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Fractional ordered Euler Riesz difference sequence spaces

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Abstract

In this article we introduce new sequence spaces ${c_0}^{(\tau)}$, $c^{(\tau)}$ and ${l_\infty}^{(\tau)}$ of fractional order τ, consisting of an operator which is a composition of Euler-Riesz operator and fractional difference operator. Certain topological properties of these spaces are investigated along with Schauder basis and $\alpha-, \beta-$ and $\gamma-$duals.

Keywords: *Euler-Riesz difference sequence space, difference operator (Δ^τ), Schauder basis, infinite matrices and $\alpha-, \beta-$ and $\gamma-$ duals.*

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1. Introduction

For all real number τ, the gamma function $\Gamma(\tau)$ is expressed as

$$\Gamma(\tau) = \int_0^\infty e^{-t}\, t^{\tau-1}\, dt \tag{1.1}$$

which is an improper integral satisfying the following properties :

1. $\Gamma(n+1) = n!$, $n \in \mathbf{N}$ set of natural numbers.
2. $\Gamma(n+1) = n\Gamma(n)$ for each real number $n \notin \{0, -1, -2, -3, ...\}$.

A paranorm on a vector space X over the real field $\mathbf{R}$ is a mapping $h : X \to \mathbf{R}$ satisfying the following conditions, for all $x, y \in X$ and a scalar λ :
(i) $h(\theta) = 0$, where $\theta = (0, 0, 0, ...)$,
(ii) $h(x) = h(-x)$,
(iii) $h(x + y) \leq h(x) + h(y)$,
(iv) $\lambda^n \to \lambda$ and $x^n \to x$ implies that $h(\lambda^n x^n) \to h(\lambda x)$ as $n \to \infty$.

i.e. scalar multiplication is continuous.

Let ω be the space of all real or complex sequences. Any subspace of ω is a sequence space. By c_0, c and l_∞ we denote the spaces of null, convergent and bounded sequences respectively, which are subspaces of ω normed by $\| x \|_\infty = \sup_k |x_k|$. By bs and cs we mean the spaces of all bounded and convergent series respectively.

For $p = (p_k)$ a bounded sequence of strictly positive real numbers, Maddox [13, 12, 11] introduced the spaces $c_0(p), c(p)$ and Simons [27] introduced the space $l_\infty(p)$ as :

$$c_0(p) = \left\{\zeta = (\zeta_k) \in \omega : \lim_{k\to\infty} |\zeta_k|^{p_k} = 0\right\},$$
$$c(p) = \left\{\zeta = (\zeta_k) \in \omega : \lim_{k\to\infty} |\zeta_k - l|^{p_k} = 0 \text{ for some } l \in R\right\},$$
$$l_\infty(p) = \left\{\zeta = (\zeta_k) \in \omega : \sup_{k\in N} |\zeta_k|^{p_k} < \infty\right\}$$

and these are complete paranormed sequence spaces with paranorm

$g(x) = \sup_{k\in N} |x_k|^{p_k/M}$ and where $M = max\{1, \sup_k\ p_k\}$.

The $\alpha-, \beta-$ and $\gamma-$ duals of sequence space X are denoted by

$$X^\alpha = \{u = (u)_k \in \omega : ux = (u_k x_k) \in l_1, \text{ for all, } x = (x_k) \in X\},$$

$$X^{\beta} = \{u = (u_k)_k \in \omega : ux = (u_k x_k) \in cs, \text{ for all, } x = (x_k) \in X\},$$
$$X^{\gamma} = \{u = (u)_k \in \omega : ux = (u_k x_k) \in bs, for\, all\, x = (x_k) \in X\}$$

respectively.

Let λ, μ be any two sequence spaces, and let $A = (a_{nk})$ be an infinite matrix of complex or real numbers, where $k, n \in \mathbf{N}$. Then, we say that A defines a matrix transformation from λ into μ, if for every sequence $x = (x_k) \in \lambda$, the sequence $Ax = (Ax)_n$, the A-transform of x, is in μ, where

$$(Ax)_n = \sum_k a_{nk} x_k \, for\, n \in \mathbf{N}. \tag{1.2}$$

When $A : \lambda \to \mu$, we write the class of matrices as $(\lambda : \mu)$. Thus, $A \in (\lambda : \mu)$ if and only if the series in (1.2) converges for each $n \in \mathbf{N}$. Also, we write $A_n = (a_{nk})$ for the sequence in the nth row of A.

The difference sequence space

$$X(\Delta) = \{x = (x_k) \in \omega : (x_k - x_{k-1}) \in X\}$$

for $X = \{c_0, c, l_\infty\}$ was introduced in 1981 by Kızmaz [9], further it generalized by Et and Çolak [17] which then attracted the attention of several mathematicians in different directions (see [16, 17, 20]).

Altay, Başar and Mursaleen [5] and Altay and Başar [2] have stuided Euler sequence spaces e_c^r, e_0^r and e_∞^r for $0 < r < 1$. The Riesz sequence spaces r_∞^q, r_c^q and r_0^q were introduced by Malkowsky [7] then Altay and Başar [4] introduced the paranorm Riesz sequence spaces $r_\infty^q(p)$, $r_c^q(p)$ and $r_0^q(p)$. For further results on Riesz sequence spaces one may refer [6, 3, 15, 10].

