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Chromatic coloring of distance graphs, III

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Abstract

A graph $G(Z, D)$ with vertex set Z is called an integer distance graph if its edge set is obtained by joining two elements of Z by an edge whenever their absolute difference is a member of D. When $D = P$ or $D \subseteq P$ where P is the set of all prime numbers then we call it a prime distance graph. After establishing the chromatic number of $G(Z, P)$ as four, Eggleton has classified the collection of graphs as belonging to class i if the chromatic number of $G(Z, D)$ is i.The problem of characterizing the family of graphs belonging to class i when D is of any given size is open for the past few decades. As coloring a prime distance graph is equivalent to producing a prime distance labeling for vertices of G, we have succeeded in giving a prime distance labeling for certain class of all graphs considered here. We have proved that if $D = \{2, 3, 5, 7, 7^{th}$ prime, 10^{th} prime, 13^{th} prime, 16^{th} prime, $(7 + \sum_{j=1}^{s-1} 4 \times 3^j)^{th}$ prime, $\ldots, (4 + \sum_{j=1}^{s} 4 \times 3^j)^{th}$ prime for any $s \in N\}$, then there exists a prime distance graph with distance set D in class 4 and if $D = \{2, 3, 5, 4^{th}$ prime, 6^{th} prime, 8^{th} prime, $(4 + \sum_{j=1}^{s-1} 3 \times 2^j)^{th}$ prime, $\ldots, (2 + \sum_{j=1}^{s} 3 \times 2^j)^{th}$ prime for any $s \in N\}$ then there exists a prime distance graphs with distance set D in class 3. Further, we have also obtained some more interesting results that are either general or existential such as a) If D is a specific sequence of integers in arithmetic progression then there exist a prime distance graph with distance set D, b) If G is any prime distance graph in class i for $1 \leq i \leq 4$ then $G \times K_2$ is also a prime distance graph in the respective class i, c) A countable union of disjoint copies of prime distance graph is again a prime distance graph, d) The Middle/Total graph of a path on n vertices is a prime distance graph. In addition we also provide a new different proof for establishing a fact that all cycles are prime distance graph.

Keywords: Chromatic Number, Distance Graph, Prime Distance Graph, Prime Distance Labeling, Unit Distance Graph.

[MSC Classification]: 05C12, 05C15.

1. Introduction

Given a graph G with finite vertex set cardinality, the task of determining a) the biggest set of elements which are non-adjacent pairwise b) The smallest set of colors used to color the vertices, so that any two of them forming an edge are colored differently are basic challenges in combinatorics. The former is called independence number α and the latter is called chromatic number χ . Several challenging problems can be cast as tasks of finding α or χ of G with finite number of vertices [1, 5].

The basic notion to this work is the distance graph DG. Suppose that (Y, σ) is a metric space. Here for $\alpha_1, \alpha_2 \in Y$ by $\sigma(\alpha_1, \alpha_2)$ we mean the separation distance SD. Let $D = \{i : 0 < i < \infty\}$. We follow the custom of calling $G(Y, D)$ a DG if $V(G) = Y$ and $\alpha_1, \alpha_2 \in Y$ is deemed to be adjacent if and only if $\sigma(\alpha_1, \alpha_2) \in D$. By $\chi(G(Y, D))$ we accept that it is the least color count used to paint the elements of Y with the attribute that every adjacent pair of elements is painted with distinct colors. We also give it an exclusive name called the chromatic number of G. In some sense this type of painting actually precludes a collection D of well-defined distances. If $D = \{\alpha\}$ with $\alpha > 0$, then the corresponding graph is understood as a DG. We agree with the practice of setting σ as Euclidean distance metric. So if Y is a subset of $\mathbf{R}^n$ for some $n \in \mathbf{Z}^+$ and if $\alpha_1, \alpha_2 \in Y$ with $\alpha_1 = (\alpha_1^1, \alpha_1^2, \alpha_1^3, \ldots, \alpha_1^n)$ and $\alpha_2 = (\alpha_2^1, \alpha_2^2, \alpha_2^3, \ldots, \alpha_2^n)$ then $\sigma(\alpha_1, \alpha_2) = \sum_{j=1}^{n} [(\alpha_1^j - \alpha_2^j)^2]^{\frac{1}{2}} = \mid \alpha_1 - \alpha_2 \mid$.

The task of finding $\chi(G(\mathbf{R}, \{1\}))$ is simple. Express $V(G(\mathbf{R}, \{1\})) = V_1 \cup V_2$ with $V_1 = \bigcup_{p=-\infty}^{\infty} [2p, 2p+1)$ and $V_2 = \bigcup_{p=-\infty}^{\infty} [2p+1, 2p+2)$. As $V_1 \cap V_2 = \emptyset$ it becomes a bipartite graph with chromatic number 2. When we attempt to find $\chi(G(\mathbf{R}^2, \{1\}))$ the task becomes extremely hard and got included as one among in the list of all time selected problems of Paul Erdos. We need not really search for words to explain the level of difficulty to find $\chi(G(\mathbf{R}^2, \{1\}))$. For more detailed discussion on the history of Euclidean DG coloring one can see [1].

We deem $G(V.E)$ the UDG if $f : V(G) \to R^2$ is an embedding with the attribute that $\mid f(\alpha) - f(\beta) \mid = 1$ whenever $(\alpha, \beta) \in E(G)$, the edge set of G. One can find in the literature a volley of unsolved problems concerning UDG. Erdos epitomized the problem due to Hadwiger-Nelson-HN regarding the chromatic number of the uncountably infinite UDG $G(R^2, \{1\})$, whose edge set is all those pairs of vertices separated by a unit-distance. The upper bound of 7 as the χ of this graph is still unchallenged. However, a long standing lower bound of 4 is improved to 5 in the recent attempt [2] by Grey. This was improved further in [3] in terms of the vertex set cardinality it should possess.

Suppose $D = \{i : 0 < i < \infty\}$ and $r \in \mathbf{Q}^+$ then for any $k \in \mathbf{Z}^+$ one can observe that $\chi(G(\mathbf{Q}^k, D)) = \chi(G(\mathbf{Q}^k, rD))$. This is because if $g : \mathbf{Q}^k \to \mathbf{Q}^k$ is defined as $g(\alpha) = (r\alpha_1, r\alpha_2, \ldots, r\alpha_k)$ for $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in \mathbf{Q}^k$. Then g is one-one as $g(\alpha) = g(\beta) \Rightarrow (r\alpha_1, r\alpha_2, \ldots, r\alpha_k)$ $= (r\beta_1, r\beta_2, \ldots, r\beta_k) \Rightarrow \alpha = \beta$; g is onto as for all $(r\alpha_1, r\alpha_2, \ldots, r\alpha_k) \in \mathbf{Q}^k$ there exist $(\alpha_1, \alpha_2, \ldots, \alpha_k) \in \mathbf{Q}^k$ such that $g(\alpha_1, \alpha_2, \ldots, \alpha_k)$ $= (r\alpha_1, r\alpha_2, \ldots, r\alpha_k)$; g is a homomorphism as $g(\alpha + \beta) = g(\alpha_1 + \beta_1, \alpha_2 + \beta_2, \ldots, \alpha_k + \beta_k) = (r(\alpha_1 + \beta_1), r(\alpha_2 + \beta_2), \ldots, r(\alpha_k + \beta_k)) = (r\alpha_1, r\alpha_2, \ldots, r\alpha_k) + (r\beta_1, r\beta_2, \ldots, r\beta_k) = g(\alpha) + g(\beta)$. So, g is an isomorphism among $V(G(\mathbf{Q}^k, D))$ and $V(G(\mathbf{Q}^k, rD))$. Moreover, we notice that for $\alpha, \beta \in \mathbf{Q}^k$, $\mid \alpha - \beta \mid = s$, for $s \in \mathbf{Z}^+ \Leftrightarrow \mid g(\alpha) - g(\beta) \mid = rs$. Hence $G(\mathbf{Q}^k, D) \cong G(\mathbf{Q}^k, rD)$ and $\chi(G(\mathbf{Q}^k, D)) = \chi(G(\mathbf{Q}^k, rD))$. Motivated by this, researchers explored the computation of χ for DGs whose vertex sets are $\mathbf{Q}^k$ for $k \geq 1$. For an interesting exposition on the Euclidean coloring over $\mathbf{Q}$ an good choice is [4] and for nice deductions such as $\chi(G(\mathbf{Q}^2, \{1\})) = 2$, $\chi(G(\mathbf{Q}^3, \{1\})) = 2$ and $\chi(G(\mathbf{Q}^4, \{1\})) = 4$ one can consult [5].

2. Coloring Integer Distance Graphs

We restrict our attention here to coloring integer DGs $G(Z, D)$ with $D \subseteq P$. The main reason that can be attributed for this work lies in the following discussion.

Consider $G(\mathbf{Z}, D)$ where $D \subseteq \mathbf{Z}^+$. It is understood that $\alpha_1, \alpha_2 \in \mathbf{Z}^+$ with $\alpha_1 < \alpha_2$ are linked by drawing an edge if and only if $\alpha_2 - \alpha_1 \in D$. Probes of such DGs were done in [6, 7] stimulated by HN Problem concerning computation of χ for two dimensional Euclidean plane $\mathbf{R}^2$. A tough task is to identify those D with $\chi(G(\mathbf{Z}, D)) < \infty$. For instance $\chi(G(\mathbf{Z}, 2\mathbf{Z})) < \infty$ due to the presence of a clique of infinite size in $G(\mathbf{Z}, D)$ and $\chi(G(\mathbf{Z}, 2\mathbf{Z} + 1)) = 2$. This actually conveys that χ varies drastically over distance sets that are translates of each other. Katznelson-Razza conjectured that $\chi(G(\mathbf{Z}, D)) < \infty \Leftrightarrow D$ can be written as the union of a finite number of lonely sets [By lonely set we mean: For $\alpha_1, \alpha_2 > 0$ with the understanding that $\mid y \mid$ means its distance from y to its closest integer

$\mid \alpha_1 y \mid \geq \alpha_2$ for every $y \in D$]. Suppose $D = 2\mathbf{Z} + 1$, then one can conveniently choose $\alpha_1 = \alpha_2 = 1/2$ and the set $2\mathbf{Z}+1$ is a lonely set. In [8, 9] the sufficiency of D being lonely to guarantee $\chi(G(\mathbf{Z}, 2\mathbf{Z} + 1)) < \infty$ is established and the respective author groups have in fact done it independently of the other. By doing this they have successfully settled a challenge thrown by Erdos regarding χ being finite for distance sets that are lacunary(sets witnessing growth in an exponential manner). In [10] the author has established the existence of a 2-coloring of $\mathbf{Z}$ that do not contain arithmetic progression that are monochromatic and long arbitrarily for each lonely set with steps $y \in D$. One can see [11, 12, 13] for more.

3. Coloring Prime Distance Graphs

Eggleton et.al coined the term PDG in 1985 [14, 15]. In the DG $G(Z, D)$ if $D = P$ or $D \subseteq P$, then we call $G(Z, P)$ a PDG. Equivalently one can also deem $G(Z, P)$ a PDG if one can produce a 1–1 labeling $g : V(G) \to Z$ with the property that any member of $E(G)$, say (α, β) possess a prime distance between them. To be precise, $\mid g(\alpha) - g(\beta) \mid = g((\alpha, \beta)) \in P$. Observe that in a PDL the labels allotted to the elements of $V(G)$ must be non-repeating. However, the labels that result out of this labeling on the elements of $E(G)$ need not be nonrepeating. Further it is to be understood that in a PDG, $G(Z, D)$ a non edge may possess a prime distance. This is because when $D \subseteq P$, the labels on $V(G)$ may produce a prime distance that is not a member of D. So, between such two vertices an edge will not be drawn. Note that this however will not happen for the PDG, $G(Z, P)$ as every prime number distance warrants an edge between them. Joshua D Leuson et. al in [16] made use of famous results and open conjectures in number theory to establish the PDG property of certain infinite families of graphs as well. For instance, using the Green Taos Theorem: For any given $k \in Z^+$ one can find a arithmetic progression of primes possessing a length k, he has proved that all bipartite graphs are PDGs. In Section 6 we give a fourth proof for “All cycles are PDGs” by not depending on any of the three proof techniques indicated in [16]. Eggleton et.al in [11, 12] also established that $\chi(G(Z, P)) = 4$ and classified the collection of graphs as belonging to class i if $\chi(G(Z, D^*)) = i$ where $D^* \subset P$ for $1 \leq i \leq 4$. One can see [17, 18, 19] for more. Motivated by the results already available in the literature for DGs and PDGs we obtain several new results concerning the existence of PDGs belonging class 2 or class 3 or class 4 whose distance sets are subsets of P with varied cardinality.

