]$ we mean the subgraph of G induced by the vertices $v_1, \ldots, v_n$. The order and size of a graph G is denoted by $o(G)$ and $s(G)$ respectively. When it is said that two graphs are different we mean they are non-isomorphic. All graphs considered in this paper are simple.

Definition 2.1. *[2] Let $G = (V, E)$ be a given graph and X be a nonempty set. Then a mapping $f : V \to 2^X$, or $f : E \to 2^X$, or $f : V \cup E \to 2^X$ is called a set-assignment or set-valuation of the vertices or edges or both.*

Definition 2.2. *[2] Let G be a given graph and X be a nonempty set. Then a set-valuation $f : V \cup E \to 2^X$ is a set-indexer of G if*

1. *$f(u, v) = f(u) \oplus f(v), \forall (u, v) \in E$, where '$\oplus$' denotes the binary operation of taking the symmetric difference of the sets in 2^X*
2. *the restriction maps $f|_V$ and $f|_E$ are both injective.*

In this case, X is called an indexing set of G. Clearly a graph can have many indexing sets and the minimum of the cardinalities of the indexing sets is said to be the set-indexing number of G, denoted by $\gamma(G)$. The set-indexing number of K_1 is defined to be zero.

Theorem 2.3. [2] Every graph has a set-indexer.

Theorem 2.4. [2] Let X be an indexing set of $G = (V, E)$. Then

1. $|E| \leq 2^{|X|} - 1$ and
2. $\lceil \log_2(|E| + 1) \rceil \leq \gamma(G) \leq |V| - 1$, where $\lceil \ \rceil$ is the ceiling function.

Theorem 2.5. [2] If G' is a subgraph of G, then $\gamma(G') \leq \gamma(G)$.

Definition 2.6. *[2] A graph G is set-graceful if $\gamma(G) = \log_2(|E| + 1)$ and the corresponding set-indexer is called a set-graceful labeling of G.*

Definition 2.7. *[2] A set-indexer f of a graph G with indexing set X is said to be a topological set-indexer (t-set-indexer) if $f(V)$ is a topology on X and X is called the topological indexing set(t-indexing set) of G. The minimum number among the cardinalities of such topological indexing sets is said to be the topological number (t-number) of G, denoted by $\tau(G)$ and the corresponding t-set indexer is called the optimal t-set-indexer of G. A graph G is said to be topologically set-graceful or t-set-graceful if $\gamma(G) = \tau(G)$.*

Theorem 2.8. [2] Every graph with at least two vertices has a t-set-indexer.

Theorem 2.9. [2] Let G be any graph with at least two vertices. Then $\gamma(G) \leq \tau(G)$.

Theorem 2.10. [12] If G' is a spanning sub graph of G, then $\tau(G') \leq \tau(G)$.

Definition 2.11. *[10] A topological space (X, τ) is said to be finite if the set X is finite.*

Definition 2.12. [14] Let X be a nonempty set. A topology τ on X is said to be graphically realizable if there exists a graph $G = (V, E)$ and a set-indexer $f : V \cup E \to 2^X$ such that $f(V) = \tau$. In this case G is said to be a graphical realization of τ. Also the topological space (X, τ) is called a graphically realizable space.
By $\mathcal{G}_\tau$, we denote the collection of all graphs which realizes (X, τ). That is, $\mathcal{G}_\tau$ is the set of all graphical realizations of the topological space (X, τ).

Remark 2.13. *In [14], it has been shown that any finite topological space (X, τ); $|\tau| = n$ is graphically realizable by $K_{1,n-1}$. It can be achieved by labeling the central vertex of $K_{1,n-1}$ with $\emptyset$ and the remaining vertices with the $n - 1$ nonempty elements of τ in any order. Here onwards we refer to it, the star realization of (X, τ).*

Theorem 2.14. [14] Every topology with finitely many open sets is graphically realizable.

Theorem 2.15. [14] A graph G of order n (> 2) realizes only chain topologies if and only if $K_n \setminus E(C_n) \subseteq G \subseteq K_n$.

Theorem 2.16. [14] Every graph of order n realizes every chain topology with n open sets.

Definition 2.17. [14] A topological space (X, τ) realized by a graph G is called an optimal space of G if $\tau(G) = |X|$. In this case G is said to be an optimal graphical realization of τ. The collection of all optimal graphical realizations of a topological space (X, τ) is denoted by $\mathcal{O}_\tau$. Evidently, $\mathcal{O}_\tau \subseteq \mathcal{G}_\tau$.

Theorem 2.18. [6] $\tau(K_n \setminus E(K_{1,3})) = \tau(K_n) - 1$; $n \geq 4$.

Theorem 2.19. [14] Every optimal topological space is T_0.

Theorem 2.20. [13] $\tau(K_{1,2^n-1}) = n$.

Definition 2.21. *[3] A topological space is called quasi-discrete if every open set is closed and vice versa.*

3. Maximal Graphical Realization of a Topology

Definition 3.1. *A graph G is said to be a maximal graphical realization of a topological space (X, τ) if*

1. *G is a graphical realization of (X, τ) and*
2. *$s(G) \geq s(H)$ whenever H is a graphical realization of (X, τ)*

Evidently, a topological space may possess many maximal graphical realizations. Obviously, spaces realized by the complete graphs have unique maximal graphical realization.

The collection of all maximal graphical realizations of a given topological space (X, τ) is denoted by $\mathcal{M}_\tau$. Clearly, $\mathcal{M}_\tau \subseteq \mathcal{G}_\tau$.