The Euler mean $E_1 = (e_{nk})$ of order one and Riesz mean $R_q = (r_{nk})$ are defined by

$$\mathrm{e}_{nk} = \begin{cases} \binom{n}{k}\frac{1}{2^n} & (0 \le k \le n), \\ 0 & (k > n) \end{cases} \quad \text{and} \quad r_{nk} = \begin{cases} \frac{q_k}{Q^n} & (0 \le k \le n), \\ 0 & (k > n), \end{cases}$$

where $q = (q_k)$ is a sequence of positive numbers and $Q_n = \sum_{k=0}^{n} q_k$ for $n, k \in \mathbf{N}_0$. And its inverses ${E_1}^{-1} = \hat{e}_{nk}$ and ${R_q}^{-1} = \hat{r}_{nk}$ are given by

$$\hat{e}_{nk} = \begin{cases} \binom{n}{k}(-1)^{n-k}2^k & (0 \le k \le n), \\ 0 & (k > n), \end{cases} \quad \text{and}$$

$$\hat{r}_{nk} = \begin{cases} (-1)^{n-k}\frac{Q_k}{q^n} & (n-1 \le k \le n), \\ 0 & (\text{otherwise}). \end{cases}$$

For a proper fraction τ, Baliarsingh [24], Baliarsingh & Dutta in a series of papers ([21, 22, 23, 26, 25]) introduced the fractional difference operator $\Delta^{(\tau)}$ as

$$\Delta^{(\tau)}x_k = \sum_i (-1)^i \frac{\Gamma(\tau+1)}{i!\Gamma(\tau-i+1)}x_{k-i}, \tag{1.3}$$

along with its inverse

$$\Delta^{(-\tau)}x_k = \sum_i (-1)^i \frac{\Gamma(-\tau+1)}{i!\Gamma(-\tau-i+1)}x_{k-i}. \tag{1.4}$$

Here the series of fractional difference operators are convergent. It is also appropriate to express the difference operator and its inverse as triangles in the following manner:

$$\Delta^{(\tau)}_{nk} = \begin{cases} {}^{n-k}\frac{\Gamma(\tau+1)}{(n-k)!\Gamma(\tau-n+k+1)} & (0 \le k \le n), \\ 0 & (k > n), \end{cases} \tag{1.5}$$

$$\Delta^{(-\tau)}_{nk} = \begin{cases} (-1)^{n-k}\frac{\Gamma(-\tau+1)}{(n-k)!\Gamma(-\tau-n+k+1)} & (0 \le k \le n), \\ 0 & (k > n). \end{cases} \tag{1.6}$$

We define the Euler Riesz matrix $\tilde{B} = (\tilde{b}_{nk})$ by the composition of matrices E_1 and R_q as

$$\tilde{b}_{nk} = \begin{cases} \sum_{i=k}^{n} \binom{n}{i} \frac{q_k}{2^n Q_i}, & 0 \le k \le n \\ 0, & k > n, \end{cases} \tag{1.7}$$

and its inverse $\tilde{B}^{-1} = (\hat{b}_{nk})$ is given by

$$\hat{b}_{nk} = \begin{cases} \sum_{i=n-1}^{n} \binom{i}{k}(-1)^{n-k}\frac{2_k Q_i}{q_n}, & if 0 \le k < n \\ \frac{2^n Q_n}{q_n}, & if k = n \\ 0, & if k > n, \end{cases} \tag{1.8}$$

for $n, k \in \mathbf{N}_0$.

Basar and Braha [8] introduced Euler-Cesaro difference sequence spaces $\check{c}$, $\check{c}_0$, $\check{l}_\infty$ of null, convergent and bounded sequences respectively. Baliarsingh and Dutta [21] introduced the fractional difference operators on various sequence spaces. For further investigations on difference operators one may refer [28, 29, 10] and many others.

Now our interest is to introduce the new paranormed difference sequence spaces of fractional order which generalizes many known spaces.

We introduce the spaces ${c_0}^{(\tau)}$, $c^{(\tau)}$ and ${l_\infty}^{(\tau)}$ by using the product of the Euler mean E_1 and Riesz mean R_q with fractional operator $\Delta^{(\tau)}$. We prove certain topological properties of these spaces and determine their $\alpha-, \beta-, \gamma-$ duals.

2. Main Results

Here we introduce the matrix $\tilde{B}\left(\Delta^{(\tau)}\right) = \tilde{B}^{(\tau)} = \tilde{b}_{nk}^{(\tau)}$ by the product of Euler-Riesz matrix $\tilde{B}$ (1.7) and fractional ordered difference operator $\Delta^{(\tau)}$ (1.3) as follows :

$$\tilde{b}_{nk}^{(\tau)} = \begin{cases} \sum_{j=k}^{n} \sum_{i=j}^{n} \binom{n}{i} \frac{\Gamma(\tau+1)(-1)^{j-k}}{(j-k)!\Gamma(\tau-j+k+1)} \frac{q_j}{2^n Q_i}, & if\, 0 \leq k \leq n \\ 0, & if\, k > n. \end{cases} \tag{2.1}$$