4. Some Motivational Results from Number Theory

Before we proceed further, we quickly give some pertinent properties of integers that has acted as sources of inspiration to obtain the PDL of certain classes of graphs and stimulated our thought process. First observe that if $1+t, 1+2t, 1+3t, \ldots, 1+it, \ldots t \in Z$ are the sequence of integers then $gcd(1+(i-1)t, 1+it) = 1$ for all i. This is because, Suppose s is a positive integer such that $1+(i-1)t \equiv 0 \pmod{s}$ and $1+it \equiv 0 \pmod{s}$. Then as $1+(i-1)t = st_1$ and $1+it = st_2$ for some $t_1, t_2 \in Z^+$ we have $t = (1+it)-(1+(i-1)t) = st_1 - st_2 = s(t_1-t_2)$. Hence $t \equiv 0 \pmod{s}$. If we set $t_1 - t_2 = t_3$ so that $t = st_3$. As $(it+1) \equiv 0 \pmod{s}$ and $(i-1)t+1 \equiv 0 \pmod{s}$, $it+1 = i(st_3)+1$. These forces $s = 1$ and $gcd(1+(i-1)t, 1+it) = 1$. Next if $\mid \alpha - \beta \mid = p$, a prime, then $gcd(\alpha, \beta) = 1$ or p. If either $\alpha \not\equiv 0 \pmod{p}$ or $\beta \not\equiv 0 \pmod{p}$, then $gcd(\alpha, \beta) = 1$. This is because, by Fundamental theorem of Arithmetic, both u and v admit prime factorization. Let $u = q_1^{m_1} q_2^{m_2} \ldots q_n^{m_n}$ and $v = s_1^{i_1} s_2^{i_2} \ldots s_w^{i_w}$. Also let $gcd(u, v) = m$. Then $u - v = (q_1^{m_1} q_2^{m_2} \ldots q_n^{m_n}) - (s_1^{i_1} s_2^{i_2} \ldots s_w^{i_w})$ and as $p = \mid u - v \mid$ we see that $p \equiv 0 \pmod{m}$. So, either $m = 1$ or $m = p$. If both u and v are not multiples of p, then gcd cannot be p. So $gcd(u, v) = 1$. Next, Suppose that $s \geq 5$ is an odd integer. If q is the least prime factor of $s-2$ then $gcd(sq-q-s+3, (q+1)s-(q+1)-s+2) = 1$. This is because, Suppose $r \in Z^+$ is such that $q \equiv 0 \pmod{r}$ and $s-2 \equiv 0 \pmod{r}$. Then we can find t_1, t_2 such that $rt_1 = qs-q-s+3$ and $rt_2 = (q+1)s-(q+1)-s+2$. Then $rt_2 - rt_1 = [(q+1)s-(q+1)-s+2] - [qs-q-s+3]$. So, $r(t_2-t_1) = [qs+s-q-1-s+2] - [qs-q-s+3] = [qs-q+1] - [qs-q-s+3] = s-2$. So $(s-2) \equiv 0 \pmod{r}$. As q is the smallest prime factor of $s-2$, we can express $s-2 = qw$ for some $w \in Z^+$. Note that w will then be deemed as a product of primes more than or equal to q as q is the smallest prime factor. Also see that $qs-q-s+3 = (q-1)(s-1)+2 = (q-1)(s-2)+(q-1)+2 = (q-1)(s-2)+(q+1)$. Now as $(q-1)(s-2)+(q+1) \equiv 0 \pmod{r}$ and $(s-2) \equiv 0 \pmod{r}$ we see that $(q+1) \equiv 0 \pmod{r}$. But as $q+1 \in 2Z$, its prime decomposition consists of powers of 2 and other prime factors less than q. However, as q is the least prime factor of $s-2$, $(q+1) \equiv 0 \pmod{r}$ and $(s-2) \equiv 0 \pmod{r}$ only if $r = 1$. Hence the only positive number that divides $qs-q-s+3$ and $(q+1)s-(q+1)-s+2$ is 1. So, when s is odd and q is the least prime divisor of $s-2$ it follows that $gcd(qs-q-s+3, (q+1)s-(q+1)-s+2) = 1$. Also, the following are true: a) The gcd of any two consecutive positive numbers is equal to 1; b) $gcd(1, s) = 1 \ \forall \ s \in N$; c) If i, $i+2 \in 2Z+1$ then $gcd(i, i+2) = 1$; d) If p

is a prime and $x \not\equiv 0 \pmod p$, then $gcd(p, x) = 1$. This is because, a) By Bezouts identity, $gcd(x, y) = 1$ for $x, y \in Z^+$ if and only if $t_1x + t_2y = 1$ for some $t_1, t_2 \in Z$. Take $x = j$, $y = j + 1$. Now if $t_1 = -1$ and $t_2 = 1$ then we see that $t_1x + t_2y = 1$ and hence the gcd of any two consecutive positive numbers is equal to 1. b) Next take $x = 1$, $y \in Z^+$. If $t_1 = 1$ and $t_2 = 0$ then $gcd(1, s) = 1$. c) Take $x = 2j + 1$ and $y = 2j + 3$, $j \in Z^+$. If $t_1 = -(j + 2)$ and $t_2 = j + 1$ then we see that Bezouts identity is satisfied and hence $gcd(2j + 1, 2j + 3) = 1$. Finally note that a prime integer p will have the gcd equal to 1 with every number less than p by the definition of prime. Moreover, the only integers larger than p with whom the prime integer p will share common factors are those that are multiples of p.

5. Some Existence Results on PDGs

Theorem 1. *Suppose that* $D = \{p-(c-1)s, p-(c-2)s, \ldots, p-s, p, p+s, p+2s, \ldots, p+(d-2)s, p+(d-1)s\}$. *Then there exists a PDG* $G(Z, D)$ *in class 2.*

Proof. Note that elements of D are all primes due to the Green-Tao Theorem said in Section 3. Clearly $\mid D \mid = c + d - 1$ and $D \underset{\neq}{\subset} P$. We now create a bipartite graph with $A \cup B$ as partite sets where $\mid A \mid = c$ and $\mid B \mid = d$. Let $V(G) = A \cup B = \{\alpha_1, \alpha_2, \ldots, \alpha_c\} \cup \{\beta_1, \beta_2, \ldots, \beta_d\}$, where $\alpha_{l+1} = ls$ with $l = 0, 1, \ldots, (c-1)$ and $v_{l+1} = p+ls$ with $l = 0, 1, \ldots, (d-1)$. Now introduce edges between A and B as we please. Then one can check that the edge labels are of the type $p + ms$ where m can be any member of the set $\{-(c-1), -(c-2), \ldots, -1, 0, 1, \ldots, d-2, d-1\}$ and all possible $p + ms$ labels are prime. As $G(Z, D)$ is bipartite, it belongs to class 2. □

Theorem 2. *Suppose that* $D = \{2, p_1\}$ *where* p_1 *is that prime with* $p_1 + p_2 = 2s - 4$ *where* $s \geq 6$ *and* $p_2 \in P$. *Then there exists a PDG,* $G(Z, D)$ *in class 2 or class 3 depending on whether* $s \in 2Z$ *or* $s \in 2Z + 1$.

Proof. We know that by Goldbachs conjecture, any even integer > 2 can be set as a sum of two primes. Assume that it is true. Then the $2s-4 \in 2Z$ can be written as $2s - 4 = p_1 + p_2$ where $p_1, p_2 \in P$. Form $G(Z, D)$ with $V(G) = \{\alpha_1, \alpha_2, \ldots, \alpha_s\}$ where $\alpha_{i-1} = 2i - 4$ for $2 \leq i \leq s$ and $\alpha_s = p_1$ or p_2. Form an edge set $E(G)$ by allowing edges between α_i and α_{i+1} for all $1 \leq i \leq s - 1$ and between α_s and α_1. Then all the edges labeled between α_i and α_{i+1} for $1 \leq i \leq s - 1$ is 2 and the edge between α_s and α_1 carry the label $(2s-4)-p_1 = p_2$ or $(2s-4)-p_2 = p_1$. One can easily see that $G(Z, D)$ constructed as above is a PDG and isomorphic to C_s, the cycle graph on s vertices. □

Theorem 3. *Suppose that $D = \{2, p_1, p_2\}$ where $p_1, p_2 \in P$ are twin primes. Then there exists a PDG, $G(Z, D)$ in class 3.*

Proof. We know that by twin prime conjecture that there are countable number of primes p_1, p_2 such that $p_2 = p_1 + 2$. Assume that the twin prime conjecture is true. Build a graph $G(Z, D)$ with $V(G) = \{\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_{2n}\}$, where $\alpha_0 = 0$, $\alpha_1 = p_1^1$ $\alpha_2 = p_2^1$, $\alpha_3 = p_1^2$, $\alpha_4 = p_2^2$, …, $\alpha_{2n-1} = p_1^n$, $\alpha_{2n} = p_2^n$ where (p_1^i, p_2^i) is i^{th} twin prime pair for $1 \leq i \leq n$. Now introduce an edge between (α_0, α_i) for $1 \leq i \leq 2n$. Also introduce an edge between α_1 and α_2, α_3 and α_4, …, α_{2n-1} and α_{2n}. Then $G(Z, D)$ is a PDG with $\mid \alpha_2 - \alpha_1 \mid = \mid \alpha_4 - \alpha_3 \mid = \ldots = \mid \alpha_{2n} - \alpha_{2n-1} \mid = 2$; $\mid \alpha_1 - \alpha_0 \mid = p_1^1$; $\mid \alpha_2 - \alpha_0 \mid = p_2^1$; …, $\mid \alpha_0 - \alpha_{2n-1} \mid = p_1^n$; $\mid \alpha_0 - \alpha_{2n} \mid = p_2^n$. Now give the color 1 to α_0 and the color 2 to $\alpha_1, \alpha_3, \alpha_5, \ldots, \alpha_{2n-1}$ and the color 3 to $\alpha_2, \alpha_4, \alpha_6, \ldots, \alpha_{2n}$. Then $G(Z, D)$ belongs to class 3. □

Theorem 4. *Suppose that $D = \{2, 5, 7\}$. Then there exists a prime distance square graph $G(Z, D)$ in class 3.*

Proof. By a square graph of G we mean a graph G^2 where V is same as G and $E(G^2) = \{(x, y) : d(x, y) \leq 2\}$. Take $G = P_n = \alpha_1, \alpha_2, \ldots, \alpha_n$. Then $E(P_n^2) = E(P_n) \cup \{(\alpha_j, \alpha_{j+2}) : 1 \leq j \leq n-2\}$. Now define a 1-1 function $g : V(G) \to Z^+$ by $g(\alpha_1) = 2$; $g(\alpha_2) = 4$; $g(\alpha_i) = g(\alpha_{i-2}) + 7$ for $3 \leq i \leq n$. Then we see that $\mid g(\alpha_1) - g(\alpha_2) \mid = 2$; $\mid g(\alpha_2) - g(\alpha_3) \mid = 5$; $\mid g(\alpha_3) - g(\alpha_4) \mid = 2$; $\mid g(\alpha_4) - g(\alpha_5) \mid = 5$; … $\mid g(\alpha_i) - g(\alpha_{i-1}) \mid = 2$ or 5 for $1 \leq i \leq n-1$; Further $\mid g(\alpha_1) - g(\alpha_3) \mid = \mid g(\alpha_2) - g(\alpha_4) \mid = \ldots = \mid g(\alpha_i) - g(\alpha_{i-2}) \mid = 7$ for $3 \leq i \leq n$. Therefore, g is a PDL for P_n^2 and P_n^2 is a prime distance square graph. Clearly $C_3 \subseteq P_n^2$ and hence $\chi(P_n^2) \geq 3$. Now color the vertices of P_n^2 as follows: Define $f : V(P_n^2) \to \{a, b, c\}$ by $f(\alpha_{3k-2}) = a$ for $1 \leq i \leq n$; $f(\alpha_{3k-1}) = b$ for $1 \leq i \leq n$; $f(\alpha_{3k}) = c$ for $1 \leq i \leq n$. Then f is a proper 3 coloring for P_n^2 and $\chi(P_n^2) \leq 3$. Hence $\chi(P_n^2) = 3$ and $P_n^2 \in$ class 3. That is $G(Z, \{2, 5, 7\}) = (P_n^2, D = \{2, 5, 7\})$ is a prime distance square graph in class 3. □

Theorem 5. *Let* $D = \{2, 3, 5, 4^{th}$*prime,* 6^{th}*prime,* 8^{th}*prime,* $(4 + \sum_{j=1}^{s-1} 3 \times 2^j)^{th}$*prime,* $\ldots$, $(2 + \sum_{j=1}^{s} 3 \times 2^j)^{th}$*prime*$\}$ *where* $s \in N$. *Then there exist a PDG G in class 3 with D as its distance set.*

Proof. We begin by constructing a family $\{T_s^*\}$ of graphs for $s = 0, 1, 2, \ldots$, as follows. We set $T_0^* = K_3$. Let $V(T_0^*) = \{u_{0,1}, u_{0,2}, u_{0,3}\}$ and $E(T_0^*) = \{(u_{0,1}, u_{0,2}), (u_{0,2}, u_{0,3}), (u_{0,3}, u_{0,1})\}$. So, the number of K_3's in T_0^* is 1.

We obtain T_1^* from the 1-crown of T_0^* by affixing a copy of K_3 on each of the pendent vertices of 1-crown of T_0^* starting from $u_{0,1}$ in the clockwise direction as shown in the Figure 5.1(a). The vertices of each copy of K_3 are $u_{1,1}^i, u_{1,2}^i, u_{1,3}^i$, $1 \le i \le 3$. Hence $V(T_1^*) = V(T_0^*) \cup \{u_{1,1}^i, u_{1,2}^i, u_{1,3}^i, | \, 1 \le i \le 3\}$ and $E(T_1^*) = E(T_0^*) \cup \{(u_{1,1}^i, u_{1,2}^i), (u_{1,2}^i, u_{1,3}^i), (u_{1,3}^i, u_{1,1}^i) \mid 1 \le i \le 3\} \cup \{(u_{0,1}, u_{1,1}^1), (u_{0,2}, u_{1,1}^2), (u_{0,3}, u_{1,1}^3)\}$. Number of K_3's in T_1^* is $1 + 3 = 4$.

Next T_2^* is obtained by affixing a copy of K_3 on each of the pendent vertices of 1-crown of T_1^* at $u_{1,2}^i, u_{1,3}^i$ $1 \le i \le 3$ by an edge starting from $u_{1,2}^1$ in the clockwise direction as shown in the Figure 5.1(b). The K_3 that is affixed on $u_{1,2}^1$ is taken as the first copy of K_3 in level 2. There will be 6 copies of K_3 in the second level. The vertices of each copy of K_3's are given by $u_{2,1}^i, u_{2,2}^i, u_{2,3}^i$ $1 \le i \le 6$. Hence $V(T_2^*) = V(T_1^*) \cup \{u_{2,1}^i, u_{2,2}^i, u_{2,3}^i \mid 1 \le i \le 6\}$ and $E(T_2^*) = E(T_1^*) \cup \{(u_{2,1}^i, u_{2,2}^i), (u_{2,2}^i, u_{2,3}^i), (u_{2,3}^i, u_{2,1}^i) \mid 1 \le i \le 6\} \cup \{(u_{1,2}^1, u_{2,1}^1), (u_{1,3}^1, u_{2,1}^2), (u_{1,2}^2, u_{2,1}^3), (u_{1,3}^2, u_{2,1}^4), (u_{1,2}^3, u_{2,1}^5), (u_{1,3}^3, u_{2,1}^6)\}$. Number of K_3's in T_2^* is $1 + (2^0 \times 3) + (2^1 \times 3) = 10$.