Remark 3.2. *In contrast with the optimal graphical realization, every finite topological space has a maximal graphical realization. Hence $\mathcal{M}_\tau \neq \emptyset$, even if $\mathcal{O}_\tau = \emptyset$. On the other hand, every element of $\mathcal{O}_\tau$ need not be in $\mathcal{M}_\tau$. For example, consider the topological space (X, τ); $X = \{a, b, c\}$, $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$. Two different elements in $\mathcal{O}_\tau$ are given below:*

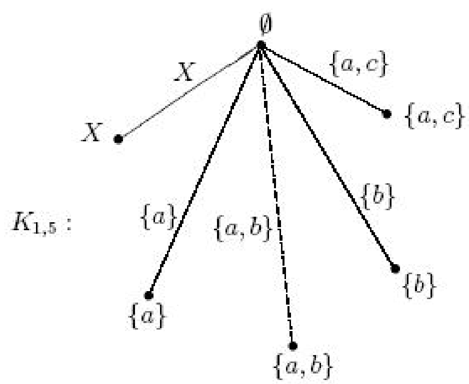

Figure 1: Optimal graphical realization

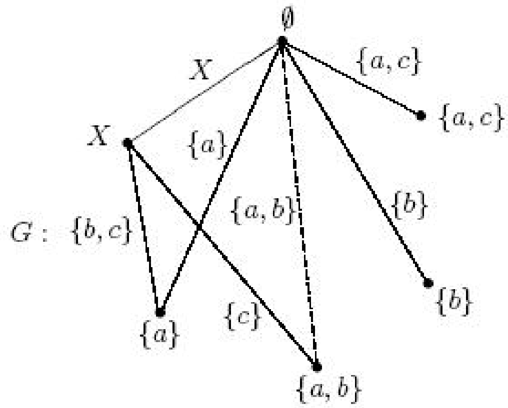

Figure 2: Optimal graphical realization

By theorem 2.4, theorem 2.9 and theorem 2.10 we have $3 \leq \gamma(K_{1,5}) \leq \tau(K_{1,5}) \leq \tau(G) \leq |X| = 3$. *Consequently* $K_{1,5}$, $G \in \mathcal{O}_\tau$. *Since all nonempty subsets of* X *are present in* G *as edge labels,* $G \in \mathcal{M}_\tau$. *Since* $s(K_{1,5}) < s(G)$ *we have* $K_{1,5} \notin \mathcal{M}_\tau$.

Theorem 3.3. *Let* $G = (V, E)$ *be a graphical realization of a topological space* (X, τ) *with t-set-indexer* f. *If no more edges can be drawn in* G *keeping* f *fixed, then* $G \in \mathcal{M}_\tau$.

Proof 1. *Let* $H = (U, F)$ *be a maximal graphical realization of* (X, τ) *with t-set-indexer* g. *Then* $g(U) = \tau = f(V)$. *Let* A *be any edge label of* H. *Then there are vertices* a *and* b *in* H *such that* $A = g(a) \oplus g(b)$. *Since* $g(a)$, $g(b) \in f(V)$, *there are vertices* c, d *in* G *such that* $f(c) = g(a)$ *and* $f(d) = g(b)$.

If $(c, d) \in E$, *then we have* $A = f(c) \oplus f(d) = f(c, d)$ *so that* A *is an edge label of* G *also.*
If $(c, d) \notin E$, *then since no more edges can be drawn in* G *keeping* f *fixed, there must be an edge* (p, q) *in* G *such that* $f(p, q) = f(c) \oplus f(d) = A$. *Thus every edge label of* H *is also an edge label of* G *so that* $s(H) \leq s(G)$. *Since* $G \in \mathcal{G}_\tau$ *and* $H \in \mathcal{M}_\tau$ *then it follows that* $G \in \mathcal{M}_\tau$. □

Definition 3.4. *Let* G *be a graph. Any graph* H *obtained from* G *by joining atleast one pair of nonadjacent vertices is called an extension of* G *and we say* G *is extendable to* H.

Following is a useful consequence of theorem 3.3.

Corollary 3.5. *Let (X, τ) be a finite topological space. Any graphical realization of (X, τ) is extendable to a maximal graphical realization of (X, τ) with the same vertex labels.*

Theorem 3.6. *Let (X, τ) be a topological space and $G \in \mathcal{M}_\tau$. Then $s(G) \geq |\tau| - 1$.*

Proof 2. *By the star realization of (X, τ) we have $K_{1,|\tau|-1} \in \mathcal{G}_\tau$. Since $G \in \mathcal{M}_\tau$ we have $s(G) \geq s(K_{1,|\tau|-1})$.* □

Remark 3.7. *The converse of the above theorem is not true, in general. For example consider the topological space (X, τ); $X = \{a, b, c, d\}$ and τ=$\{\emptyset$, $\{a\}$, $\{b\}$, $\{a, b\}$, $\{a, b, c\}$, $X\}$. From the diagram below it follows that both C_6, $G \in \mathcal{G}_\tau$ where $G = K_6 \setminus E(K_{1,3})$.*

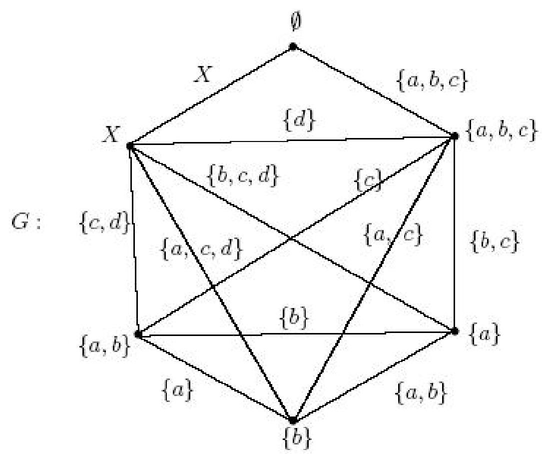

Figure 3: Graphical realization of τ by $G = K_6 \setminus E(K_{1,3})$.