Theorem 2.1. *The inverse of the fractional ordered Euler-Riesz matrix* $\left(\tilde{B}^{(\tau)}\right)$ *written as* $\left(\tilde{B}^{(-\tau)}\right) = \tilde{b}_{nk}^{(-\tau)}$ *and is given by*

$$\tilde{b}_{nk}^{(-\tau)} = \begin{cases} \sum_{j=k}^{n} \frac{\Gamma(-\tau+1)(-1)^{n-k}}{(n-j)!\Gamma(-\tau-n+j+1)} \frac{2^k}{q_j} \sum_{i=j-1}^{j} \binom{i}{k} Q_i, & if\, 0 \leq k < n \\ \frac{2^n Q_n}{q_n}, & if\, k = n \\ 0, & if\, k > n. \end{cases} \tag{2.2}$$

Proof. This theorem can be proved using equations (1.8) and (1.6), i. e.

$$\left(\tilde{B}^{(\tau)}\right)^{-1} = \left(\Delta^{(\tau)}\right)^{-1} . \left(\tilde{B}\right)^{-1},$$

and

$$\tilde{B}^{(\tau)}\tilde{B}^{(-\tau)} = \tilde{B}^{(-\tau)}\tilde{B}^{(\tau)} = I,$$

where I is an identity operator. □

For a positive real number τ, we now introduce the classes of fractional ordered Euler-Riesz difference sequence spaces ${c_0}^{(\tau)}$, $c^{(\tau)}$ and ${l_\infty}^{(\tau)}$ by

$$(a)\ {c_0}^{(\tau)} = \Big\{x=(x_k)\in w : \lim_{n\to\infty}\Big|\textstyle\sum_{l=0}^{n}\sum_{j=k}^{n}\sum_{i=j}^{n}\binom{n}{i}\frac{\Gamma(\tau+1)(-1)^{j-k}}{(j-k)!\Gamma(\tau-j+k+1)}\ \frac{q_j x_l}{2^n Q_i}\Big|^{p_k}=0\Big\},$$

$$(b)\ c^{(\tau)} = \Big\{x=(x_k)\in w : \lim_{n\to\infty}\Big|\textstyle\sum_{l=0}^{n}\sum_{j=k}^{n}\sum_{i=j}^{n}\binom{n}{i}\frac{\Gamma(\tau+1)(-1)^{j-k}}{(j-k)!\Gamma(\tau-j+k+1)}\ \frac{q_j x_l}{2^n Q_i}\Big|^{p_k}\ exists\Big\},$$

$$(c)\ {l_\infty}^{(\tau)} = \Big\{x=(x_k)\in w : \sup_{n}\Big|\textstyle\sum_{l=0}^{n}\sum_{j=k}^{n}\sum_{i=j}^{n}\binom{n}{i}\frac{\Gamma(\tau+1)(-1)^{j-k}}{(j-k)!\Gamma(\tau-j+k+1)}\ \frac{q_j x_l}{2^n Q_i}\Big|^{p_k}<\infty\Big\}.$$

These spaces can also be rewritten as :
${c_0}^{(\tau)} = (c_0(p))_{(\tilde{B}^{(\tau)})}$, $c^{(\tau)} = (c(p))_{(\tilde{B}^{(\tau)})}$ and ${l_\infty}^{(\tau)} = (l_\infty(p))_{(\tilde{B}^{(\tau)})}$.

Our introduced spaces generalize the known sequence space as follows:
1. For $\tau = 0$ and $p = (p_k) = e, q = (q_k) = e$, classes (a), (b), (c) reduce to the sequence spaces $\check{c}$, $\check{c}_0$, $\check{l}_\infty$ studied by Basar and Braha [8].
2. For $e_{nk} = I$, classes (a), (b), (c) reduce to the sequence spaces $r_0^t(p, \Delta^{(\tau)})$, $r_c^t(p, \Delta^{(\tau)})$ and $r_\infty^t(p, \Delta^{(\tau)})$ studied by Yaying [28].
3. For $r_{nk} = e_{nk} = I$, classes (a), (b), (c) reduce to the sequence spaces studied by [22].
Now with $\tilde{B}^{(\tau)}$ - transform of $x = (x_k)$ we define the sequence $y = (y_k)$ as follows :

$$(2.3)\ y_n = \left(\tilde{B}^{(\tau)}x\right)_n = \sum_{l=0}^{n}\sum_{j=k}^{n}\sum_{i=j}^{n}\binom{n}{i}\frac{\Gamma(\tau+1)(-1)^{j-k}}{(j-k)!\Gamma(\tau-j+k+1)}\ \frac{q_j x_l}{2^n Q_i}.$$

By a straightforward calculation of (2.3) it can be obtained that

$$(2.4)\, x_n = \left(\tilde{B}^{(-\tau)}y\right)_n = \sum_{l=0}^{n}\sum_{j=k}^{n}\frac{\Gamma(-\tau+1)(-1)^{n-k}}{(n-j)!\Gamma(-\tau-n+j+1)}\frac{2^k}{q_j}\sum_{i=j-1}^{j}\binom{i}{k}Q_i y_l.$$

Lemma 2.1. *The operator $\tilde{B}^{(\tau)}$ is linear.*

Proof. The proof is a routine verification, hence omitted. □

3. Topological structure

This section deals with some interesting topological results of the spaces ${c_0}^{(\tau)}$, $c^{(\tau)}$ and ${l_\infty}^{(\tau)}$.

