Proyecciones Journal of Mathematics
Vol. 42, N$^o$ 2, pp. 479-488, April 2023.
Universidad Católica del Norte
Antofagasta - Chile

10.22199/issn.0717-6279-5674

# Subspace spanning graph topological spaces of graphs

*Achu Aniyan*
*CHRIST (Deemed to be University), India*
*and*
*Sudev Naduvath*
*CHRIST (Deemed to be University), India*
*Received : October 2022. Accepted : January 2023*

**Abstract**

*A collection of spanning subgraphs $\mathcal{T}_S$, of a graph $G$ is said to be a spanning graph topology if it satisfies the three axioms: $N_n, K_0 \in \mathcal{T}_S$ where, $n = |V(G)|$, the collection is closed under any union and finite intersection. Let $(X, T)$ be a topological space in point set topology and $Y \subseteq X$ then, $T_Y = \{U \cap Y : U \in T\}$ is a topological space called a subspace topology or relative topology defined by $T$ on $Y$. In this paper we discusses the subspace spanning graph topology defined by the graph topology $\mathcal{T}_S$ on a spanning subgraph $H$ of $G$.*

**Keywords:** *Spanning graph topology, subspace spanning graph topology, d-closed spanning graphs.*

**MSC2020:** *05C62, 05C75, 54A05.*

## 1. Introduction

A graph topology on a graph is a collection of subgraphs satisfying three axioms analogous to the axioms of topology. The definition and fundamental concepts were discussed in [2] and [3].

A spanning subgraph $S$ of a graph $G$ is a subgraph of $G$ with vertex set same as the vertex set of the graph $G$ [11]. The study on spanning graph topology was initiated in the paper [8] where the author considered a topology on spanning subgraphs considering the collection of spanning subgraphs along with empty set and graph itself satisfying the properties analogous to the properties of a set topology. Since empty set itself cannot be considered as a spanning subgraph we may consider the spanning subgraph with empty edge set. Taking this into consideration in [1], the authors reformulated the definition specifically as follows :

**Definition 1.1.** [1] Let $G$ be a graph of order $n > 1$. The collection $\mathcal{T}_S$ of spanning subgraphs of $G$ is said to be a *spanning graph topology*, if it satisfies the following conditions.

1. $G, N_n \in \mathcal{T}_S$;
2. Any union of members of $\mathcal{T}_S$ is in $\mathcal{T}_S$;
3. Finite intersection of members of $\mathcal{T}_S$ is in $\mathcal{T}_S$;

where $N_n$ (or $\overline{K_n}$) is the spanning empty graph on $n$ vertices, $n$ being the order of the graph. The pair $(G, \mathcal{T}_S)$ is called the *spanning graph topological space* or *s-graph topological space*. A spanning subgraph is said to be *open* in the spanning graph topology if it is a member of $\mathcal{T}_S$.

The basic ideas and definitions of spanning graph topology has been discussed in the paper [1]. A *base* in a graph topology is the smallest subcollection of $\mathcal{T}_S$ of spanning subgraphs where an $s$-graph topology can be constructed from by this by taking arbitrary union. The following theorem characterises the basis of an $s$-graph topology.

**Theorem 1.2.** [1] *Let $(G, \mathcal{T}_S)$ be an s-graph topological space and let $G_s \subset \mathcal{T}_S$ then, $G_s$ is a basis for the s-topological space if and only if, for each $e \in E(G)$, there exist $G_j \in G_s$ such that $e \in E(G_j) \subseteq E(G)$.*

Since the vertex set remains the same, Theorem 1.2 shows that a collection of spanning subgraph is a basis for an $s$-graph topology if every edge of the graph $G$ is contained in one of the members of the basis collection.

## 2. Subspace Spanning Graph Topology

In this section, we discuss the definition of subspace $s$-graph topology of graphs.

**Definition 2.1.** Let $(G, \mathcal{T}_S)$ be a spanning graph topological space and let $S$ be a spanning subgraph of $G$. Then the spanning graph topology defined on $S$ by $\mathcal{T}_S^* = \{S \cap S_i : S_i \in \mathcal{T}_S\}$ is called the subspace $s$-graph topology.

To verify whether the collection $\mathcal{T}_S^* = \{S \cap S_i : S_i \in \mathcal{T}_S\}$ is a graph topology, we need to verify the three axioms of spanning graph topology. In order to prove that, we need the following lemma which shows that the graphs satisfies the distribution properties of union over intersection and intersection over union.