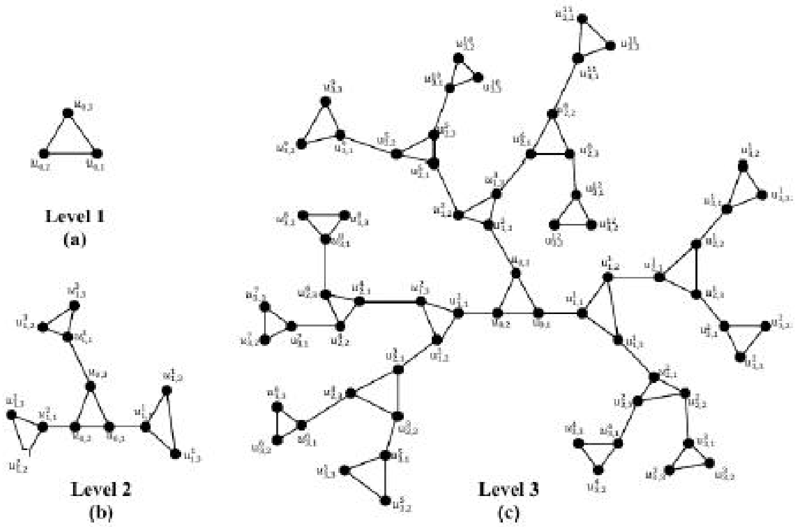

Figure 5.1(a): Level 1 T_0^*, (b) Level 2 T_1^*, (c) Level 3 T_2^*

Now T_s^* is obtained by the similar procedure of affixing a copy of K_3 on each of the pendent vertices of 1-crown of T_{s-1}^* at $u_{s-1,2}^i, u_{s-1,3}^i$ $1 \le i \le 3 \times 2^{s-1}$ by an edge starting from $u_{s-1,2}^1$ in the clockwise direction. The K_3 that is affixed on $u_{s-1,2}^1$ is called the 1^{st} copy of K_3 in the s^{th} level. The vertices of $3 \times 2^{s-1}$ copies K_3 in the s^{th} level is $u_{s,1}^i, u_{s,2}^i, u_{s,3}^i$ $1 \le i \le 3 \times 2^{s-1}$ and hence $V(T_s^*) = V(T_{s-1}^*) \cup \{u_{s,1}^i, u_{s,2}^i, u_{s,3}^i \mid 1 \le i \le 3 \times 2^{s-1}\}$ and $E(T_s^*) = E(T_{s-1}^*) \cup \{(u_{s,1}^i . u_{s,2}^i), (u_{s,2}^i, u_{s,3}^i), (u_{s,3}^i, u_{s,1}^i) \mid 1 \le i \le 3 \times 2^{s-1}\} \cup \{(u_{s-1,2}^1, u_{s,1}^1), (u_{s-1,3}^1, u_{s,1}^2), (u_{s-1,2}^2, u_{s,1}^3), (u_{s-1,3}^2, u_{s,1}^4), \ldots,$ $(u_{s-1,2}^{3\times 2^{s-2}-1}, u_{s,1}^{3\times 2^{s-1}-3})$, $(u_{s-1,3}^{3\times 2^{s-2}-1}, u_{s,1}^{3\times 2^{s-1}-2})$, $(u_{s-1,2}^{3\times 2^{s-2}}, u_{s,1}^{3\times 2^{s-1}-1})$, $(u_{s-1,3}^{3\times 2^{s-2}}, u_{s,1}^{3\times 2^{s-1}})\}$.

Next, we illustrate the process of allotting PDL of T_s^* for $s = 0, 1, 2, \ldots$. For $s = 0$, define $f_0 : V(T_0^*) \to \mathbf{Z}$ as $f_0(U_{0,1}) = a$; $f_0(u_{0,2}) = a + 2$, $f_0(u_{0,3}) = a + 5$. Then we observe that the edge labels induced by f_0 are $\mid f_0(u_{0,1}) - f_0(u_{0,2}) \mid = 2$; $\mid f_0(u_{0,2}) - f_0(u_{0,3}) \mid = 3$; $\mid f_0(u_{0,3}) - f_0(u_{0,1}) \mid = 5$ are primes and hence f_0 is a PDL of T_0^*. Moreover as $T_0^* \equiv K_3$, it is clear that $\chi(T_0^*) = \chi(K_3) = 3$ and hence T_0^* is a class 3 graph.

For $s = 1$, define $f_1 : V(T_1^*) \to \mathbf{Z}$ as $f_1(V) = f_0(V)$ if $v \in (T_0^*)$; For $v \notin V(T_0^*)$, $f_1(u_{1,1}^1) = a + 7$; $f_1(u_{1,2}^1) = a + 9$; $f_1(u_{1,3}^1) = a + 12$; $f_1(u_{1,1}^2) = a + 15$; $f_1(u_{1,2}^2) = a + 17$; $f_1(u_{1,3}^2) = a + 20$; $f_1(u_{1,1}^3) = a + 24$; $f_1(u_{1,2}^3) = a + 26$; $f_1(u_{1,3}^3) = a + 29$. Since we retain the labels of f_0, it is enough to exhibit the edge labelling of the 3 copies of K_3 in level 1 and 3 connecting edges between level 0 and level 1. The edge labels of 3 copies

of K_3's for $1 \leq i \leq 3$ are as below
$\mid f_1(u^i_{1,1}) - f_1(u^i_{1,2}) \mid= 2$; $\mid f_1(u^i_{1,2}) - f_1(u^i_{1,3}) \mid= 3$; $\mid f_1(u^i_{1,3}) - f_1(u^i_{1,1}) \mid= 5$;
The edge labels of 3 connecting edges are given below
$\mid f_1(u_{0,1}) - f_1(u^1_{1,1}) \mid= 7$; $\mid f_1(u_{0,2}) - f_1(u^2_{1,1}) \mid= 13$; $\mid f_1(u_{0,3}) - f_1(u^3_{1,1}) \mid= 19$.

Note that all these edge labels are prime numbers. Hence f_1 is a PDL for T_1^*.

Moreover, we note that one can assign colors 1, 2, 3 in a cyclic manner around the innermost copy of K_3 in the clockwise direction and then by starting at the 1^{st} copy of K_3 of level 1, we can color the vertices of the 1^{st} copy of K_3 of level 1 with the colors 2, 3, 1; we can color the vertices of 2^{nd} copy of K_3 of level 1 with the colors 3, 1, 2; we can color the vertices of 3^{rd} copy of K_3 of level 1 with the colors 1, 2, 3. Then it is easy to check that 3 colors are necessary and sufficient to color the vertices of T_1^* and hence T_1^* is a class 3 graph.

For $s = 2$, define $f_2 : V(T_2^*) \to \mathbf{Z}$ as $f_2(V) = f_1(V)$ if $v \in V(T_1^*)$; For $v \notin V(T_1^*)$,
$f_2(u^1_{2,1}) = a + 38$; $f_2(u^1_{2,2}) = a + 40$; $f_2(u^1_{2,3}) = a + 43$; $f_2(u^2_{2,1}) = a + 49$;
$f_2(u^2_{2,2}) = a + 51$; $f_2(u^2_{2,3}) = a + 54$; $f_2(u^3_{2,1}) = a + 60$; $f_2(u^3_{2,2}) = a + 62$;
$f_2(u^3_{2,3}) = a + 65$; $f_2(u^4_{2,1}) = a + 73$; $f_2(u^4_{2,2}) = a + 75$; $f_2(u^4_{2,3}) = a + 78$;
$f_2(u^5_{2,1}) = a + 87$; $f_2(u^5_{2,2}) = a + 89$; $f_2(u^5_{2,3}) = a + 92$; $f_2(u^6_{2,1}) = a + 100$;
$f_2(u^6_{2,2}) = a + 102$; $f_2(u^6_{2,3}) = a + 105$.

Since we retain the labels of f_1, it is enough to exhibit the edge labels induced by 6 copies of outermost K_3's in level 2 and the respective 6 connecting edges between level 1 and level 2 are given below.

The edge labels of K_3's of $T_2^* \backslash T_1^*$. For $1 \leq i \leq 6$ are:
$\mid f_2(u^i_{2,1}) - f_2(u^i_{2,2}) \mid= 2$; $\mid f_2(u^i_{2,2}) - f_2(u^i_{2,3}) \mid= 3$; $\mid f_2(u^i_{2,3}) - f_2(u^i_{2,1}) \mid= 5$

The edge labels of connecting edges between level 1 to level 2 are:
$\mid f_2(u^1_{1,2}) - f_2(u^1_{2,1}) \mid= 29$; $\mid f_2(u^1_{1,3}) - f_2(u^2_{2,1}) \mid= 37$; $\mid f_2(u^2_{1,2}) - f_2(u^3_{2,1}) \mid= 43$;
$\mid f_2(u^2_{1,3}) - f_2(u^4_{2,1}) \mid= 53$; $\mid f_2(u^3_{1,2}) - f_2(u^5_{2,1}) \mid= 61$; $\mid f_2(u^3_{1,3}) - f_2(u^6_{2,1}) \mid= 71$;
Hence f_2 is a PDL for T_2^*.

Define a map $g_2 : V(T_2^*) \to \{1, 2, 3\}$ such that g_2 retains the colors of the vertices of $V(T_1^*)$ as it is given at level 1. Now for the remaining outermost K_3's of T_2^*, we assign colors in the cyclic manner by proceeding in clockwise direction exactly as we did in level 1. This produces a proper 3-coloring for T_2^* and hence $\chi(T_2^*) = 3$ and T_2^* is in class 3.

We now describe the vertex and edge pattern of T_3^*. T_3^* is obtained from the 1-crown of T_2^* by affixing a copy of K_3 at the 12 pendent vertices of the 1-crown of T_2^*. The vertices of T_3^* are $V(T_3^*) = V(T_2^*) \cup \{u_{3,1}^i, u_{3,2}^i, u_{3,3}^i \mid 1 \leq i \leq 12\}$. The 12 connecting edges between level 2 and level 3 are given by $X = \{(u_{2,2}^1, u_{3,1}^1), (u_{2,3}^1, u_{3,1}^2), (u_{2,2}^2, u_{3,1}^3), (u_{2,3}^2, u_{3,1}^4), \ldots, (u_{2,2}^6, u_{3,1}^{11}), (u_{2,3}^6, u_{3,1}^{12})\}$.

Note that we have added 10^{th}, 12^{th}, 14^{th}, 16^{th}, 18^{th}, 20^{th} primes namely 29, 37, 43, 53, 61, 71 to the vertex labels of $u_{1,2}^i, u_{1,3}^i$ for $1 \leq i \leq 3$ to obtain the vertex labels of $u_{2,1}^i$ for $1 \leq i \leq 6$. Similarly, we add 22^{nd}, 24^{th}, ..., 44^{th} primes namely 79, 89, ..., 193 on the vertices $u_{2,2}^i, u_{2,3}^i$ for $1 \leq i \leq 6$ to obtain the vertex labels of $u_{3,1}^i$ for $1 \leq i \leq 12$ namely $a+119, a+132, \ldots, a+296$ respectively. Hence the vertex labels of 12 copies of K_3's of level 3 are defined through $f_3 : V(T_3^*) \to \mathbf{Z}$ as $f_3(V) = f_2(V)$ if $v \in V(T_2^*)$ and for $v \notin V(T_2^*)$

$f_3(u_{3,1}^1) = a+119$; $f_3(u_{3,2}^1) = a+121$; $f_3(u_{3,3}^1) = a+124$; $f_3(u_{3,1}^2) = a+132$; $f_3(u_{3,2}^2) = a+134$; $f_3(u_{3,3}^2) = a+137$; $f_3(u_{3,1}^3) = a+152$; $f_3(u_{3,2}^3) = a+154$; $f_3(u_{3,3}^3) = a+157$; $f_3(u_{3,1}^4) = a+161$; $f_3(u_{3,2}^4) = a+163$; $f_3(u_{3,3}^4) = a+166$; $f_3(u_{3,1}^5) = a+175$; $f_3(u_{3,2}^5) = a+177$; $f_3(u_{3,3}^5) = a+180$; $f_3(u_{3,1}^6) = a+196$; $f_3(u_{3,2}^6) = a+198$; $f_3(u_{3,3}^6) = a+201$; $f_3(u_{3,1}^7) = a+214$; $f_3(u_{3,2}^7) = a+216$; $f_3(u_{3,3}^7) = a+219$; $f_3(u_{3,1}^8) = a+229$; $f_3(u_{3,2}^8) = a+231$; $f_3(u_{3,3}^8) = a+234$; $f_3(u_{3,1}^9) = a+252$; $f_3(u_{3,2}^9) = a+254$; $f_3(u_{3,3}^9) = a+257$; $f_3(u_{3,1}^{10}) = a+265$; $f_3(u_{3,2}^{10}) = a+267$; $f_3(u_{3,3}^{10}) = a+270$; $f_3(u_{3,1}^{11}) = a+281$; $f_3(u_{3,2}^{11}) = a+283$; $f_3(u_{3,3}^{11}) = a+286$; $f_3(u_{3,1}^{12}) = a+296$; $f_3(u_{3,2}^{12}) = a+298$; $f_3(u_{3,3}^{12}) = a+301$.

The edge labels of K_3's of $T_3^* \backslash T_2^*$ for $1 \leq i \leq 12$ are:
$\mid f_3(u_{3,1}^i) - f_3(u_{3,2}^i) \mid = 2$; $\mid f_3(u_{3,2}^i) - f_3(u_{3,3}^i) \mid = 3$; $\mid f_3(u_{3,3}^i) - f_3(u_{3,1}^i) \mid = 5$.