Theorem 3.1. *The spaces* ${c_0}^{(\tau)}$, $c^{(\tau)}$ *and* ${l_\infty}^{(\tau)}$ *are paranormed spaces with the paranorm*

$$g_{\tilde{B}^{(\tau)}}(x) = \sup_{k\in N}\left|\left(\left(\tilde{B}^{(\tau)}\right)x\right)_k\right|^{\frac{p_k}{M}}$$

$$= \sup_n \left|\sum_{l=0}^{n}\sum_{j=k}^{n}\sum_{i=j}^{n}\binom{n}{i}\frac{\Gamma(\tau+1)(-1)^{j-k}}{(j-k)!\Gamma(\tau-j+k+1)}\frac{q_j x_l}{2^n Q_i}\right|^{\frac{p_k}{M}}, \tag{3.1}$$

if and only if $h = \inf_k p_k > 0$ *and* $M = max\{1, \sup_k p_k\}$.

Proof. Consider the space ${c_0}^{(\tau)}$.

Assume that $h > 0$, then $g_{\tilde{B}^{(\tau)}}(\theta) = 0$, where $\theta = (0, 0, 0, ...)$ and $g_{\tilde{B}^{(\tau)}}(-x) = g_{\tilde{B}^{(\tau)}}(x)$.
To prove the linearity of $g_{\tilde{B}^{(\tau)}}(x)$, we consider two sequences $x = (x_k), y = (y_k) \in c_0^{(\tau)}$ and any two scalars $\beta_1, \beta_2 \in \mathbf{R}$. Since $\tilde{B}^{(\tau)}$ is a linear operator consider

$$g_{\tilde{B}^{(\tau)}}(\beta_1 x + \beta_2 y)$$

$$= \sup_n \left|\sum_{l=0}^{n}\sum_{j=k}^{n}\sum_{i=j}^{n}\binom{n}{i}\frac{\Gamma(\tau+1)(-1)^{j-k}}{(j-k)!\Gamma(\tau-j+k+1)}\frac{q_j}{2^n Q_i}(\beta_1 x_l + \beta_2 y_l)\right|^{\frac{p_k}{M}}$$

$$\leq max\{1, |\beta_1|\}\sup_{k\in N}\left|\left(\left(\tilde{B}^{(\tau)}\right)x\right)_k\right|^{\frac{p_k}{M}} + max\{1, |\beta_2|\}\sup_{k\in N}\left|\left(\left(\tilde{B}^{(\tau)}\right)y\right)_k\right|^{\frac{p_k}{M}}$$

$$= max\{1, |\beta_1|\}\, g_{\tilde{B}^{(\tau)}}(x) + max\{1, |\beta_2|\}\, g_{\tilde{B}^{(\tau)}}(y).$$

Hence the subadditivity of $g_{\tilde{B}^{(\tau)}}$ i.e.

$$g_{\tilde{B}^{(\tau)}}(x+y) \leq g_{\tilde{B}^{(\tau)}}(x) + g_{\tilde{B}^{(\tau)}}(y),$$

for all $x, y \in {c_0}^{(\tau)}$.

Now consider $\{u^n\}$ is a sequence of points in ${c_0}^{(\tau)}$ then $g_{\tilde{B}^{(\tau)}}(u^n - u) \to 0$ and (λ_n) is a sequence of scalars such that $\lambda_n \to \lambda$ as $n \to \infty$. By using the subadditivity of $g_{\tilde{B}^{(\tau)}}$, we get

$$g_{\tilde{B}^{(\tau)}}(u^n) \leq g_{\tilde{B}^{(\tau)}}(u) + g_{\tilde{B}^{(\tau)}}(u^n - u).$$

Since $\{g_{\tilde{B}^{(\tau)}}(u^n)\}$ is bounded, we have
$g_{\tilde{B}^{(\tau)}}(\lambda_n u^n - \lambda u)$

$$= \sup_m \left| \sum_{l=0}^{m} \left[\sum_{j=k}^{m} \sum_{i=j}^{m} \binom{m}{i} \frac{\Gamma(\tau+1)(-1)^{j-k}}{(j-k)!\Gamma(\tau-j+k+1)} \frac{q_j}{2^m Q_i} \right] (\lambda_n {u_l}^n + \lambda u_l) \right|^{\frac{p_k}{M}}$$

$$\leq |\lambda_n - \lambda|^{\frac{p_k}{M}} g_{\tilde{B}^{(\tau)}}(u^n) + |\lambda|^{\frac{p_k}{M}} g_{\tilde{B}^{(\tau)}}(u^n - u) \to 0 \; as \; n \to \infty.$$

Hence it shows that the scalar multiplication of $g_{\tilde{B}^{(\tau)}}(x)$ is continuous and $g_{\tilde{B}^{(\tau)}}(x)$ is a paranorm on the space ${c_0}^{(\tau)}$. Proof for other spaces can be done using similar techniques. □

Theorem 3.2. *The sequence space ${c_0}^{(\tau)}$ is a complete linear space paranormed by $g_{\tilde{B}^{(\tau)}}(x)$.*