**Lemma 2.2.** *Consider three spanning subgraphs $S_1, S_2, S_3$ of a graph $G$ along with the operations union and intersection. Then,*

$$S_1 \cup (S_2 \cap S_3) = (S_1 \cup S_2) \cap (S_1 \cup S_3) \tag{2.1}$$

$$S_1 \cap (S_2 \cup S_3) = (S_1 \cap S_2) \cup (S_1 \cap S_3) \tag{2.2}$$

**Proof.** Let $S_1, S_2, S_3$ be spanning subgraphs of a graph $G$. Since the graphs are spanning subgraphs the vertex set remains the same and hence we consider the edge set of each spanning subgraphs. Now, consider the left hand side of Equation (2.1).

$$E(S_2 \cap S_3) = E(S_2) \cap E(S_2) \tag{2.3}$$

Then,

$$E(S_1 \cup (S_2 \cap S_3)) = E(S_1) \cup \Big(E(S_2) \cap E(S_2)\Big) \quad (2.4)$$

Since $E(S_i), i = 1, 2, 3$ are ordinary sets, the distribution properties of set are satisfied and hence the right hand side of Equation eqn4 becomes:

$$E(S_1) \cup \Big(E(S_2) \cap E(S_2)\Big) = \Big(E(S_1) \cup E(S_2)\Big) \cap \Big((E(S_1) \cup E(S_2)\Big) \tag{2.5}$$

From Equation (2.4) and (2.5) we have,

$$E(S_1 \cup (S_2 \cap S_3)) = \Big(E(S_1) \cup E(S_2)\Big) \cap \Big((E(S_1) \cup E(S_2)\Big) \tag{2.6}$$

The spanning subgraph induced by the edge set of the left hand side of Equation (2.5) and the edge set obtained from the right hand side of (2.5) are same.
Similarly, the Equation (2.2) can also be established. □
The Lemma 2.2 can be extended to any collection of spanning subgraphs as follows:

**Theorem 2.3.** *Let $S_1, S_2, S_3, \dots, S_j, \dots$ be spanning subgraphs of a graph $G$. Then, for a fixed integer $i \in \{1, 2, \dots, \infty\}$,*

$$S_i \bigcap \left( \bigcup_{j=1}^{\infty} S_j \right) = \bigcup_{j=1}^{\infty} (S_i \cap S_j)$$

Using Lemma 2.2 and Theorem 2.3 we prove that the collection $\mathcal{T}_S = \{S_i \cap S : S_i \in \mathcal{T}_S\}$ is an $s$-graph topology.

**Theorem 2.4.** *Consider a spanning graph topological space $(G, \mathcal{T}_S)$. Let $S$ be a spanning subgraph of $G$ and $\mathcal{T}_S^* = \{S_i \cap S : S_i \in \mathcal{T}_S\}$. Then, $\mathcal{T}_S$ is a spanning graph topological space on $S$.*

**Proof.** Consider $\mathcal{T}_S^* = \{S_i \cap S : S_i \in \mathcal{T}_S\}$ where $S_i$ is an open spanning subgraph of $G$ and $S$ is a spanning subgraph. In order to prove $\mathcal{T}_S^*$ is a spanning graph topology we verify the three axioms of spanning graph topologies.

Let $S_1, S_2, S_3, \dots$ be open spanning subgraphs in the spanning graph topological space $\mathcal{T}_S$. Then, $\mathcal{T}_S$ contains the graph $G$ and the spanning null graph $N_n$. We shall now consider the three axioms.

1. We need to show that the spanning null graph $N_n$ and graph $G$ is in the collection $\mathcal{T}_S^*$. Since $N_n, G \in \mathcal{T}_S$ and by Definition 2.1, $N_n \cap S = N_n$ and $G \cap S = S$. Hence both $N_n$ and $G$ are in $\mathcal{T}_S^*$.

2. We now need to show that any intersection of open spanning subgraphs are open. For that consider $\mathcal{T}_S^* = \{S_i \cap S, : S_i \in T\}$ for $i \in I$. Consider two spanning subgraphs $S_i$ and $S_j$ which are open in $\mathcal{T}_S$.