The edge labels of connecting edges between level 2 to level 3 are:
$\mid f_3(u_{2,2}^1) - f_3(u_{3,1}^1) \mid = 22^{nd}$ prime $= 79$;
$\mid f_3(u_{2,3}^1) - f_3(u_{3,1}^2) \mid = 24^{th}$ prime $= 89$;
$\mid f_3(u_{2,2}^2) - f_3(u_{3,1}^3) \mid = 26^{th}$ prime $= 101$;
$\mid f_3(u_{2,3}^2) - f_3(u_{3,1}^4) \mid = 28^{th}$ prime $= 107$;
$\mid f_3(u_{2,2}^3) - f_3(u_{3,1}^5) \mid = 30^{th}$ prime $= 113$;
$\mid f_3(u_{2,3}^3) - f_3(u_{3,1}^6) \mid = 32^{nd}$ prime $= 131$;
$\mid f_3(u_{2,2}^4) - f_3(u_{3,1}^7) \mid = 34^{th}$ prime $= 139$;
$\mid f_3(u_{2,3}^4) - f_3(u_{3,1}^8) \mid = 36^{th}$ prime $= 151$;

$| f_3(u^5_{2,2}) - f_3(u^9_{3,1}) |= 38^{th}$ prime $= 163$;
$| f_3(u^5_{2,3}) - f_3(u^{10}_{3,1}) |= 40^{th}$ prime $= 173$;
$| f_3(u^6_{2,2}) - f_3(u^{11}_{3,1}) |= 42^{nd}$ prime $= 181$;
$| f_3(u^6_{2,3}) - f_3(u^{12}_{3,1}) |= 44^{th}$ prime $= 193$.

Hence f_3 is a PDL for T^*_3. Next define a map $g_3 : V(T^*_3) \to \{1, 2, 3\}$ such that g_3 retains the colors of the vertices of $V(T^*_2)$ as it is given at level 2. Now for the remaining 12 outermost K_3's of T^*_3, we assign colors in the cyclic manner by proceeding in clockwise direction exactly as we did previously. This produces a proper 3-coloring for T^*_3 and hence $\chi(T^*_3) = 3$ and T^*_3 is in class 3.

Now we proceed to the higher levels with the induction process. Let us assume that T^*_{s-1} is a PDG in class 3. Consider T^*_s. Let $V(T^*_s) = V(T^*_{s-1}) \cup \{u^i_{s,1}, u^i_{s,2}, u^i_{s,3} \mid 1 \leq i \leq 3 \times 2^{s-1}\}$. These exclusive $3 \times 2^{s-1}$ outermost K_3's in s^{th} level are joined to the 1-crown of T^*_{s-1}. The vertex labeling of T^*_s is defined by $f_s : V(T^*_s) \to \mathbf{Z}$ as $f_s(V) = f_{s-1}(V)$ if $v \in V(T^*_{s-1})$. For $v \notin V(T^*_{s-1})$

$f_s(u^1_{s,1}) = f_s(u^1_{s-1,2}) + (4 + \sum_{j=1}^{s-1} 3 \times 2^j)^{th}$ prime; $f_s(u^1_{s,2}) = f_s(u^1_{s,1}) + 2$;

$f_s(u^1_{s,3}) = f_s(u^5_{s,1}) + 5$; $f_s(u^2_{s,1}) = f_s(u^1_{s-1,3}) + (4 + (\sum_{j=1}^{s-1} 3 \times 2^j) + 2)^{th}$ prime; $f_s(u^2_{s,2}) = f_s(u^2_{s,1}) + 2$; $f_s(u^2_{s,3}) = f_s(u^2_{s,1}) + 5$, ..., $f_s(u^{3\times 2^{s-1}-1}_{s,1}) = f_s(u^{3\times 2^{s-2}}_{s-1,2}) + (2 + (\sum_{j=1}^{s} 3 \times 2^j) - 2)^{th}$ prime; $f_s(u^{3\times 2^{s-1}-1}_{s,2}) = f_s(u^{3\times 2^{s-1}-1}_{s,1}) + 2$;

$f_s(u^{3\times 2^{s-1}-1}_{s,3}) = f_s(u^{3\times 2^{s-1}-1}_{s,1}) + 5$; $f_s(u^{3\times 2^{s-1}}_{s,1}) = f_s(u^{3\times 2^{s-2}}_{s-1,3}) + (2 + (\sum_{j=1}^{s} 3 \times 2^j))^{th}$ prime; $f_s(u^{3\times 2^{s-1}}_{s,2}) = f_s(u^{3\times 2^{s-1}}_{s,1}) + 2$; $f_s(u^{3\times 2^{s-1}}_{s,3}) = f_s(u^{3\times 2^{s-1}}_{s,1}) + 5$. The edge labels of $E(T^*_s) \backslash E(T^*_{s-1})$ are as follows: The edge labels of K_3's in $T^*_s \backslash T^*_{s-1}$ for $1 \leq i \leq 3 \times 2^{s-1}$
$| f_s(u^i_{s,1}) - f_s(u^i_{s,2}) |= 2$; $| f_s(u^i_{s,2}) - f_s(u^i_{s,3}) |= 3$; $| f_s(u^i_{s,3}) - f_s(u^i_{s,1}) |= 5$.

The edge labels of connecting edges between level $s - 1$ to level s are:

$$f_s(u^1_{s-1,2}) - f_s(u^1_{s,1}) \mid$$
$$=\mid f_s(u^1_{s-1,2}) - [f_s(u^1_{s-1,2}) + \left(4 + \textstyle\sum_{j=1}^{s-1} 3 \times 2^j\right)^{th} \text{ prime}] \mid$$

$$= (4 + \textstyle\sum_{j=1}^{s-1} 3 \times 2^j)^{th} \text{ prime}$$
$$\mid f_s(u^1_{s-1,3}) - f_s(u^2_{s,1}) \mid$$

$$=\mid f_s(u^1_{s-1,3}) - [f_s(u^1_{s-1,3}) + (4 + (\textstyle\sum_{j=1}^{s-1} 3 \times 2^j) + 2)^{th} \text{ prime}] \mid$$

$$= (4 + (\textstyle\sum_{j=1}^{s-1} 3 \times 2^j) + 2)^{th} \text{ prime}$$
$$etc., \mid f_s(u^{3\times 2^{s-2}}_{s-1,2}) - f_s(u^{3\times 2^{s-1}-1}_{s,1}) \mid$$

$$=\mid f_s(u^{3\times 2^{s-2}}_{s-1,2}) - [f_s(u^{3\times 2^{s-2}}_{s-1,2}) + (2 + (\textstyle\sum_{j=1}^{s-1} 3 \times 2^j) - 2)^{th} \text{ prime}] \mid$$

$$= (2 + (\textstyle\sum_{j=1}^{s} 3 \times 2^j) - 2)^{th} \text{ prime}$$
$$\mid f_s(u^{3\times 2^{s-2}}_{s-1,3}) - f_s(u^{3\times 2^{s-1}}_{s,1}) \mid$$

$$=\mid f_s(u^{3\times 2^{s-2}}_{s-1,3}) - [f_s(u^{3\times 2^{s-2}}_{s-1,3}) + (2 + \textstyle\sum_{j=1}^{s-1} 3 \times 2^j)^{th} \text{ prime}] \mid$$

$$= (2 + \textstyle\sum_{j=1}^{s} 3 \times 2^j)^{th} \text{ prime}$$

So T^*_s is a PDG. Now define a map $g_s : V(T^*_s) \to \{1, 2, 3\}$ such that g_s retains the colors of the vertices of $V(T^*_{s-1})$ as it is given at level $s-1$. Now for the remaining $3 \times 2^{s-1}$ outermost K_3's of T^*_s, we assign colors in the cyclic manner by proceeding in clockwise direction exactly as we did in previous levels. This produces a proper 3-coloring for T^*_s and hence $\chi(T^*_s) = 3$ and T^*_s is in class 3. □

Theorem 6. *Let* $D = \{2, 3, 5, 7, 7^{th}$ *prime,* 10^{th} *prime,* 13^{th} *prime,* 16^{th} *prime,* $(7 + \sum_{j=1}^{s-1} 4 \times 3^j)^{th}$ *prime, …,* $(4 + \sum_{j=1}^{s} 4 \times 3^j)^{th}$ *prime*$\}$ *where* $s \in N$*. Then there exists a PDG G in class 4 with D as its distance set.*

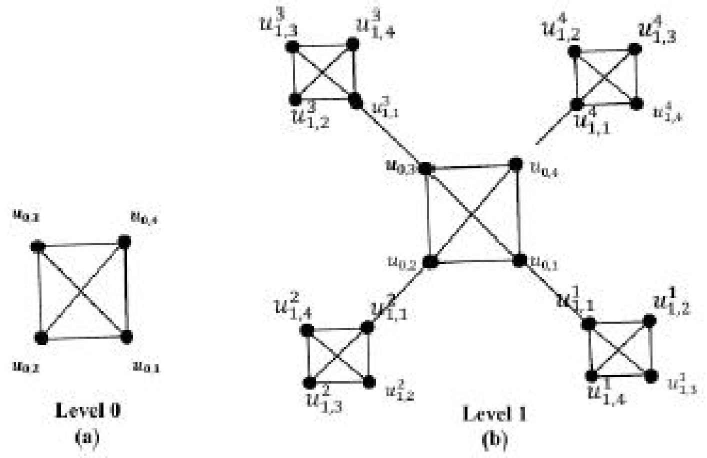

Figure 5.2(a): Level 0 K_0^*, (b) Level 1 K_1^*.

Proof. We begin by constructing a family $\{K_s^*\}$ of graphs for $s = 0, 1, 2, \ldots$ as follows. We set $K_0^* = K_4$. Let $V(K_0^*) = \{u_{0,1}, u_{0,2}, u_{0,3}, u_{0,4}\}$, $E(K_0^*) = \{(u_{0,1}, u_{0,2}), (u_{0,1}, u_{0,3}), (u_{0,1}, u_{0,4}), (u_{0,2}, u_{0,3}), (u_{0,2}, u_{0,4}), (u_{0,3}, u_{0,4})\}$. So, the number of K_4's in K_0^* is 1.

We obtain K_1^* from the 1-crown of K_0^* by affixing a copy of K_4 on each of the pendent vertices of 1-crown of K_0^* starting from $u_{0,1}$ in the clockwise direction as shown in the Figure 5.2(a). The vertices of each copy of K_4 are $u_{1,1}^i, u_{1,2}^i, u_{1,3}^i, u_{1,4}^i$, $1 \le i \le 4$. Hence $V(K_1^*) = V(K_0^*) \cup \{u_{1,1}^i, u_{1,2}^i, u_{1,3}^i, u_{1,4}^i \mid 1 \le i \le 4\}$ and $E(K_1^*) = E(K_0^*) \cup \{(u_{1,1}^i, u_{1,2}^i), (u_{1,1}^i, u_{1,3}^i), (u_{1,1}^i, u_{1,4}^i), (u_{1,2}^i, u_{1,3}^i), (u_{1,2}^i, u_{1,4}^i), (u_{1,3}^i, u_{1,4}^i) \mid 1 \le i \le 4\} \cup \{(u_{0,1}, u_{1,1}^1), (u_{0,2}, u_{1,1}^2), (u_{0,3}, u_{1,1}^3), (u_{0,4}, u_{1,1}^4)\}$. Number of K_4's in K_1^* is $1 + 2^2$.

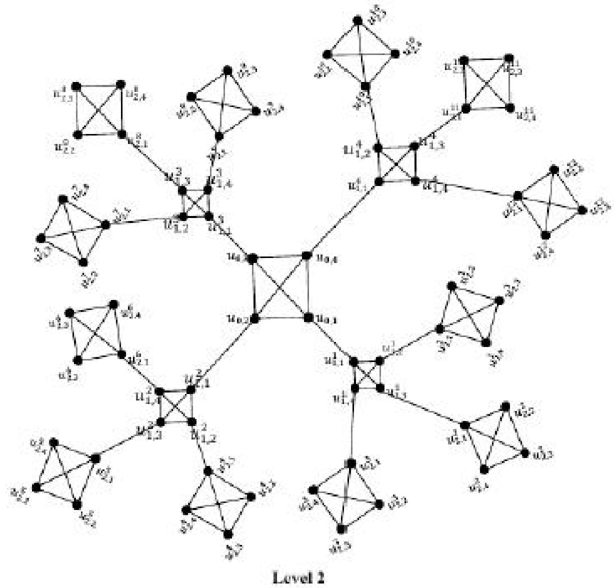

Figure 5.3: Level 2 K_2^*.

Next K_2^* is obtained by affixing a copy of K_4 on each of the 12 pendent vertices of 1-crown of K_1^* at $u_{1,2}^i, u_{1,3}^i, u_{1,4}^i$, $1 \leq i \leq 4$ by an edge starting from $u_{1,2}^1$ in the clockwise direction as shown in the Figure 5.3. The K_4's that is affixed on $u_{1,2}^1$ is taken as the first copy of K_4 in level 2. There will be 12 copies of K_4 given by $u_{2,1}^i, u_{2,2}^i, u_{2,3}^i, u_{2,4}^i$, $1 \leq i \leq 12$. Hence $V(K_2^*) = V(K_1^*) \cup \{u_{2,1}^i, u_{2,2}^i, u_{2,3}^i, u_{2,4}^i \mid 1 \leq i \leq 12\}$ and $E(K_2^*) = E(K_1^*) \cup \{(u_{1,1}^i, u_{1,2}^i), (u_{1,1}^i, u_{1,3}^i), (u_{1,1}^i, u_{1,4}^i), (u_{1,2}^i, u_{1,3}^i), (u_{1,2}^i, u_{1,4}^i), (u_{1,3}^i, u_{1,4}^i) \mid 1 \leq i \leq 12\} \cup \{(u_{1,2}^1, u_{2,1}^1), (u_{1,3}^1, u_{2,1}^2), (u_{1,4}^1, u_{2,1}^3), \ldots (u_{1,2}^4, u_{2,1}^{10}), (u_{1,3}^4, u_{2,1}^{11}), (u_{1,4}^4, u_{2,1}^{12})\}$. Here the number of K_4's in K_2^* is $1 + (3^0 \times 2^2) + (3^1 \times 2^2) = 17$.

Now K_s^* is obtained by the similar procedure of affixing a copy of K_4 on each of the pendent vertices of 1-crown of K_{s-1}^* at $u_{s-1,2}^i, u_{s-1,3}^i, u_{s-1,4}^i$, $1 \leq i \leq 4 \times 3^{s-1}$ by an edge starting from $u_{s-1,2}^1$ in the clockwise direction. The K_4 that is affixed on $u_{s-1,2}^1$ is called the 1^{st} copy of K_4 in the s^{th} level. The vertices of $4 \times 3^{s-1}$ copies K_4 in the s^{th} level is $u_{s,1}^i, u_{s,2}^i, u_{s,3}^i, u_{s,4}^i$, $1 \leq i \leq 4 \times 3^{s-1}$ and hence there will be $4 \times 3^{s-1}$ copies of K_4 are given by $u_{2,1}^i, u_{2,2}^i, u_{2,3}^i, u_{2,4}^i$ for $1 \leq i \leq 4 \times 3^{s-1}$.