By theorem 2.15, any proper spanning supergraph of G does not realize τ and hence $s(H) \leq 12$ for every $H \in \mathcal{G}_\tau$. Consequently, $G \in \mathcal{M}_\tau$ so that $C_6 \notin \mathcal{M}_\tau$.

Theorem 3.8. *Let (X, τ) be a given topological space with $|\tau| = n$ and $\mathcal{O}_\tau \neq \emptyset$. Then $K_n \in \mathcal{G}_\tau \Rightarrow K_n \in \mathcal{O}_\tau$.*

Proof 3. *Since $\mathcal{O}_\tau \neq \emptyset$, there exists a graph $G \in \mathcal{O}_\tau$ with $o(G) = n$ and $\tau(G) = |X|$. Now $K_n \in \mathcal{G}_\tau \Rightarrow \tau(K_n) \leq |X|$. But $G \subseteq K_n$ so that by theorem 2.10, $\tau(G) \leq \tau(K_n)$. Thus, we have $|X| = \tau(G) \leq \tau(K_n) \leq |X|$ so that $\tau(K_n) = |X|$. Consequently, $K_n \in \mathcal{O}_\tau$.* □

Remark 3.9. *There are topological spaces (X, τ) with $|\tau| = n$ and $\mathcal{O}_\tau \neq \emptyset$, but $K_n \notin \mathcal{O}_\tau$. For example, $K_7 \notin \mathcal{G}_\tau$; τ is the topology $\{\emptyset, X, \{a\}, \{b\}, \{a,b\}, \{a,b,c\}, \{a,b,c,d\}\}$ on $X = \{a,b,c,d,e\}$. Note that τ is not a chain topology. But it can be easily shown that $K_7 \setminus E(K_{1,3}) \in \mathcal{G}_\tau$ and by theorem 2.18 we have $\tau(K_7 \setminus E(K_{1,3})) = 5$. Thus, $K_7 \setminus E(K_{1,3}) \in \mathcal{O}_\tau$.*

By theorem 2.8 and from the definition of $\tau(G)$ we have the following:

Theorem 3.10. *Every graph G of order at least 2 is an optimal graphical realization of some topology.*

Remark 3.11. *There are graphs which are not maximal graphical realizations of any topology (X, τ). For example, $K_n \setminus E(K_2)$ and $K_n \setminus E(K_{1,2})$ are not maximal graphical realizations of any topology with n open sets (by theorem 2.15).*

Theorem 3.12. *Let G be a maximal graphical realization of a discrete space (X, τ). Then G is set-graceful as well as t-set-graceful.*

Proof 4. *For any graphical realization H of the discrete space (X, τ) we must have $n = o(H) = 2^{|X|}$ and $s(H) \leq n - 1$. By the star realization of (X, τ) we have $K_{1,n-1} \in \mathcal{G}_\tau$. Since $s(K_{1,n-1}) = n - 1$ we get $K_{1,n-1} \in \mathcal{M}_\tau$. Consequently, $o(G) = n$ and $s(G) = n - 1$, since $G \in \mathcal{M}_\tau$. Thus, there exists a t-set-indexer f of G with $f(V) = 2^X$ and $f(E) = 2^X \setminus \emptyset$. Hence, G is set-graceful and t-set-graceful.* □

Remark 3.13. *Since there are many graphs of odd order which are both set-graceful and t-set-graceful, the converse of the above theorem is not true. For, such a graph cannot even be a graphical realization of a discrete space. K_3 is one such graph.*

The following is a simple result.

Theorem 3.14. *Let G be a set-graceful graph with indexing set X and f be a t-set-indexer of G with the same indexing set X. Then G is a maximal graphical realization of $(X, f(V))$.*

Lemma 3.15. *Let (X, τ) be an optimal space of a graph G. If (X, τ) is not discrete, then $H \in \mathcal{M}_\tau \Rightarrow s(H) \geq |\tau|$.*

Proof 5. *Let A be the collection of all symmetric differences of the open sets $A_1, \ldots, A_n$ of τ; $|\tau| = n$. Without loss of generality we may assume that $A_{n-1} = X$ and $A_n = \emptyset$. Since $\emptyset \oplus A_i = A_i \in A$ for $1 \leq i \leq n-1$ we have $\tau \subseteq A$ and $|A| \geq n$. Let $|A| = n$ so that $\tau = A$. Since $X \in \tau$, then $X \oplus A_i = X \setminus A_i \in A = \tau$ for $1 \leq i \leq n-2$ and hence for any $A \in \tau$, we have $A^c \in \tau$. Clearly, n is even and every open set is closed also. Let $x, y \in X$ with $x \neq y$. By theorem 2.19, every optimal space is T_0 and hence there exists an open set U containing x but not y. Then the disjoint open sets U and $X \setminus U$ are such that $x \in U$, $y \notin U$, $x \notin X \setminus U$ and $y \in X \setminus U$. This implies that (X, τ) is a finite T_1 space and hence discrete. Then $|\tau| = 2^{|X|}$, a contradiction. Therefore, $|A| \geq n + 1$. But all the elements of A except $\emptyset$ are edge labels of H. Consequently, $s(H) \geq n$.* □

Theorem 3.16. *Let (X, τ) be a topological space. Then the star realization of (X, τ) is maximal and optimal if and only if (X, τ) is discrete*

Proof 6. *Let $K_{1,n-1} \in \mathcal{M}_\tau \cap \mathcal{O}_\tau$; $n = |\tau|$.*

If $n = 2$, then $K_{1,1} \in \mathcal{M}_\tau \cap \mathcal{O}_\tau \Rightarrow \tau(K_{1,1}) = |X| \Rightarrow 1 = |X|$ and hence (X, τ) is discrete.