Proof. Let $\left\{x^k\right\}$ be a Cauchy sequence in the space ${c_0}^{(\tau)}$ where $x^k = \left\{{x_0}^{(k)}, {x_1}^{(k)}, {x_2}^{(k)}, ...\right\}$. By definition of Cauchy sequence, there exists a positive integer $n_0(\epsilon)$ for each $\epsilon > 0$ such that

$$g_{\tilde{B}^{(\tau)}}\left(x^k - x^l\right) < \epsilon, \; for \; k, l \geq n_0(\epsilon).$$

For a fixed integer $m \in \mathbf{N}$, the sequence
$\left\{\left((\tilde{B}^{(\tau)})x^k\right)_m\right\} = \left\{\left((\tilde{B}^{(\tau)})x^1\right)_m, \left((\tilde{B}^{(\tau)})x^2\right)_m, \left((\tilde{B}^{(\tau)})x^3\right)_m, ...\right\}$ is a Cauchy sequence in $\mathbf{R}$. By completeness of $\mathbf{R}$, the sequence $\left((\tilde{B}^{(\tau)})x^k\right)_m$ converges to $\left((\tilde{B}^{(\tau)})x\right)_m$ as $k \to \infty$. For $l \to \infty$, it is clear that

$$\left|\left((\tilde{B}^{(\tau)})x^k\right)_m - \left((\tilde{B}^{(\tau)})x\right)_m\right|^{\frac{p_k}{M}} < \epsilon/2, \text{for all, } \; k \geq n_0(\epsilon). \tag{3.2}$$

Since $\left\{x^k\right\} \in {c_0}^{(\tau)}$, there exists a number $M \in \mathbf{R}$ such that

$$\sup_m \left|\left((\tilde{B}^{(\tau)})x^k\right)_m\right|^{\frac{p_k}{M}} < \epsilon/2. \tag{3.3}$$

From inequality (3.2) and (3.3) we conclude that

$$\begin{aligned}
&\sup_m \left|\left((\tilde{B}^{(\tau)})x\right)_m\right|^{\frac{p_k}{M}} \\
&\leq \sup_m \left|\left((\tilde{B}^{(\tau)})x^k\right)_m - \left((\tilde{B}^{(\tau)})x\right)_m\right|^{\frac{p_k}{M}} + \sup_m \left|\left((\tilde{B}^{(\tau)})x^k\right)_m\right|^{\frac{p_k}{M}} \\
&\leq \epsilon/2 + \epsilon/2 = \epsilon, for\, all \;\; k \geq n_0(\epsilon).
\end{aligned}$$

Hence the theorem. □

Theorem 3.3. *${c_0}^{(\tau)}$, $c^{(\tau)}$ and ${l_\infty}^{(\tau)}$ are linearly isomorphic to $c_0(p)$, $c(p)$, $l_\infty(p)$ where $0 \; < p_k \leq H \; < \infty$, respectively.*

Proof. *Now define a mapping $F : {l_\infty}^{(\tau)} \to l_\infty(p)$ by $x \to y = Fx$. Clearly, F is a linear transformation. It is obvious that $x = \theta$ whenever $Fx = \theta$, and hence F is one-one.*

Let $y = (y_n) \in l_\infty(p)$, define a sequence $x = (x_n)$ in (2.4) as

$$x_n = \sum_{l=0}^{n}\sum_{j=k}^{n} \frac{\Gamma(-\tau+1)(-1)^{n-k}}{(n-j)!\Gamma(-\tau-n+j+1)} \frac{2^k}{q_j} \sum_{i=j-1}^{j} \binom{i}{k} Q_i y_l.$$

Then

$$\begin{aligned}
g_{\tilde{B}^{(\tau)}}(x) &= \sup_n \left|\sum_{l=0}^{n}\sum_{j=k}^{n}\sum_{i=j}^{n} \binom{n}{i} \frac{\Gamma(\tau+1)(-1)^{j-k}}{(j-k)!\Gamma(\tau-j+k+1)} \frac{q_j x_l}{2^n Q_i}\right|^{\frac{p_k}{M}} \\
&= \sup_{n\in N} \left|\sum_{j=0}^{n} \delta_{nj}\; y_j\right|^{\frac{p_k}{M}} = \sup_{n\in N} |y_n|^{\frac{p_k}{M}} < \infty,
\end{aligned}$$

$$where\; \delta_{nj} = \begin{cases} 1, & if\, n = j \\ 0, & if\, n \neq j. \end{cases}$$

Thus $x \in {l_\infty}^{(\tau)}$ and F is a linear bijection and paranorm preserving. Hence the spaces ${l_\infty}^{(\tau)}$ and $l_\infty(p)$ are linearly isomorphic.
i.e. ${l_\infty}^{(\tau)} \cong l_\infty(p)$. The proof for other spaces can be obtained in a similar manner. □

4. Basis for the spaces

In this section the Schauder basis [11] for ${c_0}^{(\tau)}$, $c^{(\tau)}$ are constructed.