Hence $V(K_s^*) = V(K_{s-1}^*) \cup \{u_{s,1}^i, u_{s,2}^i, u_{s,3}^i, u_{s,4}^i \mid 1 \leq i \leq 4 \times 3^{s-1}\}$ and $E(K_s^*) = E(K_{s-1}^*) \cup \{(u_{s,1}^i, u_{s,2}^i),\ (u_{s,1}^i, u_{s,3}^i),\ (u_{s,1}^i, u_{s,4}^i),\ (u_{s,2}^i, u_{s,3}^i),$ $(u_{s,2}^i, u_{s,4}^i),\ (u_{s,3}^i, u_{s,4}^i) \mid 1 \leq i \leq 4 \times 3^{s-1}\} \cup \{(u_{s-1,2}^1, u_{s,1}^1),\ (u_{s-1,3}^1, u_{s,1}^2),$ $(u_{s-1,4}^1, u_{s,1}^3),\ \ldots\ (u_{s-1,2}^{4\times 3^{s-2}}, u_{s,1}^{4\times 3^{s-1}-2}),\ (u_{s-1,3}^{4\times 3^{s-2}}, u_{s,1}^{4\times 3^{s-1}-1}),$ $(u_{s-1,4}^{4\times 3^{s-2}}, u_{s,1}^{4\times 3^{s-1}})\}$.

Next, we illustrate the process of allotting the PDL of K_s^* for $s = 0, 1, 2, \ldots$ For $s = 0$, define $f_0 : V(K_0^*) \to \mathbf{Z}$ as $f_0(u_{0,1}) = a$; $f_0(u_{0,2}) = a+2$; $f_0(u_{0,3}) = a + 5$; $f_0(u_{0,4}) = a + 7$. Then we observe that the edge labels induced by f_0 are $\mid f_0(u_{0,1}) - f_0(u_{0,2}) \mid= 2$; $\mid f_0(u_{0,1}) - f_0(u_{0,3}) \mid= 5$; $\mid f_0(u_{0,1}) - f_0(u_{0,4}) \mid= 7$; $\mid f_0(u_{0,2}) - f_0(u_{0,3}) \mid= 3$; $\mid f_0(u_{0,2}) - f_0(u_{0,4}) \mid= 5$; $\mid f_0(u_{0,3}) - f_0(u_{0,4}) \mid= 2$ are primes and hence f_0 is a PDL for K_0^*. Moreover as $K_0^* \equiv K_4$, it is clear that $\chi(K_0^*) = \chi(K_4) = 4$ and hence K_0^* is a class 4 graph.

For $s = 1$, define $f_1 : V(K_1^*) \to \mathbf{Z}$ as $f_1(V) = f_0(V)$ if $v \in V(K_0^*)$;
For $v \notin V(K_0^*)$.
$f_1(u_{1,1}^1) = a + 17$; $f_1(u_{1,2}^1) = a + 19$; $f_1(u_{1,3}^1) = a + 22$; $f_1(u_{1,4}^1) = a + 24$;
$f_1(u_{1,1}^2) = a + 31$; $f_1(u_{1,2}^2) = a + 33$; $f_1(u_{1,3}^2) = a + 36$; $f_1(u_{1,4}^2) = a + 38$;
$f_1(u_{1,1}^3) = a + 46$; $f_1(u_{1,2}^3) = a + 48$; $f_1(u_{1,3}^3) = a + 51$; $f_1(u_{1,4}^3) = a + 53$;
$f_1(u_{1,1}^4) = a + 60$; $f_1(u_{1,2}^4) = a + 62$; $f_1(u_{1,3}^4) = a + 65$; $f_1(u_{1,4}^4) = a + 67$;

Since we retain the labels of f_0, it is enough to exhibit the edge labels of the copies of K_4 in level 1and 4 connecting edges between level 0 and level 1. The edge labels of 4 copies of K_4's for $1 \leq i \leq 4$ are as given below:
$\mid f_1(u_{1,1}^i) - f_1(u_{1,2}^i) \mid= 2$; $\mid f_1(u_{1,1}^i) - f_1(u_{1,3}^i) \mid= 5$; $\mid f_1(u_{1,1}^i) - f_1(u_{1,4}^i) \mid= 7$;
$\mid f_1(u_{1,2}^i) - f_1(u_{1,3}^i) \mid= 3$; $\mid f_1(u_{1,2}^i) - f_1(u_{1,4}^i) \mid= 5$; $\mid f_1(u_{1,3}^i) - f_1(u_{1,4}^i) \mid= 2$.

The edge labels of 4 connecting edges between level 0 and level 1 are given below:
$\mid f_1(u_{0,1}) - f_1(u_{1,1}^1) \mid= 17$; $\mid f_1(u_{0,2}) - f_1(u_{1,1}^2) \mid= 29$; $\mid f_1(u_{0,3}) - f_1(u_{1,1}^3) \mid= 41$; $\mid f_1(u_{0,4}) - f_1(u_{1,1}^4) \mid= 53$

Note that the above edge labels are prime numbers. Hence f_1 is a PDL for K_1^*. Moreover, we note that one can assign colors 1, 2, 3, 4 in a cyclic manner around the innermost copy of K_4 in the clockwise direction and then by starting at the level 1 we can color the vertices of the 1^{st} copy of K_4 with the colors 2, 3, 4, 1; we can color the vertices of the 2^{nd} copy of K_4 with the colors 3, 4, 1, 2; we can color the vertices of the 3^{rd} copy of K_4 with the colors 4, 1, 2, 3; we can color the vertices of the 4^{th} copy of K_4 with the colors 1, 2, 3, 4. Then it is easy to check that 4 colors are necessary and sufficient to color the vertices of K_1^* and hence K_1^* is a class 4 graph.

For $s = 2$, define $f_2 : V(K_2^*) \to \mathbf{Z}$ as $f_2(V) = f_1(V)$ if $v \in V(K_1^*)$;
For $v \notin V(K_1^*)$.
$f_2(u_{2,1}^1) = a + 86$; $f_2(u_{2,2}^1) = a + 88$; $f_2(u_{2,3}^1) = a + 91$; $f_2(u_{2,4}^1) = a + 93$;
$f_2(u_{2,1}^2) = a+101$; $f_2(u_{2,2}^2) = a+103$; $f_2(u_{2,3}^2) = a+106$; $f_2(u_{2,4}^2) = a+108$;
$f_2(u_{2,1}^3) = a+121$; $f_2(u_{2,2}^3) = a+123$; $f_2(u_{2,3}^3) = a+126$; $f_2(u_{2,4}^3) = a+128$;
$f_2(u_{2,1}^4) = a+140$; $f_2(u_{2,2}^4) = a+142$; $f_2(u_{2,3}^4) = a+145$; $f_2(u_{2,4}^4) = a+147$;
$f_2(u_{2,1}^5) = a+163$; $f_2(u_{2,2}^5) = a+165$; $f_2(u_{2,3}^5) = a+168$; $f_2(u_{2,4}^5) = a+170$;
$f_2(u_{2,1}^6) = a+177$; $f_2(u_{2,2}^6) = a+179$; $f_2(u_{2,3}^6) = a+182$; $f_2(u_{2,4}^6) = a+184$;
$f_2(u_{2,1}^7) = a+205$; $f_2(u_{2,2}^7) = a+207$; $f_2(u_{2,3}^7) = a+210$; $f_2(u_{2,4}^7) = a+212$;
$f_2(u_{2,1}^8) = a+224$; $f_2(u_{2,2}^8) = a+226$; $f_2(u_{2,3}^8) = a+229$; $f_2(u_{2,4}^8) = a+231$;
$f_2(u_{2,1}^9) = a+244$; $f_2(u_{2,2}^9) = a+246$; $f_2(u_{2,3}^9) = a+249$; $f_2(u_{2,4}^9) = a+251$;
$f_2(u_{2,1}^{10}) = a+261$; $f_2(u_{2,2}^{10}) = a+263$; $f_2(u_{2,3}^{10}) = a+266$; $f_2(u_{2,4}^{10}) = a+268$;
$f_2(u_{2,1}^{11}) = a+292$; $f_2(u_{2,2}^{11}) = a+294$; $f_2(u_{2,3}^{11}) = a+297$; $f_2(u_{2,4}^{11}) = a+299$;
$f_2(u_{2,1}^{12}) = a+306$; $f_2(u_{2,2}^{12}) = a+308$; $f_2(u_{2,3}^{12}) = a+311$; $f_2(u_{2,4}^{12}) = a+313$.

Since we retain the labels of f_1, we observe that the edge labels induced by 12 copies of outermost K_4's and the connecting edges between level 1 and level 2 are given below:
The edge labels of K_4's of $K_2^* \backslash K_1^*$ for $1 \le i \le 12$ are:
$| f_2(u_{2,1}^i) - f_2(u_{2,2}^i) |= 2$; $| f_2(u_{2,1}^i) - f_2(u_{2,3}^i) |= 5$; $| f_2(u_{2,1}^i) - f_2(u_{2,4}^i) |= 7$;
$| f_2(u_{2,2}^i) - f_2(u_{2,3}^i) |= 3$; $| f_2(u_{2,2}^i) - f_2(u_{2,4}^i) |= 5$; $| f_2(u_{2,3}^i) - f_2(u_{2,4}^i) |= 2$.

The edge labels of connecting edges between level 1 to level 2 are:
$| f_2(u_{1,2}^1) - f_2(u_{2,1}^1) |= 67$; $| f_2(u_{1,3}^1) - f_2(u_{2,1}^2) |= 79$; $| f_2(u_{1,4}^1) - f_2(u_{2,1}^3) |= 97$;
$| f_2(u_{1,2}^2) - f_2(u_{2,1}^4) |= 107$; $| f_2(u_{1,3}^2) - f_2(u_{2,1}^5) |= 127$; $| f_2(u_{1,4}^2) - f_2(u_{2,1}^6) |= 139$;
$| f_2(u_{1,2}^3) - f_2(u_{2,1}^7) |= 157$; $| f_2(u_{1,3}^2) - f_2(u_{2,1}^8) |= 173$; $| f_2(u_{1,4}^3) - f_2(u_{2,1}^9) |= 191$;
$| f_2(u_{1,2}^4) - f_2(u_{2,1}^{10}) |= 199$; $| f_2(u_{1,3}^3) - f_2(u_{2,1}^{11}) |= 227$; $| f_2(u_{1,4}^4) - f_2(u_{2,1}^{12}) |= 239$

So f_2 is a PDL for K_2^*. Now define a map $g_2 : V(K_2^*) \to \{1, 2, 3, 4\}$ such that g retains the colors of the vertices of $V(K_1^*)$ as it is given at level 1. Now for the remaining outermost K_4's of K_2^*, we assign colors in the cyclic manner by proceeding in clockwise direction exactly as we did in level 1. This produces a proper 4-coloring for K_2^* and hence $\chi(K_2^*) = 4$ and hence K_2^* is in class 4.

We now describe the vertex and edge pattern of K_3^*. K_3^* is obtained from the 1-crown of K_2^* by affixing a copy of K_4 at the 36 pendent vertices of the 1-crown of K_2^*. The vertices of K_3^* are $V(K_3^*) = V(K_2^*) \cup \{u_{3,1}^i, u_{3,2}^i, u_{3,3}^i, u_{3,4}^i \mid 1 \leq i \leq 36\}$. The 36 connecting edges between level 2 and level 3 connecting each of the 36 K_4's are given by $X = \{(u_{2,2}^1, u_{3,1}^1),$ $(u_{2,3}^1, u_{3,1}^2)$, $(u_{2,4}^1, u_{3,1}^3)$, $(u_{2,2}^2, u_{3,1}^4)$, $(u_{2,3}^2, u_{3,1}^5)$, $(u_{2,4}^2, u_{3,1}^6)$, ..., $(u_{2,2}^{11}, u_{3,1}^{31})$, $(u_{2,3}^{11}, u_{3,1}^{32})$, $(u_{2,4}^{11}, u_{3,1}^{33})$, $(u_{2,2}^{12}, u_{3,1}^{34})$,
$(u_{2,3}^{12}, u_{3,1}^{35})$, $(u_{2,4}^{12}, u_{3,1}^{36})\}$.

Note that we have added 19^{th}, 22^{nd} 25^{th}, ..., 52^{nd} primes namely 67, 79, 97, ..., 239 to obtain the vertex labels of $u_{1,2}^i, u_{1,3}^i, u_{1,4}^i$ for $1 \leq i \leq 4$. Similarly, we have added 55^{th}, 58^{th}, 61^{st}, ..., 160^{th} primes namely 257, 271, 283, ..., 941 to obtain the vertex labels of $u_{2,2}^i, u_{2,3}^i, u_{2,4}^i$ for $1 \leq i \leq 12$ namely $a+88, a+91, a+93, \ldots, a+313$ respectively.