If $n = 3$, then the star realization $K_{1,2}$ of (X, τ) is not maximal.

If $n = 4$, then $K_{1,3} \in \mathcal{M}_\tau \cap \mathcal{O}_\tau \Rightarrow \tau(K_{1,3}) = 2 = |X|$ by theorem 2.20 and hence (X, τ) is discrete.

Now suppose $n \geq 5$ so that $|X| \geq 3$. To prove that (X, τ) is discrete we need only to show that $n = 2^{|X|}$.

Suppose $n \neq 2^{|X|}$. Then by lemma 3.15, we have $s(K_{1,n-1}) \geq |\tau| \Rightarrow n - 1 \geq n$, a contradiction. Consequently, (X, τ) is discrete.

Conversely let (X, τ) be a discrete space so that $|\tau| = 2^{|X|}$. Then the star realization $K_{1,|\tau|-1}$ has $2^{|X|} - 1$ edges so that it is a maximal graphical realization of (X, τ). Now the optimality of $K_{1,|\tau|-1}$ follows from theorem 2.20. □

Theorem 3.17. *Let f be a t-set-indexer of the graph $G = K_n \setminus E(K_{1,3})$ with indexing set X. Then the removal of atmost one set from $f(V)$ results in a chain topology on X.*

Proof 7. *Let $V = \{v_1, \ldots, v_n\}$; $d(v_1) = n - 4$, $d(v_2) = d(v_3) = d(v_4) = n - 2$. If $f(V)$ is a chain topology, then there is nothing to be proved. If $f(V)$ is not a chain topology, then there exists A, $B \in f(V)$ such that $A \cup B \neq A$ and $A \cup B \neq B$. Since $f(V)$ is a topology on X, there exists four distinct vertices say v_i, v_j, v_k and v_l in G such that $f(v_i) = A$, $f(v_j) = B$, $f(v_k) = A \cup B$ and $f(v_l) = A \cap B$. Since*

$$f(v_i) \oplus f(v_j) = A \oplus B = f(v_k) \oplus f(v_l),$$
$$f(v_i) \oplus f(v_k) = B \setminus A = f(v_j) \oplus f(v_l) \text{ and}$$
$$f(v_i) \oplus f(v_l) = A \setminus B = f(v_j) \oplus f(v_k),$$

at least three possible edges are absent in $G[v_i, v_j, v_k, v_l]$. Consequently, we must have $v_i, v_j, v_k, v_l \in \{v_1, v_2, v_3, v_4\}$. Otherwise,

$$G[v_i, v_j, v_k, v_l] = \begin{cases} K_4, & \text{if } v_1 \notin \{v_i, v_j, v_k, v_l\} \\ K_4, & \text{if } v_1 \in \{v_i, v_j, v_k, v_l\} \text{ and } v_2, v_3, v_4 \notin \{v_i, v_j, v_k, v_l\} \\ K_4 \setminus E(K_2), & \text{if exactly one of } \{v_1, v_2\} \text{ or } \{v_1, v_3\} \text{ or } \{v_1, v_4\} \\ & \subseteq \{v_i, v_j, v_k, v_l\} \\ K_4 \setminus E(K_{1,2}), & \text{if exactly one of } \{v_1, v_2, v_3\} \text{ or } \{v_1, v_2, v_4\} \text{ or } \{v_1, v_3, v_4\} \\ & \subseteq \{v_i, v_j, v_k, v_l\}. \end{cases}$$

- a contradiction.

Let C and D be any two sets in $f(V) \setminus A$. It is claimed that $C \cup D = C$ or $C \cup D = D$.
Suppose, on the contrary $C \cup D \notin \{C, D\}$. Let u, v, w and x be the vertices of G with $f(u) = C$, $f(v) = D$, $f(w) = C \cup D$ and $f(x) = C \cap D$. But

$$C \oplus D = (C \cup D) \oplus (C \cap D),$$
$$C \oplus (C \cup D) = D \setminus C = D \oplus (C \cap D) \text{ and}$$
$$C \oplus (C \cap D) = C \setminus D = D \oplus (C \cup D)$$

so that atleast three edges are absent in $G[u, v, w, x]$. Consequently, we must have $G[u, v, w, x] \subseteq K_{1,3}$ or $G[u, v, w, x] \subseteq K_3 \cup K_1$. Now we claim that $\{u, v, w, x\} \neq \{v_1, v_2, v_3, v_4\}$. Otherwise, $s(G[u, v, w, x]) \geq 4$, a contradiction. Thus, we have $\{C, D, C \cup D, C \cap D\} = \{A, B, A \cup B, A \cap B\}$. Since $A \notin \{C, D\}$ we must have $A \in \{C \cup D, C \cap D\}$. If $A = C \cup D$, then $B \in \{C, D, C \cap D\}$ so that $B \subseteq A$, which is not possible. Therefore, we must have $A = C \cap D$. Then $B \in \{C, D, C \cup D\}$ so that $A \subseteq B$, which is also not true. Hence we must have $C \cup D = C$ or $C \cup D = D$ as claimed and $f(V) \setminus A$ is a chain topology. □

Theorem 3.18. *Let f be a t-set-indexer of the graph $G = K_n \setminus E(K_3)$; $n \geq 4$ with indexing set X. Then the removal of atmost one set from $f(V)$ results in a chain topology.*

Proof 8. *Let $V = \{v_1, \ldots, v_n\}$; $d(v_1) = d(v_2) = d(v_3) = n - 3$. If $f(V)$ is a chain topology, then there is nothing to be proved. If $f(V)$ is not a chain topology, then there exists A, $B \in f(V)$ such that $A \cup B \neq A$ and*