Theorem 4.1. *For* $0 < p_k \leq H < \infty$, *let* $\mu_k(q) = \left(\left(\tilde{B}^{(\tau)}\right)x\right)_k$. *For* $k \in N_0$ *define* $b^{(k)}(q) = \left\{{b_n}^{(k)}(q)\right\}_{n \in N_0}$ *by*

$$\left\{{b_n}^{(k)}(q)\right\} = \begin{cases} \sum_{j=k}^{n} \frac{\Gamma(-\tau+1)(-1)^{n-k}}{(n-j)!\Gamma(-\tau-n+j+1)} \frac{2^k}{q_j} \sum_{i=j-1}^{j} \binom{i}{k} Q_i, & if\, 0 \leq k < n \\ \frac{2^n Q_n}{q_n}, & if\, k = n \\ 0, & if\, k > n. \end{cases}$$

(i) $\left\{{b_n}^{(k)}(q)\right\}$ *is a basis for* ${c_0}^{(\tau)}$ *and each* $x \in {c_0}^{(\tau)}$ *and* x *has unique representation*

$$x = \sum_k \mu_k(q){b_n}^{(k)}(q).$$

(ii) $\left\{\left(\tilde{B}^{(-\tau)}\right)e, {b_n}^{(k)}(q)\right\}$ *is a basis for* $c^{(\tau)}$, *and each* $x \in c^{(\tau)}$ *and* x *has unique representation*

$$x = le + \sum_k (\mu_k - l)\, b^{(k)}, \quad \text{where } l = \lim_{k\to\infty} \mu_k.$$

Proof. (*i*) By the definition of $\left(\tilde{B}^{(\tau)}\right)$ and ${b_n}^{(k)}(q)$,

$$\tilde{B}^{(\tau)}{b_n}^{(k)}(q) = e^{(k)} \in c_0,$$

Let $x \in {c_0}^{(\tau)}$, then

$$x^{[s]} = \sum_{k=0}^{s} \mu_k(q)b^{(k)}(q)$$

for an integer $s \geq 0$.

By applying $\tilde{B}^{(\tau)}$ we get

$$\tilde{B}^{(\tau)}x^{[s]} = \sum_{k=0}^{s} \mu_k(q)\tilde{B}^{(\tau)}b^{(k)}(q)$$

$$= \sum_{k=0}^{s} \mu_k(q)e^{(k)} = \left(\left(\tilde{B}^{(\tau)}\right)x\right)_k e^{(k)}$$

and

$$\tilde{B}^{(\tau)}\left(x - x^{[s]}\right)_r = \begin{cases} 0, & if\, 0 \leq r \leq s \\ \left(\left(\tilde{B}^{(\tau)}\right)x\right)_k, & if\, r > s; \end{cases} \tag{4.1}$$

where $r, s \in \mathbf{N}_0$. For $\epsilon > 0$ there exist an integer m_0 s.t.

$$\sup_{r \geq s}\left|\left(\left(\tilde{B}^{(\tau)}\right)x\right)_r\right|^{\frac{p_k}{M}} < \frac{\epsilon}{2} \; for\ all\ s \geq m_0.$$

Hence

$$g_{\tilde{B}}\left(x - x^{[s]}\right) = \sup_{r \geq s}\left|\left(\left(\tilde{B}^{(\tau)}\right)x\right)_r\right|^{\frac{p_k}{M}} < \frac{\epsilon}{2} < \epsilon, for\ all\ s \geq m_0.$$

Assume that $x = \sum_k \eta_k(q) b^{(k)}(q)$. Since the linear mapping F from ${c_0}^{(\tau)}$ to $c_0\left(p\right)$ is continuous we have

$$\left(\left(\tilde{B}^{(\tau)}\right)x\right)_k = \sum_k \eta_k(q)\left(\left(\tilde{B}^{(\tau)}\right)b^{(k)}(q)\right)_n$$

$$= \sum_k \eta_k(q) e^{(k)} = \eta_n(q).$$

This contradicts to our assumption that $\left(\left(\tilde{B}^{(\tau)}\right)x\right)_k = \mu_k(q)$ for each $k \in \mathbf{N}_0$. Thus the representation is unique.
(ii) The proof as it is similar to previous one. □

5. $\alpha-, \beta-$ and $\gamma-$ duals

Here we determine $\alpha-, \beta-$ and $\gamma-$ duals of ${c_0}^{(\tau)}$, $c^{(\tau)}$ and ${l_\infty}^{(\tau)}$. Throughout the collection of all finite subsets of $\mathbf{N}$ is denoted by κ. We consider $K \in \kappa$.

Lemma 5.1. *[14] Let $A = (a_{nk})$ be an infinite matrix. Then,*

1. *$A \in (l_\infty(p), l(q))$ if and only if*

$$\sup_{k \in \kappa} \sum_n \left|\sum_{k \in K} a_{nk} B^{\frac{1}{p_k}}\right|^{q_n} < \infty, \; for\ all\ integers,\ B > 1 and\, q_n \geq 1\ for\ all,\ n; \tag{5.1}$$

2. $A \in (l_\infty(p), l_\infty(q))$ *if and only if*

$$(5.2)\quad \sup_{n\in N}\left(\sum_k |a_{nk}|\, B^{\frac{1}{p_k}}\right)^{q_n} < \infty, \qquad \text{for all integers}, B > 1;$$

