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Sustainability of a system of two competing prey and a predator in polluted environment

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Abstract

In this study, a general model of interacting species consisting of two competing prey and a predator under the presence of pollution is formed. Criteria for the existence of equilibria and their (local and global) stability are derived. The conditions for persistence and bifurcation have also been derived. With the help of numerical simulation, it is shown how the change in the pollution level results in species extinction.

Keywords: *Competition, Pollution, Equilibria, Stability, Bifurcation, Persistence.*

AMS: *92D25 and 93A30.*

1. Introduction

The species in the ecosystem are interdependent for their survival. They interact with each other for different reasons, like food and shelter. The type of interaction depends upon their biological needs and surrounding conditions. The interacting species in a particular ecosystem form food chains or food webs. The last several decades have seen tremendous growth in industrialisation and urbanisation. The extensive use of non-biodegradable products has caused a damaging impact on our environment. The excess exploitation of nature has severely disturbed the ecological balance. It has forced the species, including humans, to be exposed to anthropogenic substances through different sources, including air, water and food. All these have resulted in the extinction of many species from the earth. The international union for the conservation of nature published a list of 160 species going extinct from 2010-2019 [**15**]. Keeping environment pollution free is one of the most challenging problems today. The pollutant moving upward through food chains becomes more hazardous for the species at the higher trophic level. For example, the DDT from prey breaks down into DDE in predators [**17**].

Mathematical models can help analyse the present situation and predict the future so that anyone can take necessary measures to control pollution levels. Some of the mathematical models formed to analyse the dynamics of two or more species are of the following types:

Freedman and Waltman [**7**] studied the system of two predators competing with each other, feeding on a single prey and a single predator feeding on two competing prey species.

Kar and Batabyal [**13**] proposed a mathematical model to analyse the dynamics of a system having two prey and one predator in the presence of a time delay due to gestation.

Ali and Chakravarthy [**2**] analysed a model with the intra-specific competition among predator populations consists of two competing prey and one predator.

Daga et. al. [**6**] proposed a predator-prey model with Holling type III functional response and analysed the dynamics of a two prey, one predator system.

The problem of approximating the effect of a toxicant on a population by mathematical models began in the early 1980s. **Ma and Hallam [16]** and **Hauping and Ma [9]** obtained a survival threshold for single and two species, respectively, under the effect of pollution. **Chauhan and Mishra [3]** studied a single species model under the combined impact of toxicant and infection. **Sinha et.al. [18]** studied the predator-prey model under the influence of toxicants. **Chauhan et. al. [4]** studied the predatorprey model under the effect of infection for prey and predator species, respectively. **Huang et.al. [10]** formulated a toxin-dependent aquatic population model and connected the model to the experimental data via model parameterisation. **Huang et al. [11]** developed a toxin-dependent Predator-prey model for two species and discussed the effect of pollution on species. **Lawaniya et.al. [14]** developed a generalised model for two species under the effect of pollution and obtained conditions regarding stability and persistence.

In this study dynamics of three species consisting of two competing prey and one predator is carried out under the effect of pollution. In next three sections the model has been formulated and has been shown to be viable. Criteria for the existence of equilibria are derived in section 5, and criteria for local and global stability have been carried out in section 6. The conditions for persistence and bifurcation have been discussed in sections 7 and 8, respectively. In section 9, the results regarding the existence of equilibria and bifurcation have been validated through the numerical examples In section 10, the results are discussed.

2. Model formation

We have Gause type model for three species consisting of two competing prey species and one predator of the form;

$$
\begin{aligned}
&x_1' = x_1 \tau_1 g_1(x_1) - q_1 x_1 x_2 - y p_1(x_1)\\
&x_2' = x_2 \tau_2 g_2(x_2) - q_2 x_1 x_2 - y p_2(x_2)\\
&y' = y(-s(y) + c_1 p_1(x_1) + c_2 p_2(x_2))
\end{aligned}
$$

where x_i for $i = 1, 2$ are density of prey species population, y denotes the predator population, $s(y)$ is the death rate of the predator, $g_i(x_i)$ for $i = 1, 2$ are growth rates of prey species, τ_i for $i = 1, 2$ are growth rate coefficients, $p_i(x_i)$ for $i = 1, 2$ are predation functions, constants q_i for $i = 1, 2$ are interspecies competition coefficients and c_i are the coefficients

for conversion of prey biomass into predator biomass.
Under the effect of environmental pollution the model is of the form;