$A \cup B \neq B$. Since $f(V)$ is a topology on X, there exists four distinct vertices say v_i, v_j, v_k and v_l in G such that $f(v_i) = A$, $f(v_j) = B$, $f(v_k) = A \cup B$ and $f(v_l) = A \cap B$. Since

$$\begin{gathered} f(v_i) \oplus f(v_j) = A \oplus B = f(v_k) \oplus f(v_l), \\ f(v_i) \oplus f(v_k) = B \setminus A = f(v_j) \oplus f(v_l) \text{ and} \\ f(v_i) \oplus f(v_l) = A \setminus B = f(v_j) \oplus f(v_k), \end{gathered}$$

at least three possible edges are absent in $G[v_i, v_j, v_k, v_l]$. Consequently, we must have $v_1, v_2, v_3 \in \{v_i, v_j, v_k, v_l\}$. Otherwise,

$$G[v_i, v_j, v_k, v_l] = \begin{cases} K_4, & \text{if exactly one of } v_1, v_2, v_3 \text{ belongs to } \{v_i, v_j, v_k, v_l\} \\ K_4 \setminus E(K_2), & \text{if exactly one of } \{v_1, v_2\} \text{ or } \{v_1, v_3\} \text{ or } \{v_2, v_3\} \\ & \subseteq \{v_i, v_j, v_k, v_l\} \end{cases}$$

– a contradiction.

Let C and D be any two sets in $f(V) \setminus A$. It is claimed that $C \cup D = C$ or $C \cup D = D$.

Suppose on the contrary, $C \cup D \notin \{C, D\}$. Let u, v, w and x be the vertices of G with $f(u) = C$, $f(v) = D$, $f(w) = C \cup D$ and $f(x) = C \cap D$. But

$$\begin{gathered} C \oplus D = (C \cup D) \oplus (C \cap D), \\ C \oplus (C \cup D) = D \setminus C = D \oplus (C \cap D) \text{ and} \\ C \oplus (C \cap D) = C \setminus D = D \oplus (C \cup D) \end{gathered}$$

so that atleast three edges are absent in $G[u, v, w, x]$. Consequently, we must have $G[u, v, w, x] \subseteq K_{1,3}$ or $G[u, v, w, x] \subseteq K_3 \cup K_1$. Now we claim that $\{u, v, w, x\} \neq \{v_1, v_2, v_3, v_4\}$. Otherwise, $s(G[u, v, w, x]) \geq 4$, a contradiction. Thus, we have $\{C, D, C \cup D, C \cap D\} = \{A, B, A \cup B, A \cap B\}$. Since $A \notin \{C, D\}$ we must have $A \in \{C \cup D, C \cap D\}$. If $A = C \cup D$, then $B \in \{C, D, C \cap D\}$ so that $B \subseteq A$, which is not possible. Therefore, we must have $A = C \cap D$. Then $B \in \{C, D, C \cup D\}$ so that $A \subseteq B$, which is also not true. Hence, we must have $C \cup D = C$ or D, as claimed and $f(V) \setminus A$ is a chain topology. □

Theorem 3.19. *Let $G = (V, E)$ be the graph $K_n \setminus E(K_{1,3})$ and $H = (U, F)$ be the graph $K_n \setminus E(K_3)$. Then for every t-set indexer f of G, there exists a t-set indexer g of H satisfying $f(V) = g(U)$ and conversely.*

Proof 9. *Let $V = \{v_1, \ldots, v_n\}$ with $d(v_1) = n-4$, $d(v_2) = d(v_3) = d(v_4) = n-2$ and $U = \{u_1, \ldots, u_n\}$ with $d(u_1) = d(u_2) = d(u_3) = n-3$. Since f is*

a t-set-indexer of G, by theorem 3.17, the removal of atmost one set from $f(V)$ results in a chain topology. If $f(V)$ itself is a chain topology, then we can define a set-indexer say g_1 of H by $g_1(u_i) = f(v_i)$ for $i = 1, \ldots, n$. Clearly, $g_1(U) = f(V)$ and then g_1 is a t-set-indexer of H.
Otherwise, there exists $A \in f(V)$ such that $f(V) \setminus A$ is a chain topology. Clearly there exists $B(\neq A)$ in $f(V)$ such that $A \cup B \neq A$, B. Now consider a set-valuation, say g_2 of H defined by $g_2(u_i) = f(v_i)$ for $i = 1, \ldots, n$. Clearly, $g_2(U) = f(V)$. Obviously, $A, B, A \cup B, A \cap B \in g_2(U)$. But

$$\begin{gathered} A \oplus B = (A \cup B) \oplus (A \cap B) \\ A \oplus (A \cup B) = B \setminus A = B \oplus (A \cap B) \\ A \oplus (A \cap B) = A \setminus B = B \oplus (A \cup B). \end{gathered}$$

Hence atleast 3 edges should be absent in the subgraph of H induced by the vertices with labels A, B, $A \cup B$ and $A \cap B$. Note that $(v_2, v_3) \in E$ but $(u_2, u_3) \notin F$. Also $(v_1, v_4) \notin E$ but $(u_1, u_4) \in F$. Consequently, we must have A, B, $A \cup B$, $A \cap B \in \{g_2(u_1), g_2(u_2), g_2(u_3), g_2(u_4)\}$. Also $g_2(u_1, u_4) = f(v_1) \oplus f(v_4) = f(v_2, v_3)$ and $g_2(u_i, u_j) = f(v_i, v_j)$, for all $(u_i, u_j) \in F \setminus \{(u_1, u_4)\}$. Therefore, the edge labels of H under g_2 are distinct so that g_2 is a t-set-indexer of H.