Since $\mathcal{T}_S$ is a spanning graph topology on $G$, $S_i \cap S_j$ is open in $\mathcal{T}_S$ and so is their union $\mathcal{T}_{\mathcal{S}}$. Now consider, $(S_i \cap S) \cap (S_j \cap S) = (S_i \cap S_j) \cap S \in \mathcal{T}_S^*$. Since $\bigcup_i S_i$ is open in $\mathcal{T}_S$ we have,

$$\begin{aligned}(S_i \cap S) \cap (S_j \cap S) \quad &= S_i \cap S \cap S_j \cap S \\ &= (S_i \cap S_j) \cap S \in \mathcal{T}_S^*.\end{aligned}$$

3. The third axiom is that $\mathcal{T}_S^*$ is closed under union. Let us consider two spanning subgraphs $S_i \cap S$ and $S_j \cap S$ in $\mathcal{T}_{\mathcal{S}}{}^*$. In order to prove that $(S_i \cap S) \cup (S_j \cap S) \in \mathcal{T}_S^*$. By Lemma 2.2 for any three spanning subgraphs,

   $$(S_i \cap S) \cup (S_j \cap S) \quad = (S_i \cup S_j) \cap S \in \mathcal{T}_S^*.$$

   By Theorem 2.3, we can say that any union of open spanning subgraphs is in $\mathcal{T}_S^*$. That is,

   $$\bigcup_i \Big(S_i \cap S\Big) = \Big(\bigcup_i S_i\Big) \cap S \in \mathcal{T}_S^*$$

   □

**Theorem 2.5.** *Let $(G, \mathcal{T}_S)$ be a spanning graph topological space and let $G_s = \{\kappa_i, i \in I\}$ be the basis of $\mathcal{T}_S$. Consider a spanning subgraph $S$ of $G$ with subspace $s$-graph topology $\mathcal{T}_S^* = \{S_i^* = S_i \cap S : S_i \in \mathcal{T}_S\}$. Then, $H = \{\kappa_i \cap S\}$ is a basis for the subspace $s$-graph topology on $S$.*

**Proof.** Consider an $s$-graph topology $(G, \mathcal{T}_S)$. Let $(S, \mathcal{T}_S^*)$ be the subspace $s$-graph topological space on the spanning subgraph $S$ of $G$. Given that $G_s = \{\kappa_1, \kappa_2, \kappa_3, \ldots\}$ is the basis for the spanning graph topological space $(G, \mathcal{T}_S)$. In order to prove that $H = \{\kappa_i \cap S\}$ is a basis for $\mathcal{T}_S^*$. We need to show that for every edge in the spanning graph $S$ there exists a spanning subgraph in $H$ containing them. Let $e$ be an arbitrary edge in $E(S)$, then we need to find a spanning subgraph in $H$. Let $S_i$ be an open spanning subgraph of $S$ containing the edge $e$. Since $S_i \in \mathcal{T}_S$, $S_i^* = S_i \cap S$ where $S_i \in \mathcal{T}_S$ by Definition 2.1. Since $e \in S_i$ by Theorem 1.2 there exist a graph $\kappa_i \in G_s$ containing $e$. That is, $e \in \kappa_i \cap S \in H$. Hence, for every edge in the basis of the $s$-graph topology $G_s$, there exist a spanning subgraph in the basis $H$ of the subspace $s$-graph topology $G_s$. □

It is not always true that the spanning subgraph which are open in the subspace space $(S, \mathcal{T}_S^*)$ are open in the ambient space $(G, \mathcal{T}_S)$. The following proposition shows the condition for an open spanning subgraph to be open in the ambient space.

**Proposition 2.6.** *If $S^*$ is an open spanning subgraph in a subspace s-graph topology $\mathcal{T}_S^*$ and $S$ be an open subgraph of $\mathcal{T}_S$ then, $S^*$ is an open spanning subgraph in the ambient space $(G, \mathcal{T}_S)$.*

**Proof.** Let $(G, \mathcal{T}_S)$ be an $s$-graph topology and for a spanning subgraph $S$, let $(S, \mathcal{T}_S^*)$ be the subspace $s$-graph topological space. Given that $S^*$ is open in $\mathcal{T}_S^*$. Then, $S^* = S_i \cap S$, where $S_i \in \mathcal{T}_S$. Since $S_i$ and $S$ are open by the third axiom of Definition 1.1, $S_i \cap S$ is open. That is, $S^*$ is open, $S^* \in \mathcal{T}_S$. Hence the claim is proved. □

**Proposition 2.7.** *Let $(G, \mathcal{T}_S)$ be an s-graph topological space and $(S, \mathcal{T}_S^*)$ be the subspace s-graph topological space of $(G, \mathcal{T}_S)$. Let $S^*$ be a subgraph of $S$. Then, the subspace s-graph topology on $S^*$ inherited from the graph topology $(S, \mathcal{T}_S^*)$ is same as the subspace s-graph topology on $S^*$ inherited from the graph topology $(G, \mathcal{T}_S)$.*