$$\left.\begin{array}{rcl} x_1' &=& x_1\tau_1 g_1(x_1) - q_1 x_1 x_2 - y p_1(x_1) - r_1 O x_1 \\ x_2' &=& x_2\tau_2 g_2(x_2) - q_2 x_1 x_2 - y p_2(x_2) - r_2 O x_2 \\ y' &=& y(-s(y) + c_1 p_1(x_1) + c_2 p_2(x_2) - r_3 O) \\ E' &=& -Eh(E) + Q_E \\ O' &=& -d_1 O + p(E) \end{array}\right\} \tag{2.1}$$

where $E(t)$ is the concentration of toxicant in environment, $O(t)$ is the concentration of the toxicant in the organism, r_i for $i = 1, 2, 3$ are the rates of loss of biomass for three species x_1, x_2, y respectively due to environmental pollution, $h(E)$ is the loss rate function of environmental pollution, Q_E is the input rate of pollutant in the environment, d_1 is the coefficient of depuration of organismal pollution and $p(E)$ denotes the conversion function of environmental pollution into organismal pollution.

Assumptions: All the functions $g_i, p_i, s, h(E), p(E)$ are smooth enough so that the solutions of the system exist, are unique and continuous for all $t > 0$ **Ahmad and Rao [1]** and $Q_E > 0, d_1 > 0$.

G1) Growth rate functions $\mathbf{g_i(x_i)}$: It is assumed that $g_i(0) > 0$, $\frac{dg_i}{dx_i} < 0$ and $\exists$ a $K_i > 0$ for which $g_i(K_i) = 0$ and $0 < \tau_i$ for $i = 1, 2$.

P1) Predation function $\mathbf{p_i(x_i)}$: It is assumed that $p_i(0) = 0$, $\frac{dp_i}{dx_i} > 0$ for $i = 1, 2$.

S1) Death rate function $\mathbf{s(y)}$: It is assumed that $s(0) > 0$, and $s'(y) > 0$. These conditions interpreted that the death rate always remains positive and is density dependent.

Q1) Competition coefficients $\mathbf{q_i}$: The coefficients q_1, q_2 represent competition between the prey species x_1, x_2 respectively, $q_i > 0$ for $i = 1, 2$.

H1) Environmental Pollution loss rate: $h(0) > 0$ and $h'(E) \geq 0$.

P2) Conversion function from environmental pollution E to organismal pollution O: $p(0) > 0$ and $p'(E) \geq 0$.

As $\lim\limits_{E\to\infty} Eh(E) > Q_E$, $\exists$ E^* such that $-E^*h(E^*) + Q_E = 0$.

3. Invariant Region [14]

The solutions of (2.1) with non-negative initial conditions stay non-negative for all time $t > 0$ and the region,
$S = \{(x_1, x_2, y, E, O) \mid 0 \le x_1 \le \hat{x}_1+\epsilon, 0 \le x_2 \le \hat{x}_2+\epsilon, 0 \le c_1x_1+c_2x_2+y \le \frac{T}{s(0)} + \epsilon, 0 \le E \le \frac{Q_E}{h(0)} + \epsilon, 0 \le O \le O^* + \epsilon\}$
is a positively invariant and attracting region for system (2.1)
where $\epsilon > 0, \hat{x}_i=$ carrying capacity of i^{th} prey under the effect of pollution, $O^* = \frac{p(E^*)}{d_1}$ and $T = c_1(\hat{x}_1 + \epsilon)(g_1(0) + s(0)) + c_2(\hat{x}_2 + \epsilon)(g_2(0) + s(0))$.

4. Possible Equilibria

There are seven possible equilibria for the system

(i) $E_0(0, 0, 0, E^*, O^*)$

(ii) $E_1(\hat{x}_1, 0, 0, E^*, O^*)$

(iii) $E_2(0, \hat{x}_2, 0, E^*, O^*)$

(iv) $E_3(l_1, l_2, 0, E^*, O^*)$

(v) $E_4(x_4, 0, y_4, E^*, O^*)$

(vi) $E_5(0, x_5, y_5, E^*, O^*)$

(vii) $E_6(x_1^*, x_2^*, y^*, E^*, O^*)$

5. Existence of Equilibria

The equilibrium $E_0(0, 0, 0, E^*, O^*)$ always exists and equilibria $E_1(\hat{x}_1, 0, 0, E^*, O^*)$ exists if $\tau_1 g_1(0) - r_1 O^* > 0$ and $E_2(0, \hat{x}_2, 0, E^*, O^*)$ exists if $\tau_2 g_2(0) - r_2 O^* > 0$.