Conversely, let g be a t-set-indexer of H. Then by theorem 3.18, the removal of atmost one set from $g(U)$ results in a chain topology. If $g(U)$ itself is a chain topology, then we can define a set-indexer say f_1 of G by $f_1(v_i) = g(u_i)$ for $i = 1, \ldots, n$. Clearly $f_1(V) = g(U)$ so that f_1 is a t-set-indexer of G.
Otherwise, there exists $C \in g(U)$ such that $g(U) \setminus C$ is a chain topology. Evidently, there exists $D(\neq C)$ in $g(U)$ such that $C \cup D \neq C$, D. Now consider a set-valuation say f_2 of G defined by $f_2(v_i) = g(u_i)$ for $i = 1, \ldots, n$. Clearly, $f_2(V) = g(U)$. Obviously, $C, D, C \cup D, C \cap D \in f_2(V)$. But

$$\begin{gathered} C \oplus D = (C \cup D) \oplus (C \cap D) \\ C \oplus (C \cup D) = D \setminus C = D \oplus (C \cap D) \\ C \oplus (C \cap D) = C \setminus D = D \oplus (C \cup D). \end{gathered}$$

Hence atleast 3 edges should be absent in the subgraph of G induced by the vertices with labels C, D, $C \cup D$ and $C \cap D$. Note that $(u_1, u_4) \in F$ but $(v_1, v_4) \notin E$. Also $(u_2, u_3) \notin F$ but $(v_2, v_3) \in E$. Consequently we must have C, D, $C \cup D$, $C \cap D \in \{f_2(v_1), f_2(v_2), f_2(v_3), f_2(v_4)\}$. Also $f_2(v_2, v_3) = g(u_2) \oplus g(u_3) = g(u_1, u_4)$ and $f_2(v_i, v_j) = g(u_i, u_j)$, for all

$(v_i, v_j) \in E \setminus \{(v_2, v_3)\}$. Therefore, the edge labels of G under f_2 are distinct so that f_2 is a t-set-indexer of G. □

Remark 3.20. *From the above theorem it follows that there is a one-to-one correspondence between the t-set indexers of $K_n \setminus E(K_{1,3})$ and $K_n \setminus E(K_3)$, in such a way that the corresponding t-set indexers induce the same topology on the common indexing set.*

An immediate consequence of the above theorem is:

Corollary 3.21. *Let (X, τ) be a topological space. Then $K_n \setminus E(K_{1,3}) \in \mathcal{G}_\tau$(or $\mathcal{M}_\tau$) if and only if $K_n \setminus E(K_3) \in \mathcal{G}_\tau$(or $\mathcal{M}_\tau$).*

Theorem 3.22. *Let (X, τ); $|\tau| = n(\geq 4)$ be a given topological space such that the removal of exactly one set from τ results in a chain topology. Then a graph G of order n is a graphical realization of (X, τ) if and only if $G \notin A = \{H : K_n \setminus E(C_n) \subseteq H \subseteq K_n\}$.*

Proof 10. *We have to prove that if $G \in \mathcal{G}_\tau$, then $G \notin A$. Suppose $G \in A$. Then by theorem 2.15 we have $G \notin \mathcal{G}_\tau$.*

Conversely, let $G \notin A$. Then $G \subseteq K_n \setminus E(K_{1,3})$ or $G \subseteq K_n \setminus E(K_3)$. By theorem 3.17 and theorem 3.18 we have $K_n \setminus E(K_{1,3})$ and $K_n \setminus E(K_3) \in \mathcal{G}_\tau$. Since every spanning subgraph of any graph in $\mathcal{G}_\tau$ is also in $\mathcal{G}_\tau$ we have $G \in \mathcal{G}_\tau$. □

The following is a consequence of theorem 2.15 and theorem 2.16.

Theorem 3.23. *No graph $G \in A = \{H : K_n \setminus E(C_n) \subseteq H \subset K_n\}$ is a maximal graphical realization of any topological space (X, τ); $|\tau| = n > 2$.*

Theorem 3.24. *Let (X, τ) be a topological space. Then $\mathcal{M}_\tau = \{K_{|\tau|}\}$ if and only if τ is a chain topology on X.*

Proof 11. *Follows from theorem 2.15 and theorem 2.16.* □

Conjecture 3.25. *If the topological space (X, τ) has a unique maximal graphical realization, then τ is a chain topology.*

Theorem 3.26. *Let (X, τ) be a finite topological space. If $\mathcal{M}_\tau$ contains a tree, then τ is quasi-discrete.*

Proof 12. *Let T be a tree and $T \in \mathcal{M}_\tau$. Then there exists a t-set-indexer say, f of T with $f(V) = \tau$ and let $|\tau| = n$. By the star realization of (X, τ) we have $K_{1,n-1} \in \mathcal{G}_\tau$. Since $s(T) = n - 1 = s(K_{1,n-1})$ we have $K_{1,n-1} \in \mathcal{M}_\tau$. Let $V(K_{1,n-1}) = \{v_0, v_1, \ldots, v_{n-1}\}$; $d(v_0) = n - 2$. Now by assigning $\emptyset$ to v_0 and the remaining distinct nonempty elements of $f(V(T))$ to the pendant vertices of $K_{1,n-1}$ in any order we get a t-set-indexer say g of $K_{1,n-1}$ with t-indexing set X. Without loss of generality assume that $g(v_1) = X$. Then we have $g(v_1) \oplus g(v_j) = X \setminus g(v_j)$ for all $j \in \{2, \ldots, n-1\}$ so that $X \setminus g(v_j)$ is an edge label of $K_{1,n-1}$ for all $j \in \{2, \ldots, n-1\}$, since $K_{1,n-1} \in \mathcal{M}_\tau$. But the central vertex v_0 is of label $\emptyset$. Consequently, $X \setminus g(v_j)$ is also a vertex label of $K_{1,n-1}$ under g for every $j \in \{2, \ldots, n-1\}$. Thus, $X \setminus g(v_j) \in \tau$ whenever $g(v_j) \in \tau$. Hence, τ is quasi-discrete.* □

It is already shown that every topology admits a star realization. Then the natural question, what about the maximality of these star realizations, arises.The following theorem answers this.