Hence the vertex labels of 36 copies of K_4's of level 3 are defined through $f_3 : V(K_3^*) \to \mathbf{Z}$ as $f_3(V) = f_2(V)$ if $v \in V(K_2^*)$ and for $v \notin V(K_2^*)$
$f_3(u_{3,1}^1) = a+345$; $f_3(u_{3,2}^1) = a+347$; $f_3(u_{3,3}^1) = a+350$; $f_3(u_{3,4}^1) = a+352$;
$f_3(u_{3,1}^2) = a+362$; $f_3(u_{3,2}^2) = a+364$; $f_3(u_{3,3}^2) = a+367$; $f_3(u_{3,4}^2) = a+369$;
$f_3(u_{3,1}^3) = a+376$; $f_3(u_{3,2}^3) = a+378$; $f_3(u_{3,3}^3) = a+381$; $f_3(u_{3,4}^3) = a+383$;
$f_3(u_{3,1}^4) = a+414$; $f_3(u_{3,2}^4) = a+416$; $f_3(u_{3,3}^4) = a+419$; $f_3(u_{3,4}^4) = a+421$;
$f_3(u_{3,1}^5) = a+437$; $f_3(u_{3,2}^5) = a+439$; $f_3(u_{3,3}^5) = a+442$; $f_3(u_{3,4}^5) = a+444$;
$f_3(u_{3,1}^6) = a+457$; $f_3(u_{3,2}^6) = a+459$; $f_3(u_{3,3}^6) = a+462$; $f_3(u_{3,4}^6) = a+464$;
$f_3(u_{3,1}^7) = a+490$; $f_3(u_{3,2}^7) = a+492$; $f_3(u_{3,3}^7) = a+495$; $f_3(u_{3,4}^7) = a+497$;
$f_3(u_{3,1}^8) = a+509$; $f_3(u_{3,2}^8) = a+511$; $f_3(u_{3,3}^8) = a+514$; $f_3(u_{3,4}^8) = a+516$;
$f_3(u_{3,1}^9) = a+529$; $f_3(u_{3,2}^9) = a+531$; $f_3(u_{3,3}^9) = a+534$; $f_3(u_{3,4}^9) = a+536$;
$f_3(u_{3,1}^{10}) = a+563$; $f_3(u_{3,2}^{10}) = a+565$; $f_3(u_{3,3}^{10}) = a+568$; $f_3(u_{3,4}^{10}) = a+570$;
$f_3(u_{3,1}^{11}) = a+584$; $f_3(u_{3,2}^{11}) = a+586$; $f_3(u_{3,3}^{11}) = a+589$; $f_3(u_{3,4}^{11}) = a+591$;
$f_3(u_{3,1}^{12}) = a+604$; $f_3(u_{3,2}^{12}) = a+606$; $f_3(u_{3,3}^{12}) = a+609$; $f_3(u_{3,4}^{12}) = a+611$;
$f_3(u_{3,1}^{13}) = a+632$; $f_3(u_{3,2}^{13}) = a+634$; $f_3(u_{3,3}^{13}) = a+637$; $f_3(u_{3,4}^{13}) = a+639$;
$f_3(u_{3,1}^{14}) = a+659$; $f_3(u_{3,2}^{14}) = a+661$; $f_3(u_{3,3}^{14}) = a+664$; $f_3(u_{3,4}^{14}) = a+666$;
$f_3(u_{3,1}^{15}) = a+679$; $f_3(u_{3,2}^{15}) = a+681$; $f_3(u_{3,3}^{15}) = a+684$; $f_3(u_{3,4}^{15}) = a+686$;
$f_3(u_{3,1}^{16}) = a+720$; $f_3(u_{3,2}^{16}) = a+722$; $f_3(u_{3,3}^{16}) = a+725$; $f_3(u_{3,4}^{16}) = a+727$;
$f_3(u_{3,1}^{17}) = a+745$; $f_3(u_{3,2}^{17}) = a+747$; $f_3(u_{3,3}^{17}) = a+750$; $f_3(u_{3,4}^{17}) = a+752$;
$f_3(u_{3,1}^{18}) = a+761$; $f_3(u_{3,2}^{18}) = a+763$; $f_3(u_{3,3}^{18}) = a+766$; $f_3(u_{3,4}^{18}) = a+768$;
$f_3(u_{3,1}^{19}) = a+806$; $f_3(u_{3,2}^{19}) = a+808$; $f_3(u_{3,3}^{19}) = a+811$; $f_3(u_{3,4}^{19}) = a+813$;
$f_3(u_{3,1}^{20}) = a+823$; $f_3(u_{3,2}^{20}) = a+825$; $f_3(u_{3,3}^{20}) = a+828$; $f_3(u_{3,4}^{20}) = a+830$;
$f_3(u_{3,1}^{21}) = a+843$; $f_3(u_{3,2}^{21}) = a+845$; $f_3(u_{3,3}^{21}) = a+848$; $f_3(u_{3,4}^{21}) = a+850$;
$f_3(u_{3,1}^{22}) = a+873$; $f_3(u_{3,2}^{22}) = a+875$; $f_3(u_{3,3}^{22}) = a+878$; $f_3(u_{3,4}^{22}) = a+880$;
$f_3(u_{3,1}^{23}) = a+890$; $f_3(u_{3,2}^{23}) = a+892$; $f_3(u_{3,3}^{23}) = a+895$; $f_3(u_{3,4}^{23}) = a+897$;

$f_3(u^{24}_{3,1}) = a+914$; $f_3(u^{24}_{3,2}) = a+916$; $f_3(u^{24}_{3,3}) = a+919$; $f_3(u^{24}_{3,4}) = a+921$;
$f_3(u^{25}_{3,1}) = a+955$; $f_3(u^{25}_{3,2}) = a+957$; $f_3(u^{25}_{3,3}) = a+960$; $f_3(u^{25}_{3,4}) = a+962$;
$f_3(u^{26}_{3,1}) = a+982$; $f_3(u^{26}_{3,2}) = a+984$; $f_3(u^{26}_{3,3}) = a+987$; $f_3(u^{26}_{3,4}) = a+989$;
$f_3(u^{27}_{3,1}) = a+1002$; $f_3(u^{27}_{3,2}) = a+1004$; $f_3(u^{27}_{3,3}) = a+1007$; $f_3(u^{27}_{3,4}) = a+1009$;
$f_3(u^{28}_{3,1}) = a+1032$; $f_3(u^{28}_{3,2}) = a+1034$; $f_3(u^{28}_{3,3}) = a+1037$; $f_3(u^{28}_{3,4}) = a+1039$;
$f_3(u^{29}_{3,1}) = a+1063$; $f_3(u^{29}_{3,2}) = a+1065$; $f_3(u^{29}_{3,3}) = a+1068$; $f_3(u^{29}_{3,4}) = a+1070$;
$f_3(u^{30}_{3,1}) = a+1089$; $f_3(u^{30}_{3,2}) = a+1091$; $f_3(u^{30}_{3,3}) = a+1094$; $f_3(u^{30}_{3,4}) = a+1096$;
$f_3(u^{31}_{3,1}) = a+1123$; $f_3(u^{31}_{3,2}) = a+1125$; $f_3(u^{31}_{3,3}) = a+1128$; $f_3(u^{31}_{3,4}) = a+1130$;
$f_3(u^{32}_{3,1}) = a+1154$; $f_3(u^{32}_{3,2}) = a+1156$; $f_3(u^{32}_{3,3}) = a+1159$; $f_3(u^{32}_{3,4}) = a+1161$;
$f_3(u^{33}_{3,1}) = a+1176$; $f_3(u^{33}_{3,2}) = a+1178$; $f_3(u^{33}_{3,3}) = a+1181$; $f_3(u^{33}_{3,4}) = a+1183$;
$f_3(u^{34}_{3,1}) = a+1195$; $f_3(u^{34}_{3,2}) = a+1197$; $f_3(u^{34}_{3,3}) = a+1200$; $f_3(u^{34}_{3,4}) = a+1202$;
$f_3(u^{35}_{3,1}) = a+1230$; $f_3(u^{35}_{3,2}) = a+1232$; $f_3(u^{35}_{3,3}) = a+1235$; $f_3(u^{35}_{3,4}) = a+1237$;
$f_3(u^{36}_{3,1}) = a+1254$; $f_3(u^{36}_{3,2}) = a+1256$; $f_3(u^{36}_{3,3}) = a+1259$; $f_3(u^{36}_{3,4}) = a+1261$;

Here the edge labels of $K_3^*\backslash K_2^*$ are as follows:
The edge labels of K_4's in $K_3^*\backslash K_2^*$ for $1 \le i \le 36$ are as below:
$\mid f_3(u^i_{3,1}) - f_3(u^i_{3,2}) \mid= 2$; $\mid f_3(u^i_{3,1}) - f_3(u^i_{3,3}) \mid= 5$; $\mid f_3(u^i_{3,1}) - f_3(u^i_{3,4}) \mid= 7$;
$\mid f_3(u^i_{3,2}) - f_3(u^i_{3,3}) \mid= 3$; $\mid f_3(u^i_{3,2}) - f_3(u^i_{3,4}) \mid= 5$; $\mid f_3(u^i_{3,3}) - f_3(u^i_{3,4}) \mid= 2$.

The edge labels of connecting edges between level 2 to level 3 are:
$\mid f_3(u^1_{2,2}) - f_3(u^1_{3,1}) \mid= 55^{th}$ prime $= 257$;
$\mid f_3(u^1_{2,3}) - f_3(u^2_{3,1}) \mid= 58^{th}$ prime $= 271$;
$\mid f_3(u^1_{2,4}) - f_3(u^3_{3,1}) \mid= 61^{st}$ prime $= 283$;
$\mid f_3(u^2_{2,2}) - f_3(u^4_{3,1}) \mid= 64^{th}$ prime $= 311$;
$\mid f_3(u^2_{2,3}) - f_3(u^5_{3,1}) \mid= 67^{th}$ prime $= 331$;
$\mid f_3(u^2_{2,4}) - f_3(u^6_{3,1}) \mid= 70^{th}$ prime $= 349$;
$\mid f_3(u^3_{2,2}) - f_3(u^7_{3,1}) \mid= 73^{rd}$ prime $= 367$;
$\mid f_3(u^3_{2,3}) - f_3(u^8_{3,1}) \mid= 76^{th}$ prime $= 383$;
$\mid f_3(u^3_{2,4}) - f_3(u^9_{3,1}) \mid= 79^{th}$ prime $= 401$;
$\mid f_3(u^4_{2,2}) - f_3(u^{10}_{3,1}) \mid= 82^{nd}$ prime $= 421$;

$| f_3(u^4_{2,3}) - f_3(u^{11}_{3,1}) |= 85^{th}$ prime $= 439$;
$| f_3(u^4_{2,4}) - f_3(u^{12}_{3,1}) |= 88^{th}$ prime $= 457$;
$| f_3(u^5_{2,2}) - f_3(u^{13}_{3,1}) |= 91^{st}$ prime $= 467$;
$| f_3(u^5_{2,3}) - f_3(u^{14}_{3,1}) |= 94^{th}$ prime $= 491$;
$| f_3(u^5_{2,4}) - f_3(u^{15}_{3,1}) |= 97^{th}$ prime $= 509$;
$| f_3(u^6_{2,2}) - f_3(u^{16}_{3,1}) |= 100^{th}$ prime $= 541$;
$| f_3(u^6_{2,3}) - f_3(u^{17}_{3,1}) |= 103^{rd}$ prime $= 563$;
$| f_3(u^6_{2,4}) - f_3(u^{18}_{3,1}) |= 106^{th}$ prime $= 577$;
$| f_3(u^7_{2,2}) - f_3(u^{19}_{3,1}) |= 109^{th}$ prime $= 599$;
$| f_3(u^7_{2,3}) - f_3(u^{20}_{3,1}) |= 112^{th}$ prime $= 613$;
$| f_3(u^7_{2,4}) - f_3(u^{21}_{3,1}) |= 115^{th}$ prime $= 631$;
$| f_3(u^8_{2,2}) - f_3(u^{22}_{3,1}) |= 118^{th}$ prime $= 647$;
$| f_3(u^8_{2,3}) - f_3(u^{23}_{3,1}) |= 121^{st}$ prime $= 661$;
$| f_3(u^8_{2,4}) - f_3(u^{24}_{3,1}) |= 124^{th}$ prime $= 683$;
$| f_3(u^9_{2,2}) - f_3(u^{25}_{3,1}) |= 127^{th}$ prime $= 709$;
$| f_3(u^9_{2,3}) - f_3(u^{26}_{3,1}) |= 130^{th}$ prime $= 733$;
$| f_3(u^9_{2,4}) - f_3(u^{27}_{3,1}) |= 133^{rd}$ prime $= 751$;
$| f_3(u^{10}_{2,2}) - f_3(u^{28}_{3,1}) |= 136^{th}$ prime $= 769$;
$| f_3(u^{10}_{2,3}) - f_3(u^{29}_{3,1}) |= 139^{th}$ prime $= 797$;
$| f_3(u^{10}_{2,4}) - f_3(u^{30}_{3,1}) |= 142^{nd}$ prime $= 821$;
$| f_3(u^{11}_{2,2}) - f_3(u^{31}_{3,1}) |= 145^{th}$ prime $= 829$;
$| f_3(u^{11}_{2,3}) - f_3(u^{32}_{3,1}) |= 148^{th}$ prime $= 857$;
$| f_3(u^{11}_{2,4}) - f_3(u^{33}_{3,1}) |= 151^{st}$ prime $= 877$;
$| f_3(u^{12}_{2,2}) - f_3(u^{34}_{3,1}) |= 154^{th}$ prime $= 887$;
$| f_3(u^{12}_{2,3}) - f_3(u^{35}_{3,1}) |= 157^{th}$ prime $= 919$;
$| f_3(u^{12}_{2,4}) - f_3(u^{36}_{3,1}) |= 160^{th}$ prime $= 941$.

Hence f_3 is a PDL for K_3^*. Define a map $g_3 : V(K_3^*) \to \{1, 2, 3, 4\}$ such that g_3 retains the colors of the vertices of $V(K_2^*)$ as it is given at level 2. Now for the remaining 36 outermost K_4's of K_3^*, we assign colors in the cyclic manner by proceeding in clockwise direction exactly as we did previously. This produces a proper 4-coloring for K_3^* and hence $\chi(K_3^*) = 4$ and K_3^* is in class 4.