Further throughout our analysis we assume that E_1 and E_2 exist. The conditions for existence of other equilibria are as follows;

(i) $E_3(l_1, l_2, 0, E^*, O^*)$ exists if

$$\left.\begin{array}{llll} & \hat{x}_1 < \frac{(\tau_2 g_2(0) - r_2 O^*)}{q_2} & \text{and} & \hat{x}_2 < \frac{(\tau_1 g_1(0) - r_1 O^*)}{q_1} \\ \text{or} & \hat{x}_1 > \frac{(\tau_2 g_2(0) - r_2 O^*)}{q_2} & \text{and} & \hat{x}_2 > \frac{(\tau_1 g_1(0) - r_1 O^*)}{q_1} \end{array}\right\} \tag{5.1}$$

(ii) $E_4(x_4, 0, y_4, E^*, O^*)$ exists if $0 < x_6 < \hat{x}_1$.

(iii) $E_5(0, x_5, y_5, E^*, O^*)$ exists if $0 < x_7 < \hat{x}_2$ such that $s_2(0) + c_2p_2(x_7) - r_3O^* = 0$.

(iv) $E_6(x_1^*, x_2^*, y^*, E^*, O^*)$ exists if our system is uniformly persistent. Conditions for uniform persistence are given in section 7.

6. Stability

6.1. Local Stability

With the help of jacobian matrix we determine the local stability of the system corresponding to equilibria **Ahmad and Rao** [1].
The jacobian matrix corresponding to the system (2.1) is

$$J = \begin{bmatrix} j_{11} & -q_1x_1 & -p_1(x_1) & 0 & -r_1x_1 \\ -q_2x_2 & j_{22} & -p_2(x_2) & 0 & -r_2x_2 \\ c_1yp_1'(x_1) & c_2yp_2'(x_2) & j_{33} & 0 & -r_3y \\ 0 & 0 & 0 & -Eh'(E) - h(E) & 0 \\ 0 & 0 & 0 & p'(E) & -d_1 \end{bmatrix}$$

where, $j_{11} = x_1\tau_1g_1'(x_1) + \tau_1g_1(x_1) - yp_1'(x_1) - q_1x_2 - r_1O$, $j_{22} = x_2\tau_2g_2'(x_2) + \tau_2g_2(x_2) - yp_2'(x_2) - q_2x_1 - r_2O$ and $j_{33} = -s(y) - ys'(y) + c_1p_1(x_1) + c_2p_2(x_2) - r_3O$.

(i) The equilibrium E_0 is unstable.

(ii) $E_1(\hat{x}_1, 0, 0, E^*, O^*)$ is stable if $(\tau_2g_2(0) - \hat{x}_1q_2 - r_2O^*) < 0$ and $(-s(0) + c_1p_1(\hat{x}_1) - r_3O^*) < 0$.

(iii) $E_2(0, \hat{x}_2, 0, E^*, O^*)$ is stable if $(\tau_1g_1(0) - \hat{x}_2q_1 - r_1O^*) < 0$ and $(-s(0) + c_2p_2(\hat{x}_2) - r_3O^*) < 0$.

(iv) $E_3(l_1, l_2, 0, E^*, O^*)$ is stable if $\tau_1\tau_2g_1'(l_1)g_2'(l_2) - q_1q_2 > 0$ and $-s(0) + c_1p_1(l_1) + c_2p_2(l_2) - r_3O^* < 0$.

(v) $E_4(x_4, 0, y_4, E^*, O^*)$ is stable if $j_{22}(4) < 0, j_{11}(4) + j_{33}(4) < 0$ and $j_{11}(4).j_{33}(4) + p_1(x_4)(c_1y_4p_1'(x_4)) > 0$,
where $j_{22}(4) = \tau_2g_2(0) - y_4p_2'(0) - q_2x_4 - r_2O^*, j_{11}(4) = x_4\tau_1g_1'(x_4) + \tau_1g_1(x_4) - y_4p_1'(x_4) - r_1O^*$ and $j_{33}(4) = -y_4s'(y_4)$.

(vi) E_5 is stable if $j_{11}(5) < 0, j_{22}(5) + j_{33}(5) < 0$ and $j_{22}(5).j_{33}(5) + p_2(x_5).c_2y_5p_2'(x_5) > 0$,
where, $j_{11}(5) = \tau_1 g_1(0) - q_1x_5 - r_1O^*, j_{22}(5) = x_5\tau_2 g_2'(x_5) + \tau_2 g_2(x_5) - y_5p_2'(x_5) - r_2O^*$ and $j_{33}(5) = -y_5s'(y_5)$.