3. $A \in (l_\infty(p), c(q))$ *and* $q = (q_n)$ *be a bounded sequence of strictly positive real numbers if and only if*

$$(5.3)\qquad \sup_{n\in N}\sum_k |a_{nk}|\, B^{\frac{1}{p_k}} < \infty, \qquad \text{for all}, B > 1,$$

there exists $(\tau_k) \subset \mathbf{R}$ *such that* $\lim_{n\to\infty}\left(\sum_k |a_{nk} - \tau_k|\, B^{\frac{1}{p_k}}\right)^{q_n} = 0,$ *for all*, $B > 1$;

4. $A \in (l_\infty(p), c_0(q))$ *if and only if*

$$\lim_{n\to\infty}\left(\sum_k |a_{nk}|\, B^{\frac{1}{p_k}}\right)^{q_n} = 0, \qquad \text{for all}, B > 1,$$

.

Lemma 5.2. *[14] Let $A = (a_{nk})$ be an infinite matrix. Then,*

1. $A \in (c_0(p), l_\infty(q))$ *if and only if*

$$(5.4)\qquad \sup_{n\in N}\left(\sum_k |a_{nk}|\, B^{\frac{-1}{p_k}}\right)^{q_n} < \infty, \qquad \text{for all}, B > 1;$$

2. $A \in (c_0(p), c(q))$ *if and only if*

$$(5.5)\qquad \sup_{n\in N}\sum_k |a_{nk}|\, B^{\frac{-1}{p_k}} < \infty, \qquad \text{for all}, B > 1,$$

$$(5.6)\text{there exists } (\tau_k) \subset R \text{ such that } \sup_{n\in N}\sum_k |a_{nk} - \tau_k|\, M^{\frac{-1}{p_k}} B^{\frac{-1}{p_k}} < \infty,$$

for all integers $M, B > 1$;

$$\text{there exists } (\tau_k) \subset R \text{ such that } \lim_{n\to\infty}\sum_k |a_{nk} - \tau_k|^{q_n} = 0, \text{for all}, k \in N;$$
(5.7)

3. $A \in (c_0(p), c_0(q))$ *if and only if*

$$\text{(5.8)} \qquad there\, exists\, (\tau_k) \subset R,\ such\ that\ \sup_{n\in N} \sum_k |a_{nk}|\, M^{\frac{-1}{p_k}} B^{\frac{-1}{p_k}} < \infty,$$

for all integers $M, B > 1$;

$$\text{(5.9)}\ there\, exists\, (\tau_k) \subset R\ \ such\ that\ \lim_{n\to\infty} \sum_k |a_{nk}|^{q_n} = 0, for\ all, k \in N.$$

Lemma 5.3. *[14] Let $A = (a_{nk})$ be an infinite matrix. Then,*

1. $A \in (c(p), l_\infty(q))$ *if and only if equation (5.4) holds and*

$$\text{(5.10)} \qquad \sup_{n\in N} \left|\sum_k a_{nk}\right|^{q_n} < \infty;$$

2. $A \in (c(p), c(q))$ *if and only if equations (5.5), (5.6), (5.7) hold:*

$$\text{(5.11)}\ there\, exists\, (\tau) \subset R\ \ such\ that\ \lim_{n\to\infty} \left|\sum_k a_{nk} - \tau\right|^{q_n} = 0;$$

3. $A \in (c(p), c_0(q))$ *if and only if equations (5.8), (5.9) hold and*

$$\text{(5.12)} \qquad \lim_{n\to\infty} \left|\sum_k a_{nk}\right|^{q_n} = 0.$$

Theorem 5.1. *The $\alpha-, \beta-$ and $\gamma-$ duals of ${c_0}^{(\tau)}$, $c^{(\tau)}$ and ${l_\infty}^{(\tau)}$ are the following defined sets*

$$D_1^{(\tau)}(p) = \bigcap_{M>1} \left\{ a = (a_k) : \sup_{k\in\tau} \sum_n \left| \sum_{k\in K} \left[\tilde{B}^{(\tau)}\left(\frac{a_k}{q_k}\right) Q_k \right] \right| M^{\frac{1}{p_k}} < \infty \right\},$$

$$D_2^{(\tau)}(p) = \bigcap_{M>1} \left\{ a = (a_k) : \sum_k \left| \tilde{B}^{(\tau)}\left(\frac{a_k}{q_k}\right) Q_k \right| M^{\frac{1}{p_k}} < \infty, \left(\frac{a_k}{q_k} Q_k M^{\frac{1}{p_k}}\right) \in c_0 \right\},$$

$$D_3^{(\tau)}(p) = \bigcap_{M>1} \left\{ a = (a_k) : \sum_k \left| \tilde{B}^{(\tau)}\left(\frac{a_k}{q_k}\right) Q_k \right| M^{\frac{1}{p_k}} < \infty, \left\{ \tilde{B}^{(\tau)}\left(\frac{a_k}{q_k}\right) Q_k \right\} \in l_\infty \right\},$$

$$D_4^{(\tau)}(p) = \bigcup_{M>1} \left\{ a = (a_k) : \sup_{k\in\tau} \sum_n \left| \sum_{k\in K} \left[\tilde{B}^{(\tau)} \left(\frac{a_k}{q_k} \right) Q_k \right] \right| M^{\frac{-1}{p_k}} < \infty \right\},$$