Now we proceed to the higher levels with the induction process. Let us assume that K_{s-1}^* is a PDG in class 4. Let $V(K_s^*) = V(K_{s-1}^*) \cup \{u^i_{s,1}, u^i_{s,2}, u^i_{s,3}, u^i_{s,4} \mid 1 \leq i \leq 4 \times 3^{s-1}\}$. These exclusive $4 \times 3^{s-1}$ outermost K_4's in s^{th} level are joined to the 1-crown of K_{s-1}^*. The vertex labeling of

K_s^* is given by $f_s : V(K_s^*) \to \mathbf{Z}$ as $f_s(V) = f_{s-1}(V)$ if $v \in V(K_{s-1}^*)$ and for $v \notin V(K_{s-1}^*)$
$f_s(u_{s,1}^1) = f_s(u_{s-1,2}^1) + (7 + (\sum_{j=1}^{s-1} 4 \times 3^j))^{th}$ prime;
$f_s(u_{s,2}^1) = f_s(u_{s,1}^1) + 2$; $f_s(u_{s,3}^1) = f_s(u_{s,1}^1) + 5$; $f_s(u_{s,4}^1) = f_s(u_{s,1}^1) + 7$;
$f_s(u_{s,1}^2) = f_s(u_{s-1,3}^1) + (7 + (\sum_{j=1}^{s-1} 4 \times 3^j) + 3)^{th}$ prime;
$f_s(u_{s,2}^2) = f_s(u_{s,1}^2) + 2$; $f_s(u_{s,3}^2) = f_s(u_{s,1}^2) + 5$; $f_s(u_{s,4}^2) = f_s(u_{s,1}^2) + 7$;
$\ldots\ f_s(u_{s,1}^{4\times 3^{s-1}-1}) = f_s(u_{s-1,3}^{4\times 3^{s-2}}) + (4 + (\sum_{j=1}^{s} 4 \times 3^j) - 3)^{th}$ prime;
$f_s(u_{s,2}^{4\times 3^{s-1}-1}) = f_s(u_{s,1}^{4\times 3^{s-1}-1}) + 2$; $f_s(u_{s,3}^{4\times 3^{s-1}-1}) = f_s(u_{s,1}^{4\times 3^{s-1}-1}) + 5$;
$f_s(u_{s,4}^{4\times 3^{s-1}-1}) = f_s(u_{s,1}^{4\times 3^{s-1}-1}) + 7$; $f_s(u_{s,1}^{4\times 3^{s-1}}) = f_s(u_{s-1,4}^{4\times 3^{s-2}}) + (4 + (\sum_{j=1}^{s} 4 \times 3^j))^{th}$ prime;
$f_s(u_{s,2}^{4\times 3^{s-1}}) = f_s(u_{s,1}^{4\times 3^{s-1}}) + 2$; $f_s(u_{s,3}^{4\times 3^{s-1}}) = f_s(u_{s,1}^{4\times 3^{s-1}}) + 5$;
$f_s(u_{s,4}^{4\times 3^{s-1}}) = f_s(u_{s,1}^{4\times 3^{s-1}}) + 7$.

The edge labels of $E(K_s^*) \backslash E(K_{s-1}^*)$ are as follows:
The edge labels of K_4's in $K_s^* \backslash K_{s-1}^*$ for $1 \leq i \leq 4 \times 3^{s-1}$ are as below:
$\mid f_s(u_{s,1}^i) - f_{s-1}(u_{s,2}^i) \mid = 2$; $\mid f_s(u_{s,1}^i) - f_{s-1}(u_{s,3}^i) \mid = 5$; $\mid f_s(u_{s,1}^i) - f_{s-1}(u_{s,4}^i) \mid = 7$;
$\mid f_s(u_{s,2}^i) - f_{s-1}(u_{s,3}^i) \mid = 3$; $\mid f_s(u_{s,2}^i) - f_{s-1}(u_{s,4}^i) \mid = 5$; $\mid f_s(u_{s,3}^i) - f_{s-1}(u_{s,4}^i) \mid = 2$.

The edge labels of connecting edges between level $s-1$ to level s are:

$$
\begin{aligned}
\mid f_s(u_{s-1,2}^1) - f_s(u_{s,1}^1) \mid &= \mid f_s(u_{s-1,2}^1) - [f_s(u_{s-1,2}^1) + \left(7 + \textstyle\sum_{j=1}^{s-1} 4 \times 3^j\right)^{th} \text{ prime}] \mid \\
&= (7 + \textstyle\sum_{j=1}^{s-1} 4 \times 3^j)^{th} \text{ prime} \\
\mid f_s(u_{s-1,3}^1) - f_s(u_{s,1}^2) \mid &= \mid f_s(u_{s-1,3}^1) - [f_s(u_{s-1,3}^1) + (7 + \textstyle\sum_{j=1}^{s-1} 4 \times 3^j + 3)^{th} \text{ prime}] \mid \\
&= (7 + \textstyle\sum_{j=1}^{s-1} 4 \times 3^j + 3)^{th} \text{ prime etc.,}
\end{aligned}
$$

$$
\begin{aligned}
&\mid f_s(u_{s-1,3}^{4\times 3^{s-2}}) - f_s(u_{s,1}^{4\times 3^{s-1}-1}) \mid \\
&\quad = \mid f_s(u_{s-1,3}^{4\times 3^{s-2}}) - [f_s(u_{s-1,3}^{4\times 3^{s-2}}) + (4 + (\textstyle\sum_{j=1}^{s} 4 \times 3^j) - 3)^{th} \text{ prime}] \mid \\
&\quad = (4 + (\textstyle\sum_{j=1}^{s} 4 \times 3^j) - 3)^{th} \text{ prime} \\
&\mid f_s(u_{s-1,4}^{4\times 3^{s-2}}) - f_s(u_{s,1}^{4\times 3^{s-1}}) \mid \\
&\quad = \mid f_s(u_{s-1,4}^{4\times 3^{s-2}}) - [f_s(u_{s-1,4}^{4\times 3^{s-2}}) + (4 + (\textstyle\sum_{j=1}^{s} 4 \times 3^j))^{th} \text{ prime}] \mid \\
&\quad = (4 + (\textstyle\sum_{j=1}^{s} 4 \times 3^j))^{th} \text{ prime}
\end{aligned}
$$

This clearly shows that K_s^* is a PDG. Define a map $g_s : V(K_s^*) \to \{1, 2, 3, 4\}$ such that g_s retains the colors of the vertices of $V(K_{s-1}^*)$ as it is given at level $s-1$. Now for the remaining $4 \times 3^{s-1}$ outermost K_4's of K_s^*, we assign colors in the cyclic manner by proceeding in clockwise direction exactly as we did in previous levels. This produces a proper 4-coloring for K_s^* and hence $\chi(K_s^*) = 4$ and K_s^* is in class 4. □

Note 5.1. A Similar construction of family of graphs like the one in Theorem 5 and Theorem 6 with base graph K_n for $n \geq 5$ is not considered here due to the fact that K_n admits no PDL for $n \geq 5$. Moreover in such constructions a non-PDG K_n with $n \geq 5$ sits as an induced subgraph precluding the possibility of a PDL for the bigger graphs.

6. Some General Results

Theorem 1. *For any two G and H, $\chi(G \times H) = \max\{\chi(G), \chi(H)\}$. Here $\times$ stands for the cartesian product.*

Proof. First $\chi(G \times H)$ is at least $\chi(G)$ as the G portion of $G \times H \cong G$. Similarly, $\chi(G \times H) \geq \chi(H)$. So, we deduce that $\chi(G \times H) \geq \max\{\chi(G), \chi(H)\}$. Let $\chi(G) \geq \chi(H)$. Let $f_1 : V(G) \to \{1, 2, \ldots, \chi(G)\}$ be a proper vertex coloring of $V(G)$ and $f_2 : V(H) \to \{1, 2, \ldots, \chi(H)\}$ be a proper vertex coloring of $V(H)$. Suppose we define $g : V(G \times H) \to \{1, 2, \ldots, \chi(G)\}$ as $g(\alpha, \beta) = f_1(\alpha) + f_2(\beta)(\text{mod } \chi(G) + 1)$ then we can deduce the following. If $((\alpha, \beta_1), (\alpha, \beta_2)) \in E(G \times H)$ with $(\beta_1, \beta_2) \in E(H)$ then $g(\alpha, \beta_1) \neq g(\alpha, \beta_2)$. Similarly, if $((\alpha_1, \beta), (\alpha_2, \beta)) \in E(G \times H)$ with $(\alpha_1, \beta_2) \in E(G)$ then $g(\alpha_1, \beta) \neq g(\alpha_2, \beta)$. Hence g is a $\chi(G)$ vertex coloring of $G \times H$ and so $\chi(G \times H)$ is at most $\max\{\chi(G), \chi(H)\}$. □

Theorem 2. *Let G be any PDG in class i for $1 \leq i \leq 4$. Then $G \times K_2$ is also a PDG of the respective class.*

Proof. Given that G is a PDG. Then G has a PDL $g : V(G) \to Z^+$. Let $V(G) = \{\beta_1, \beta_2, \ldots, \beta_n\}$. Now consider the graph $G \times K_2$. Let $V(G \times K_2) = \{\beta_i, \beta_i' \text{ for } 1 \leq i \leq n\}$. Suppose that β_r is that vertex of G with the largest label among the vertices of G under g. Pick the first prime number q that is greater than $g(\beta_r)$. Now define a 1-1 function $g^* : V(G \times K_2) \to Z^+$ by $g^*(\beta_i) = g(\beta_i)$; $g^*(\beta_i') = g(\beta_i) + q$ for $1 \leq i \leq n$. Note that $g^*(\beta_r') = g(\beta_r) + q$ and $g^*(\beta_s') = g(\beta_s) + q$. Therefore whenever (β_r', β_s') belongs to second copy of G we see that $| g^*(\beta_r') - g^*(\beta_s') | = | (g(\beta_r) + q) - (g(\beta_s) + q) | = | g(\beta_r) - g(\beta_s) | =$ a prime. Also $| g^*(\beta_i) - g^*(\beta_i') | = | g(\beta_i) - (g(\beta_i) + q) | = q$, a prime. So $G \times K_2$ admits a PDL provided by g^*. This means that $G \times K_2$ is a PDG. By Theorem 1, we see that $\chi(G \times K_2) = \max\{\chi(G), \chi(K_2)\}$. Clearly as $K_2 \subseteq G$ for any connected graph G and χ is a monotone function $\chi(K_2) \leq \chi(G)$ and $\max\{\chi(G), \chi(K_2)\} = \chi(G)$. Thus $\chi(G \times K_2) = \chi(G)$. Now if $\chi(G) \in$ *Class* i for $1 \leq i \leq 4$ then $G \times K_2$ also belongs to class i, for $1 \leq i \leq 4$. □

Theorem 3. *Let G be any PDG. Then any countable union of disjoint copies of G is a PDG. Moreover both G and nG, $n \in Z^+$ belongs to the same class i, for $1 \le i \le 4$.*

Proof. Let G be any PDG with PDL g. Let $V(G) = \{\beta_1, \ldots, \beta_n\}$. Consider nG. Let $V(nG) = \{\beta_1^1, \beta_2^1, \ldots, \beta_n^1; \beta_1^2, \beta_2^2, \ldots, \beta_n^2; \ldots; \beta_1^n, \beta_2^n, \ldots, \beta_n^n\}$. We proceed by the principle of mathematical induction. Let $n = 2$, choose the first prime say q larger than the $\max\{g(\beta_i) : 1 \le i \le n\}$ and define $h(\beta_i^2) = g(\beta_i) + q$ for $1 \le i \le n$. Then h is a PDL for the second copy of G and if we let $h(\beta_i^1) = g(\beta_i)$ for $1 \le i \le n$ then $h : V(2G) \to Z^+$ is a PDL for $2G$ and $2G$ is a PDG. Next assume that for $n = r$ the result is true and let $n = r + 1$. Now consider $(r + 1)G$. Let $g^* : V(rG) \to Z^+$ be a PDL of rG. Now define $h^* : V((r + 1)G) \to Z^+$ by $h^*(\beta) = g^*(\beta)$ if $\beta \in V(rG)$ and $h^*(\beta_j^{r+1}) = g^*(\beta_j^r) + q$ if $\beta_j^{r+1} \in V((r + 1)G)$ for $1 \le j \le n$. Also, q is the first prime larger than the $\max\{g^*(\beta)\}$ where $\beta \in V(rG)$. Then one can check that h^* is a PDL of $(r + 1)G$ and $(r + 1)G$ is a PDG. Hence, we deduce that nG is a PDG for $n \in Z^+$ by the principle of mathematical induction. Moreover $\chi(nG) = \chi(G)$ as the same color can be retained in all copies of G. So, both G and nG for $n \in Z^+$ belongs to the same class i, for $1 \le i \le 4$. □

Theorem 4. *The middle graph of a path on n vertices is a PDG and it belongs to class 3.*

Proof. Let $P_n = \beta_1\beta_2 \ldots \beta_n$ be the path on n vertices. Then the middle graph of P_n denoted $M(P_n)$ has $V(M(P_n)) = \{\beta_1,\ \beta_2,\ \ldots,\ \beta_n,\ \varphi_1,\ \varphi_2,\ \ldots,\ \varphi_{n-1}\}$ where φ_i is the edge between β_i and β_{i+1} for $1 \le i \le n-1$ and $E(M(P_n)) = E_1 \cup E_2$ where $E_1 = \{(\varphi_r, \varphi_s) : \varphi_r$ and φ_s are adjacent in $P_n\}$ and $E_2 = \{(a, b) : a$ is an edge and b is a vertex or a is a vertex and b is an edge in P_n and one is incident on the other$\}$. Clearly K_3 is an induced subgraph of $M(P_n)$ and hence $\chi(M(P_n)) \ge 3$. Moreover define $g : V(P_n) \to \{a, b, c\}$ such that $g(\beta_j) = c$, if $1 \le j \le n$ and $g(\varphi_{2j-1}) = a$ if $1 \le j \le n-1$; $g(\varphi_{2j}) = b$ if $1 \le j \le n-1$. Then g is a chromatic 3-coloring of $M(P_n)$ and hence $\chi(M(P_n)) \le 3$ and so $\chi(M(P_n)) = 3$. Also define $g^* : V(M(P_n)) \to Z$ as follows: $g^*(\beta_{3j-2}) = x$; $g^*(\beta_{3j-1}) = x + 3$; $g^*(\beta_{3j}) = x + 5$ for $1 \le j \le n$ where $x \in Z$; $g^*(\varphi_{3r-2}) = x + 5$; $g^*(\varphi_{3r-1}) = x$; $g^*(\varphi_{3r}) = x + 7$ $1 \le r \le n-1$ for $x \in Z$ Then one can check that $\mid g^*(\beta_{3r-1}) - g^*(\varphi_{3j-2}) \mid = \mid g^*(\beta_{3r}) - g^*(\varphi_{3j}) \mid = \mid g^*(\varphi_{3j}) - g^*(\varphi_{3j-2}) \mid = 2$; $\mid g^*(\beta_{3r-1}) - g^*(\varphi_{3j-1}) \mid = \mid g^*(\beta_{3r-2}) - g^*(\varphi_{3j}) \mid = \mid g^*(\varphi_{3j-1}) - g^*(\varphi_{3j}) \mid = 3$;