**Proof.** Let $(G, \mathcal{T}_S)$ be a graph topological space and $(S, \mathcal{T}_S^*)$ be a subspace $s$-graph topological space of $G$. Let $S^*$ be a spanning subgraph of $S$. Consider the subspace $s$-graph topological space of $S^*$ inherited from $S$ that is, $\mathcal{T}_S^* = \{S_i \cap S^* : S_i \in \mathcal{T}_S\}$. Since $S_i \in \mathcal{T}_S^*$, by Definition 2.1, we have $S_i^* = S_i \cap S$ for some $i \in I$. Therefore,

$$\begin{aligned} \mathcal{T}_S^* &= \{(S_i \cap S) \cap S^* : S_i \in T\} \\ &= \{S_i \cap (S \cap S^*) : S_i \in T\} \\ &= \{S_i \cap S^* : S_i \in T\} \end{aligned}$$

Hence, we can say that the subspace topology on $S^*$ obtained from the subspace graph topological space on $S$ is same as the subspace graph topology on $S^*$ inherited from the graph topology on $G$. □

**Proposition 2.8.** *Any open spanning subgraph of $S$ in the spanning graph topological space $(G, \mathcal{T}_S)$ is open in the subspace s-graph topological space $(S, \mathcal{T}_S^*)$.*

**Proof.** Let $S_i^*$ be a spanning subgraph of $S$ which is open in the $s$-graph topological space. Then by Definition 2.1, $S_i^* \cap S \in \mathcal{T}_S^*$. Since $S_i^*$ is a subgraph of $S$, $S_i^* \cap S = S_i^* \in \mathcal{T}_S^*$. Hence $S_i^*$ is open in subspace $s$-graph topology. □

## 3. Closed Graphs in Spanning $s$-Graph Topological Space

In set topology, a set is said to be closed if its complement is open in the topology. In $s$-graph topology a spanning subgraph is either $d$-closed or $n$-closed if corresponding $d$-complement or $n$-complement is open [1,2]. The $d$-complement of a spanning subgraph is defined as follows:

**Definition 3.1.** Consider the spanning graph topological space $(G, \mathcal{T}_S)$ and $H = (V, E_H)$ be a spanning subgraph of the graph $G$, where $V = V(G)$. The decomposition complement of $H$ with respect to $G$ is the graph $H^* = (V, E^*)$ where, $E^* = E - E_H$ is the decomposition-complement of the spanning subgraph $H$ (see [1]). Since we consider the spanning subgraphs, the vertex set remains same as the vertex set of $G$.

**Definition 3.2.** [1]In a spanning graph topological space, a spanning subgraph is said to be *d-closed* if the decomposition complement is open in $\mathcal{T}_S$.

**Theorem 3.3.** *Let $(S, \mathcal{T}_S^*)$ be a spanning s-graph topological space of a spanning graph topological space $(G, \mathcal{T}_S)$ for a subgraph $S$. Suppose that a spanning subgraph $Z$ is d-closed in $(S, \mathcal{T}_S^*)$. Then, the edge set of $Z$ is, $E(Z) = E(S) - E(S_i \cap S)$ where $S_i \in \mathcal{T}_S$.*

**Proof.** Let $(G, \mathcal{T}_S)$ be a spanning graph topology and $(S, \mathcal{T}_S^*)$ be a subspace $s$-graph topology for a spanning subgraph $S$ of $G$. Let $Z$ be a $d$-closed spanning subgraph in $(S, \mathcal{T}_S^*)$. Then, by Definition 3.1, the decomposition complement $Z^*$ of $Z$ is open in $(S, \mathcal{T}_S^*)$. Since $Z^*$ is open in $\mathcal{T}_S$, by Definition 2.1, $Z^* = S_i \cap S$ where $S_i \in \mathcal{T}_S$.

$$\begin{aligned}
E(\overline{Z^*}) &= E(S) - E(Z) \\
E(S_i \cap S) &= E(S) - E(Z) \\
E(Z) &= E(S) - E(S_i \cap S)
\end{aligned}$$

When $|S_i \cap S| < |E(S)|$, and $|S_i \cap S| \neq \emptyset$ then, $E(S^*)$ will be a proper subset of $E(S)$ and hence the spanning subgraph with this edge set will be a proper spanning subgraph of $S$. □

**Proposition 3.4.** *In a subspace s-graph topological space, the graph $S$ and the spanning null graph $N_n$ where $n$ is the order of the graph, are d-closed.*

**Proof.** Let $(S, \mathcal{T}_S^*)$ be a spanning $s$-graph topological space of a spanning graph topological space $(G, \mathcal{T}_S)$ for a subgraph $S$. We need to prove that the graph $S$ and the spanning null graph $N_n$ where $n$ is the order of the graph, are $d$-closed. By Theorem 3.3, for any $d$-closed spanning subgraph graph $Z$ of subspace $s$-topological space, $E(Z) = E(S) - E(S_i \cap S)$ where $S_i \in \mathcal{T}_S$. Suppose $S_i \cap S = S$ then, $E(\overline{Z^*}) = E(H) - E(H) = \emptyset$. Then,the spanning subgraph induced by empty edge set becomes $N_n$. Hence, $Z$ will be $N_n$.