$$D_5^{(\tau)}(p) = \bigcup_{M>1} \left\{ a = (a_k) : \sum_n \left| \sum_k \left[\tilde{B}^{(\tau)} \left(\frac{a_k}{q_k} \right) Q_k \right] \right| < \infty \right\},$$

$$D_6^{(\tau)}(p) = \bigcap_{M>1} \left\{ a = (a_k) : \sum_k \left| \tilde{B}^{(\tau)} \left(\frac{a_k}{q_k} \right) Q_k \right| M^{\frac{-1}{p_k}} < \infty \right\},$$

where

$$\tilde{B}^{(\tau)} \left(\frac{a_k}{q_k} \right) = \sum_{j=k}^{n} \frac{\Gamma(-\tau+1)(-1)^{n-k}}{(n-j)!\Gamma(-\tau-n+j+1)} \frac{2^k}{q_j} \sum_{i=j-1}^{j} \binom{i}{k} a_k \tag{5.13}$$

Then

$$\left\{ l_\infty{}^{(\tau)} \right\}^\alpha = D_1^{(\tau)}(p),\ \left\{ l_\infty{}^{(\tau)} \right\}^\beta = D_2^{(\tau)}(p),\ \left\{ l_\infty{}^{(\tau)} \right\}^\gamma = D_3^{(\tau)}(p),$$

$$\left\{ c^{(\tau)} \right\}^\alpha = D_4^{(\tau)}(p) \cap D_5^{(\tau)}(p),\ \left\{ c^{(\tau)} \right\}^\beta = D_6^{(\tau)}(p) \cap cs,\ \left\{ c^{(\tau)} \right\}^\gamma = D_6^{(\tau)}(p) \cap bs,$$

$$\left\{ c_0{}^{(\tau)} \right\}^\alpha = \left\{ c_0{}^{(\tau)} \right\}^\beta = \left\{ c_0{}^{(\tau)} \right\}^\gamma = D_6^{(\tau)}(p).$$

Proof. Consider the space $l_\infty{}^{(\tau)}$. Now let $x = (x_k)$ as in (2.4), and $a = (a_k) \in \omega$, define

$$a_n x_n = \sum_{l=0}^{n} \sum_{j=k}^{n} \frac{\Gamma(-\tau+1)(-1)^{n-k}}{(n-j)!\Gamma(-\tau-n+j+1)} \frac{2^k}{q_j} \sum_{i=j-1}^{j} \binom{i}{k} Q_i a_n y_l$$

$$= (Uy)_n, \qquad for\, n \in N,$$

where matrix $U = (u_{nk})$ is defined as

$$u_{nk} = \begin{cases} \sum_{j=k}^{n} \frac{\Gamma(-\tau+1)(-1)^{n-k}}{(n-j)!\Gamma(-\tau-n+j+1)} \frac{2^k}{q_j} \sum_{i=j-1}^{j} \binom{i}{k} Q_i a_n, & if\, 0 \le k < n \\ \frac{2^n Q_n}{q_n} a_n, & if\, k = n \\ 0, & if\, k > n. \end{cases}$$

Therefore we conclude that $ax = (a_n x_n) \in l_1$ whenever $x = (x_k) \in l_\infty{}^{(\tau)}$ if and only if $Uy \in l_1$ as $y = (y_k) \in l_\infty(p)$. By lemma (5.1) we conclude that $\left\{ l_\infty{}^{(\tau)} \right\}^\alpha = D_1^{(\tau)}(p)$.

Now

$$\sum_{k=0}^{n} a_k x_k = \sum_{k=0}^{n} a_k \left[\sum_{l=0}^{n} \sum_{j=k}^{n} \frac{\Gamma(-\tau+1)(-1)^{n-k}}{(n-j)!\Gamma(-\tau-n+j+1)} \frac{2^k}{q_j} \sum_{i=j-1}^{j} \binom{i}{k} Q_i y_l \right]$$

$$= \sum_{k=0}^{n} y_k Q_k \tilde{B}^{(\tau)} \left(\frac{a_k}{q_k} \right) = (Vy)_n ,$$

where matrix $V = (v_{nk})$ is defined as

$$v_{nk} = \begin{cases} \tilde{B}^{(\tau)}\left(\frac{a_k}{q_k}\right) Q_k, & if\, 0 \leq k \leq n \\ \frac{2^n Q_n}{q_n} a_n, & if\, k = n \\ 0, & if\, k > n. \end{cases} \tag{5.14}$$

Therefore we deduce that $ax = (a_n x_n) \in cs$ whenever $x = (x_k) \in l_\infty{}^{(\tau)}$ if and only if $Vy \in c$ whenever $y = (y_k) \in l_\infty(p)$. By using lemma (5.1) with $q = q_n = 1$ we conclude that $\left\{ l_\infty{}^{(\tau)} \right\}^\beta = D_2^{(\tau)}(p)$. Similarly by using lemma (5.1) with $q = q_n = 1$ for all n, we conclude that $\left\{ l_\infty{}^{(\tau)} \right\}^\gamma = D_3^{(\tau)}(p)$. Hence the theorem proved and the duals of other spaces can be obtained in a similar manner using lemma (5.2) and lemma (5.3). □

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