Theorem 3.27. *Let (X, τ) be a finite topological space. Then $K_{1,|\tau|-1} \in \mathcal{M}_\tau$ if and only if (X, τ) is quasi-discrete.*

Proof 13. *The necessary part follows from theorem 3.26. On the other hand, let (X, τ) be a finite quasi-discrete space. Then $|\tau|$ is an even number . If $|\tau| = 2$, then $\mathcal{M}_\tau$ contains only one graph namely the tree K_2. If $|\tau| = 4$, then $\tau = \{X, \emptyset, A, X \setminus A\}$; $A \subset X$. Let G be a graphical realization of (X, τ). By theorem 2.14 such a graph exists. Since*

$$\emptyset \oplus A = A = X \oplus (X \setminus A),$$
$$\emptyset \oplus (X \setminus A) = (X \setminus A) = X \oplus A \text{ and}$$
$$\emptyset \oplus X = X = A \oplus (X \setminus A)$$

we have either $G \subseteq K_{1,3}$ or $G \subseteq K_3 \cup K_1$. Consequently $s(G) \leq 3$ and therefore the star realization $K_{1,3}$ of (X, τ) belongs to $\mathcal{M}_\tau$.

Suppose $|\tau| \geq 6$ and $A, B \in \tau$. Since (X, τ) is a quasi-discrete, τ is closed under symmetric difference. By the star realization of (X, τ) we have $K_{1,|\tau|-1} \in \mathcal{G}_\tau$ and let g be the corresponding t-set-indexer with $g(V) = \tau$ and $g(v_0) = \emptyset$ where v_0 is the central vertex of $K_{1,|\tau|-1}$. Let v_i and v_j be any two distinct pendant vertices of $K_{1,|\tau|-1}$. Then we have $g(v_i) \oplus g(v_j) \in \tau$. Since $g(v_i) \oplus g(v_j)$ is different from both $g(v_i)$ and $g(v_j)$, there exists another pendant vertex v_k in $K_{1,|\tau|-1}$ such that $g(v_k) = g(v_i) \oplus g(v_j)$ and $k \in \{1, \ldots, |\tau| - 1\} \setminus \{i, j\}$. Then we have $g(v_0) \oplus g(v_k) =$

$g(v_k) = g(v_i) \oplus g(v_j)$. Consequently, no more edges can be drawn in $K_{1,|\tau|-1}$ such that the resulting graph realizes (X, τ) keeping g fixed. Then by theorem 3.3 we have $K_{1,|\tau|-1} \in \mathcal{M}_\tau$. □

A consequence of the above theorem is,

Corollary 3.28. *No tree of odd order is a maximal graphical realization of a topological space.*

Theorem 3.29. *Maximal graphical realization of a topology is a cycle if and only if it is C_3.*

Proof 14. *Let (X, τ); $|\tau| = n$ be a given topological space and suppose $C_n \in \mathcal{M}_\tau$ where $n \geq 3$. By the star realization of (X, τ) we have $K_{1,n-1} \in \mathcal{G}_\tau$ and let g be the corresponding t-set-indexer. Let $V = \{v_0, v_1, \ldots, v_{n-1}\}$ with $d(v_0) = n - 1$ be the vertex set of $K_{1,n-1}$. Then we have $g(v_0) = \emptyset$ and $g(v_k) = X$ for some $v_k \neq v_0$. Since $C_n \in \mathcal{M}_\tau$, by corollary 3.5, we can extend $K_{1,n-1}$ to a maximal graphical realization $G = (V, E)$ of (X, τ) just by joining two distinct pendant vertices v_i and v_j, keeping all the vertex labels fixed. Let f be the corresponding t-set-indexer which is an extension of g.*

It is claimed that either $v_i = v_k$ or $v_j = v_k$.

Suppose $v_k \notin \{v_i, v_j\}$ and let $f(v_i) = A$ and $f(v_j) = B$. Obviously A and B are nonempty and $B \neq X \setminus A$. Since (v_k, v_i), $(v_k, v_j) \notin E$ and $G \in \mathcal{M}_\tau$, both $X \setminus A$ and $X \setminus B$ are edge labels of G under f. But $f|_{K_{1,n-1}} = g$ and g corresponds to the star realization of $K_{1,n-1}$. Consequently, there are vertices $v_a, v_b \in V \setminus \{v_i, v_j, v_k, v_0\}$ such that $f(v_a) = X \setminus A$ and $f(v_b) = X \setminus B$.

Now A, B, $X \setminus A$, $X \setminus B \in f(V) = \tau$

$\Rightarrow A \cap (X \setminus B), B \cap (X \setminus A) \in \tau$

$\Rightarrow (A \cap (X \setminus B)) \cup (B \cap (X \setminus A)) = (A \setminus B) \cup (B \setminus A) = A \oplus B \in \tau$.

$\Rightarrow$ there exists $v_l \in V$ such that $f(v_l) = A \oplus B$.

Because of obvious reasons, $v_l \notin \{v_0, v_i, v_j, v_k, v_a, v_b\}$.

But then $f(v_0, v_l) = A \oplus B = f(v_i, v_j)$ – a contradiction. Hence, we must have $v_k = v_i$ or $v_k = v_j$ as claimed.