Now, suppose that $E(S_i \cap S) = \emptyset$ then, $E(\overline{Z^*}) = E(S) - \emptyset = E(S)$ and the subgraph induced by the edge set $E(Z)$ will be $S$. Hence, $S$ is closed. □

**Theorem 3.5.** *Let $(G, T)$ be a graph topological space and $(S, \mathcal{T}_S)$ be a subspace graph topological space. Let $S_i$ be a $d$-closed subgraph in $\mathcal{T}_S$, then $S_i \cap S$ is a $d$-closed subgraph in the subspace $s$-graph topological space $(S, \mathcal{T}_S^*)$.*

**Proof.** Let $(G, T)$ be a graph topological space and $S_i$ be a $d$-closed graph in the subspace graph topological space. Then, by Definition 3.2, the decomposition complement of $S_i$ is open in $\mathcal{T}_S$ and

$$E(S_i^*) = E(G) - E(S_i)$$

The subgraph induced the edge set $E(S_i^*)$ is open in $\mathcal{T}_S$.

$$\begin{aligned} E(S_i^*) &= E(G) - E(S_i) \\ E(S_i^*) \cap E(S) &= \Big(E(G) - E(S_i)\Big) \cap E(S) \\ &= \Big(E(G) \cap E(S)\Big) - \Big(E(S_i) \cap E(S)\Big) \\ &= E(S) - \Big(E(S_i) \cap E(S)\Big) \end{aligned}$$

Since the $S$ is $d$-closed $E(S) = E(S^*)$ and $E(S_i) \cap E(S) = E(S_i \cap S)$, we have

$$\begin{aligned} E(S_i^*) \cap E(S^*) &= E(S) - \Big(E(S_i \cap S)\Big) \\ E(S_i^* \cap S) &= E(S) - \Big(E(S_i \cap S)\Big) \end{aligned}$$

By Definition 2.1, $S_i^* \cap S \in \mathcal{T}_S^*$ and hence $S_i \cap S$ is closed in $\mathcal{T}_S^*$. □

## Conclusion

The paper discusses about the subspace spanning graph topological spaces and their properties. The topic is promising for further investigation as topological structures of graphs and their analyses are relatively new and have enough within themselves. Further studies in this area are yet to be explored. Many topological properties can be extended to the spanning graph topological spaces. All these facts highlight a wide scope for further studies in this area.

## Acknowledgement

We would like to thank the unknown reviewers for taking the time and effort necessary to review the manuscript. We sincerely thank them for their insightful comments and recommendations, which allowed us to increase the quality of the manuscript.

## References

[1] A. Aniyan and S. Naduvath, "A study on graph topology", *Communications in Combinatorics and Optimization,* vol. 8, no. 2, pp. 397-409, 2023. doi: 10.22049/CCO.2022.27399.1253

[2] A. Aniyan and S. Naduvath, "Subspace graph topologies of graphs", *Proyecciones (Antofagasta, On line),* vol. 42, no. 2, pp. 521-532, 2022.

[3] D. B. West, *Introduction to graph theory,* vol. 2. New Delhi: Prentice-Hall of India, 2001.

[4] K. Karunakaran, "Topics in graph theory topological approach", Ph.D. Thesis, University of Kerala, 2007.

[5] A. Aniyan and S. Naduvatah, "Spanning graph topology of graphs", *Proyecciones (Antofagasta, On line),* vol. 42, no. 2, pp. 479-488, 2023.

[6] W.V. Kandasamy and F. Smarandache, *Strong neutrosophic graphs and subgraph topological subspaces.* Europa Nova, 2016.

[7] J. R. Munkres, *Topology.* Pearson Education, 2014.

[8] K. D. Joshi, *Introduction to general topology.* New Age International, 1983.

[9] F. Harary, *Graph theory*. New Delhi: Narosa Publications, 1969.

[10] J. A. Bondy and U.S.R. Murty, *Graph theory.* New York: Springer, 2008.

**Achu Aniyan**
Department of Mathematics
CHRIST (Deemed to be University)
Bengaluru,
India
e-mail: achu.aniyan@res.christuniversity.in

and

**Sudev Naduvath**
Department of Mathematics
CHRIST (Deemed to be University)
Bengaluru,
India
e-mail: sudev.nk@christuniversity.in
Corresponding author