$| g^*(\beta_{3r-2}) - g^*(\varphi_{3j-2}) | = | g^*(\beta_{3r}) - g^*(\varphi_{3j-1}) | = | g^*(\varphi_{3j-2}) - g^*(\varphi_{3j-1}) | = 5$ for $1 \le r \le n$ and $1 \le j \le n-1$. So $M(P_n)$ is a PDG with PDL g^* and $M(P_n)$ belongs to class 3. □

Theorem 5. *The total graph of a path on n vertices is a PDG and it belongs to class 3.*

Proof. Let $P_n = \beta_1\beta_2 \ldots \beta_n$ be the path on n vertices. Then the total graph of P_n denoted by $T(P_n)$ has vertex set $V(T(P_n)) = \{\beta_1, \beta_2, \ldots, \beta_n, \varphi_1, \varphi_2, \ldots, \varphi_{n-1}\}$ where φ_i is the edge between β_i and β_{i+1} for $1 \le i \le n-1$. The edge set of $T(P_n)$ is given by $E(T(P_n)) = \{(a, b) : a, b$ are vertices adjacent in P_n or a, b are edges adjacent in P_n or a is a vertex and b is an edge or a is an edge and b is vertex with one incident on the other$\}$. Clearly K_3 is an induced subgraph of $T(P_n)$ and hence $\chi(T(P_n)) \ge 3$. Moreover define $g : V(T(P_n)) \to \{a, b, c\}$ by $g(\beta_{3j-2}) = a$; $g(\beta_{3j-1}) = b$; $g(\beta_{3j}) = c$ for $1 \le j \le n$; $g(\varphi_{3r-2}) = c$; $g(\varphi_{3r-1}) = a$; $g(\varphi_{3r}) = b$ for $1 \le r \le n-1$. Then g is a proper 3-coloring of $T(P_n)$ and hence $\chi(T(P_n)) \le 3$ and so $\chi(T(P_n)) = 3$. Also define $g^* : V(T(P_n)) \to Z$ as follows: $g^*(\beta_{3j-2}) = x$; $g^*(\beta_{3j-1}) = x+3$; $g^*(\beta_{3j}) = x+5$; $g^*(\varphi_{3r-2}) = x+5$; $g^*(\varphi_{3r-1}) = x$; $g^*(\varphi_{3r}) = x+3$ for $x \in Z$ $1 \le r \le n-1$, $1 \le j \le n$. Then one can check that $| g^*(\beta_{3j-1}) - g^*(\varphi_{3r-2}) | = | g^*(\beta_{3j}) - g^*(\varphi_{3r}) | = | g^*(\varphi_{3r}) - g^*(\varphi_{3r-2}) | = | g^*(\beta_{3j-1}) - g^*(\beta_{3j}) | = 2$; $| g^*(\beta_{3j-1}) - g^*(\varphi_{3r-1}) | = | g^*(\beta_{3j-2}) - g^*(\varphi_{3r}) | = | g^*(\varphi_{3r-1}) - g^*(\varphi_{3r}) | = | g^*(\beta_{3j-2}) - g^*(\beta_{3j-1}) | = 3$; $| g^*(\beta_{3j-2}) - g^*(\varphi_{3r-2}) | = | g^*(\beta_{3j}) - g^*(\varphi_{3r-1}) | = | g^*(\varphi_{3r-2}) - g^*(\varphi_{3r-1}) | = | g^*(\beta_{3j}) - g^*(\beta_{3j-2}) | = 5$ for $1 \le r \le n-1$ and $1 \le j \le n$. So, $T(P_n)$ is a PDG with PDL g^* and $T(P_n)$ belongs to class 3. □

Theorem 6. *Let G and H be any two PDGs belonging to class i for $3 \le i \le 4$. Then $G \vee H$ is not a PDG.*

Proof. $\chi(G(Z, P)) = 4$ by a result in [15]. Therefore if $G(Z, D)$ is any PDG with $D \subseteq P$ then $\chi(G(Z, D))$ is at most 4. A contrapositive of this statement reveals that if the χ of any DG, $G(Z, D)$ at least 5 then $G(Z, D)$ is not a PDG. Note that $\chi(G \vee H) = \chi(G) + \chi(H)$. Hence if $\chi(G) = 3$ or 4 and $\chi(H) = 3$ or 4 then obviously $\chi(G \vee H) = \chi(G) + \chi(H)$ is at least 6 and hence $G \vee H$ is not a PDG. □

Corollary 7. *If G and H are any two PDGs with either $G \in$ class 3 and $H \in$ class 2 or vice-versa then $G \vee H$ is not a PDG.*

Observation 1. *If $G(Z, D)$ with $D \subset P$ is a PDG then $\chi(G(Z, D))$ is at most 4. But the reverse implication is not necessarily be true. For instance, the wheel graph $W_n = C_{n-1} \vee K_1$ has chromatic number 3. But if $n \geq 9$ then W_n admits no PDL. This is because one can find only three consecutives odd labels induced by a twin prime triple (3, 5, 7). One another set of such consecutive odd labels are induced by $(-3, -5, -7)$. Also, as 2 is the only even prime, we see that it is induced by an edge of a wheel only by two vertex labels namely $\alpha + 2$ or $\alpha - 2$ with α as any label for the vertex of K_1 in $C_{n-1} \vee K_1$. That is, there can be at most 8 labels namely $\alpha+3, \alpha+5, \alpha+7, \alpha-3, \alpha-5, \alpha-7, \alpha+2$ and $\alpha-2$ that can appear as vertex labels of the vertices of $W_n = C_{n-1} \vee K_1$ with α as the label for the vertex of K_1 to produce prime edge labels on the edges of W_n. Hence W_n admits no PDL for $n \geq 9$.*

Observation 2. *It is easy to see that any subgraph of a PDG is a PDG. The same can be said differently using contrapositive statement that if any subgraph of a graph admits no PDL then the graph itself admits no PDL. In view of this and above observation we note that Helm graph constructed out of a wheel graph by attaching a pendant edge on each of the vertices of the cycle C_{n-1}, admits no PDL and hence it is not a PDG. Also Helm graph is another instance of a graph with chromatic number 3 possessing no PDL.*

Lemma 8. *Any PDL f of K_4 allots to the vertices of K_4 the labels in any order either of the form $x, x+2, x+5, x+7$ or $x, x-2, x-5, x-7$ for $x \in Z$*

Proof. $V(K_4) = \{u_1, u_2, u_3, u_4\}$ and $E(K_4) = \{u_i u_{i+1}$ for $1 \leq i \leq 3$, $u_1 u_3, u_2 u_4\}$. Suppose that f is a PDL of K_4. As $f(u_i)$ is distinct for $1 \leq i \leq 4$, we have either $f(u_i) < f(u_j)$ or $f(u_i) > f(u_j)$ for any $i < j$. Without lose of generality assume that $f(u_i) < f(u_j)$ for any $i < j$. As f is a PDL it is clear that $\mid f(u_i) - f(u_{i+1}) \mid, \mid f(u_1) - f(u_3) \mid, \mid f(u_2) - f(u_4) \mid$ are all prime numbers. Let $\mid f(u_1) - f(u_2) \mid = p_1$, $\mid f(u_1) - f(u_3) \mid = p_2$ and $\mid f(u_1) - f(u_4) \mid = p_3$. Then all p_i's are distinct. This is because, if any two p_i's are equal say $p_1 = p_2$. Then the distance between u_2 and u_3 is 0, a contradiction. Now we claim that $p_i \notin 2Z + 1$ for all i with $1 \leq i \leq 3$. Suppose not, then as $\mid \{f(u_1) - f(u_4)\} - \{f(u_1) - f(u_3)\} \mid = \mid f(u_3) - f(u_4) \mid = p_3 - p_2$; $\mid f(u_3) - f(u_2) \mid = p_2 - p_1$; $\mid f(u_4) - f(u_2) \mid = p_3 - p_1$ are all in $2Z$ we infer that $p_1 = p_2$, a contradiction. Hence it follows that $p_1 = 2$ and $p_2, p_3 \in 2Z + 1$. Moreover $p_3 = p_2 + 1$. Next if $p_2 \in 3Z$ then $p_2 = 3t$ for some $t \in Z$. Now t divides p_2 implies $t = 1$ or $t = p_2$. As

$t = p_2$ is not possible we get $p_2 = 3$. But then $p_2 - p_1 = 1$, a contradiction. Similarly if $p_2 \in 3Z + 1$ then one can derive a contradiction with similar reasoning. So $p_2 \in 3Z + 2$. As $p_2 - p_1 = 3t$ for some $t \in Z$ and $3t$ is composite for all $t \geq 2$ we infer that $p_2 - p_1$ is a prime only when $t = 1$. So $p_2 = 5$. Further $p_3 = p_2 + 2$ implies $p_3 = 7$. Also $| f(u_1) - f(u_2) |= p_1 = 2$ implies $f(u_1) = 0$. Hence one sequence of PDL allotted for K_4 is 0, 2, 5, 7. One can obtain different sequence of such PDLs by giving a uniform shift of x to the above labels. Thus $x, x+2, x+5, x+7$ for any x is a PDL for K_4. Further one can argue in a similar manner that $x, x-2, x-5, x-7$ for any $x \in Z$ is a PDL for K_4. □

Theorem 9. *$K_5 - e$ is not a PDG.*

Proof. Let $V(K_5 - e) = \{u_1, u_2, u_3, u_4, u_5\}$ and $E(K_5 - e) = \{u_i u_{i+1}$ for $1 \leq i \leq 4, u_1u_3, u_1u_4, u_1u_5, u_2u_4, u_3u_5\}$. We claim that $K_5 - e$ is not a PDG. Suppose that $K_5 - e$ admits a PDL f with $f(u_1) = x$; $f(u_2) = x+2$; $f(u_3) = x+5$; $f(u_4) = x+7$ and $f(u_5) = y$ by Lemma 8 as $\{u_1, u_3, u_4, u_5\}$ induces a K_4 in $K_5 - e$. Here two cases arise.

Case 1: $x \in 2Z$ Now the label y can be either odd or even. If $y \in 2Z$ then as $| f(u_1) - f(u_5) |$ is a prime one can deduce that $y = x + 2$. Observe that y cannot lie between x and $x+7$. So, either $y < x$ or $y > x+7$. This means y cannot be $x+2$. Hence $y \notin 2Z$. If $y \in 2Z+1$ and $y > x+7$ then we derive a contradiction as $| f(u_5) - f(u_3) |$ is a prime and $| y - x + 5 |$ is even and 2 is the only even prime. Again if $y \in 2Z+1$ and $y < x$ then also one can derive a contradiction as $| f(u_3) - f(u_5) |=| (x+5) - y |$ is an even prime and 2 is the only even prime.

Case 2: $x \in 2Z+1$ A similar argument as in Case 1 yields a contradiction.

This means in both Case 1 and Case 2 one cannot give a label for y which yields a PDL for $K_5 - e$. □

Corollary 10. *If G and H are any two bipartite graphs then $G \vee H$ admits no PDL and hence there exist a graph which is not a member of class 4 but has chromatic number 4.*

Proof. Note that $K_5 - e$ is an induced subgraph of $G \vee H$ and $\chi(G \vee H) = \chi(G) + \chi(H) = 2 + 2 = 4$. Also, it is a fact that a subgraph of a PDG is a PDG and hence if the subgraph of a graph is not a PDG then the graph itself is not a PDG. So, we are done by Theorem 9. □

Theorem 11. *All cycles are PDGs*

Proof. Let $C_n = \alpha_1, \alpha_2, \ldots, \alpha_{n-1}, \alpha_n$ be the cycle graph on n-vertices then the execution of the following procedure yields a PDL for C_n

Step 1:
(i) If $n = 2t$ for some $t \in N$, then label the vertices α_1 and α_{2t} as x and $x + 2$
(ii) If $n = 2t + 1$ for some $t \in N$, then label the vertices α_1 and α_{2t+1} as x and $x + 2$

Step 2:
Choose any twin prime pair p_1 and p_2
(i) If $n = 2t$, for some $t \in N$, then label the vertices α_2 and α_{2t-1} as $x + p_1$ and $x + 2 + p_1$
(ii) If $n = 2t + 1$, for some $t \in N$, then label the vertices of α_2 and α_{2t} as $x + p_1$ and $x + 2 + p_1$

Step 3:
(i) If $n = 2t$, for some $t \in N$ then label the vertices $\alpha_3, \alpha_4, \ldots, \alpha_t$ as $(x + p_1) + 3, (x + p_1) + 2(3), \ldots, (x + p_1) + 3(t - 2)$ in order
(ii) If $n = 2t + 1$, for some $t \in N$ then label the vertices $\alpha_3, \alpha_4, \ldots \alpha_t$ as $(x + p_1) + 3, (x + p_1) + 2(3), \ldots, (x + p_1) + 3(t - 2)$ in order

Step 4:
(i) If $n = 2t$, for some $t \in N$ then label the vertices $\alpha_{2t-2}, \alpha_{2t-3}, \ldots \alpha_{t+1}$ as $(x+2+p_1)+3, (x+2+p_1)+2(3), (x+2+p_1)+3(3), \ldots, (x+2+p_1)+3(t-2)$ in order.
(ii) If $n = 2t+1$, for some $t \in N$, then label the vertices $\alpha_{2t-1}, \alpha_{2t-2}, \ldots, \alpha_{t+1}$ as $(x + 2 + p_1) + 3, (x + 2 + p_1 + 2(3)), \ldots, (x + 2 + p_1) + p(t - 1)$

Step 5:
(i) if $n = 2t$, $t \in N$, then check whether the edge labels in the clockwise direction are $p_1, 3, 3, \ldots 3(2t - 3)$ times, p_2, 2. If so, then go to Step 6

(ii) if $n = 2t$ $t \in N$, then check whether edge labels in the clockwise direction are $p_1, 3, 3, \ldots 3(2t-2)$ times, p_2, 2. If so, then go to Step 6

Step 6:
Declare the above labeling as PDL and call C_n as PDG for all n and go to Step 7

Step 7:
Stop

$\square$

7. Conclusion

While attempting the problem of characterizing the family of graphs belonging to class i when D is of any given size we have somehow succeeded in obtaining one family each of graphs in class 3 and 4 whose distance set consists of countably many elements in Theorem 5 and Theorem 6. We also obtained certain interesting general results and existential results regarding class i collection of graphs.

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9. Data Availability Statement

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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