Without loss of generality assume that $v_k = v_i$ so that $f(v_i) = A = X$. Then $f(v_i, v_j) = X \setminus B$ and $X \setminus B \notin f(V)$. Suppose G contains a fourth vertex v_x other than v_0, v_i and v_j. Let $f(v_x) = D$. Clearly, $D \notin \{X, B, \emptyset, X \setminus B\}$. Since $(v_i, v_x) \notin E$ and $G \in \mathcal{M}_\tau$, there exists a vertex $v_y \in V \setminus \{v_0, v_i, v_j, v_x\}$ such that $f(v_y) = X \setminus D$.

But $B, D, X \setminus D \in f(V)$ *and* $G \in \mathcal{M}_\tau$
$\Rightarrow (B \oplus D), (B \oplus (X \setminus D)) \in f(V)$
$\Rightarrow (B \oplus D) \cup (B \oplus (X \setminus D)) \in \tau = f(V)$
$\Rightarrow B \oplus (D \cup (X \setminus D)) = B \oplus X = X \setminus B \in \tau = f(V)$.
$\Rightarrow$ *there exists* $v_z \in V \setminus \{v_0, v_i, v_j, v_x, v_y\}$ *such that* $f(v_z) = X \setminus B$.
$\Rightarrow f(v_0, v_z) = X \setminus B = f(v_i, v_j)$ *– a contradiction. Consequently,* $V = \{v_0, v_i, v_j\}$ *and* $G = C_3$.

Converse part follows from theorem 3.24. □

Theorem 3.30. *Let* (X, τ) *be a topological space with* $|\tau| \geq 4$ *and* $G \in \mathcal{M}_\tau$. *Then* $s(G) = |\tau| - 1$ *if* τ *is quasi-discrete and* $s(G) > |\tau|$ *otherwise.*

Proof 15. *If* τ *is quasi-discrete, by theorem 3.27,* $K_{1,|\tau|-1} \in \mathcal{M}_\tau$ *so that* $s(G) = s(K_{1,|\tau|-1}) = |\tau| - 1$.

If τ *is not quasi-discrete, then again by theorem 3.27,* $K_{1,|\tau|-1} \notin \mathcal{M}_\tau$. *But by the star realization,* $K_{1,|\tau|-1} \in \mathcal{G}_\tau$ *so that* $s(G) \geq |\tau|$. *Suppose* $s(G) = |\tau|$ *and let* $H = (V, E)$ *be a graph obtained from* $K_{1,|\tau|-1}$ *by joining two nonadjacent vertices of* $K_{1,|\tau|-1}$ *by an edge such that* $H \in \mathcal{M}_\tau$ *and* f *be the corresponding t-set-indexer of* H *with* $f(V) = \tau$. *Let* $V = \{v_0, v_1, \ldots, v_n\}$; $n = |\tau| - 1$, $d(v_0) = n$ *and* $d(v_1) = 2 = d(v_2)$. *Clearly,* $f(v_0) = \emptyset$. *Suppose* $f(v_1) = A$ *and* $f(v_2) = B$; $A, B \subset X$. *Obviously,* $A \oplus B \notin f(V)$.
Suppose $X \notin \{A, B\}$. *Since* $f(V) = \tau$ *and* $n \geq 3$, *there exists a vertex* $v_3 \in V \setminus \{v_0, v_1, v_2\}$ *such that* $f(v_3) = X$. *Also* $B \neq X \setminus A$. *Otherwise,* $f(v_1, v_2) = X = f(v_0, v_3)$ *– a contradiction. Obviously,* $A \cup B, A \cap B \in f(V)$. *Since* $G \in \mathcal{M}_\tau$, $f(v_0) = \emptyset$ *and* $f(v_3) = X$, *we must have* $X \setminus A, X \setminus B, X \setminus (A \cap B), X \setminus (A \cup B) \in f(V)$. *Now* $A \oplus B = (A \cap (X \setminus B)) \cup (B \cap (X \setminus A)) \in f(V)$ *– a contradiction. Consequently,* $X \in \{A, B\}$.

Without loss of generality, assume that $B = X$. *Then* $f(v_1, v_2) = X \setminus A$ *and since* $G \in \mathcal{M}_\tau$ *and* $f(v_0) = \emptyset$ *it follows that* $X \setminus A \notin f(V)$. *Since* $n \geq 3$, *there exists a vertex* v_3 *in* $V \setminus \{v_0, v_1, v_2\}$ *and let* $f(v_3) = C$. *Note that* $C \neq X \setminus A$, *otherwise* $f(v_1, v_2) = X \setminus A = f(v_0, v_3)$ *– a contraction. Obviously,* $A \cup C$, $A \cap C \in f(V)$. *Since* $G \in \mathcal{M}_\tau$ *and* $f(v_0) = \emptyset$ *we must have* $X \setminus C$, $X \setminus (A \cup C)$, $X \setminus (A \cap C)$, $A \oplus C$, $X \setminus (A \oplus C) \in f(V)$. *But*

$$X \setminus A = \begin{cases} (X \setminus C) \cup (X \setminus (A \oplus C)); & C \subset A \\ (X \setminus (A \cup C)) \cup (C \cap (X \setminus (A \cap C)); & C \not\subset A \end{cases}.$$

This shows that $X \setminus A \in f(V)$ *– a contradiction. Consequently,* $s(G) > |\tau|$, *if* τ *is not quasi-discrete.* □

Obviously, $\mathcal{O}_\tau$ is non empty whenever $\mathcal{M}_\tau \subseteq \mathcal{O}_\tau$. There is every reason to believe that the converse is also true and the study puts forward the following:

Conjecture 3.31. *Let* (X, τ) *be a topological space. Then* $\mathcal{M}_\tau \subseteq \mathcal{O}_\tau$ *whenever* $\mathcal{O}_\tau$ *is non empty.*

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