Theorem 6.1.1. *The equilibrium $E_6(x_1^*, x_2^*, y^*, E^*, O^*)$ is asymptotically stable in $R^+_{(x_1,x_2,y,E,O)}$ if $a_{11}, a_{22} < 0, a_{12}a_{21} < min\{a_{11}a_{22}, -a_{13}a_{31}, -a_{23}a_{32}\}$, $a_{21}a_{32} > max\{a_{31}a_{22}, -a_{12}a_{23}\}$ and $a_{11}a_{32} - a_{12}a_{31} < 0$, where $a_{11} = x_1^*\tau_1 g_1'(x_1^*) + \tau_1 g_1(x_1^*) - y^*p_1'(x_1^*) - q_1x_2^* - r_1O^*, a_{12} = -q_1x_1^*, a_{13} = -p_1(x_1^*), a_{21} = -q_2x_2^*, a_{22} = x_2^*\tau_2 g_2'(x_2^*) + \tau_2 g_2(x_2^*) - y^*p_2'(x_2^*) - q_2x_1^* - r_2O^*, a_{23} = -p_2(x_2^*), a_{31} = c_1y^*p_1'(x_1^*), a_{32} = c_2y^*p_2'(x_2^*)$ and $a_{33} = -y^*s'(y^*)$.*

Proof: The jacobian corresponding to E_6 is given by:

$$J(6) = \begin{bmatrix} j_{11}(6) & -q_1x_1^* & -p_1(x_1^*) & 0 & -r_1x_1^* \\ -q_2x_2^* & j_{22}(6) & -p_2(x_2^*) & 0 & -r_2x_2^* \\ c_1y^*p_1'(x_1^*) & c_2y^*p_2'(x_2^*) & -y^*s'(y^*) & 0 & -r_3y^* \\ 0 & 0 & 0 & -E^*h'(E^*) - h(E^*) & 0 \\ 0 & 0 & 0 & p'(E^*) & -d_1 \end{bmatrix}$$

where, $j_{11}(6) = x_1^*\tau_1 g_1'(x_1^*) + \tau_1 g_1(x_1^*) - y^*p_1'(x_1^*) - q_1x_2^* - r_1O^*, j_{22}(6) = x_2^*\tau_2 g_2'(x_2^*) + \tau_2 g_2(x*_2) - y^*p_2'(x_2^*) - q_1x_1^* - r_2O^*$.
$j(6)$ can be reduced to $\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$.

In E, O directions the eigen values are negative.
Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$,
where $a_{11} = x_1^*\tau_1 g_1'(x_1^*) + \tau_1 g_1(x_1^*) - y^*p_1'(x_1^*) - q_1x_2^* - r_1O^*, a_{12} = -q_1x_1^*$, $a_{13} = -p_1(x_1^*), a_{21} = -q_2x_2^*, a_{22} = x_2^*\tau_2 g_2'(x_2^*) + \tau_2 g_2(x_2^*) - y^*p_2'(x_2^*) - q_2x_1^* - r_2O^*, a_{23} = -p_2(x_2^*), a_{31} = c_1y^*p_1'(x_1^*), a_{32} = c_2y^*p_2'(x_2^*)$ and $a_{33} = -y^*s'(y^*)$.

The characteristic equation of j_6 is $\lambda^3 + \lambda^2 A_1 + \lambda A_2 + A_3 = 0$,

where $A_1 = -(a_{11} + a_{22} + a_{33}), A_2 = a_{22}a_{33} + a_{11}a_{22} + a_{11}a_{33} - a_{12}a_{21} - a_{23}a_{32} - a_{13}a_{31}$ and $A_3 = a_{12}a_{21}a_{33} + a_{11}a_{23}a_{32} + a_{13}a_{31}a_{22} - a_{11}a_{22}a_{33} - a_{12}a_{31}a_{23} - a_{13}a_{21}a_{32}$.

According to Routh-Hurwitz Criteria **Ahmad and Rao [1]** the equilibrium $E_6(x_1^*, x_2^*, y^*, E^*, O^*)$ is locally asymptotically stable when $A_1 > 0, A_3 > 0$ and $A_1A_2 > A_3$.
Hence $E_6(x_1^*, x_2^*, y^*, E^*, O^*)$ is locally asymptotically stable.

6.2. Global Stability

$E_0(0, 0, 0, E^*, O^*)$ is globally stable in $R^+_{0,0,y,E,O}$.

Theorem 6.2.1. *Let* $\xi_1 = (\tau_1 g_1(0) - \hat{x}_2 q_1 - r_1 O^*)$ *and* $\xi_2 = (\tau_2 g_2(0) - \hat{x}_1 q_2 - r_2 O^*)$*. Then*

Case I: *If* $\xi_1 < 0, \xi_2 < 0$ *then* E_3 *exists and is a saddle in* $R^+_{x_1,x_2,0,E,O}$.

Case II: *If* $\xi_1 > 0, \xi_2 > 0$ *then* E_3 *exists and is globally stable in* $R^+_{x_1,x_2,0,E,O}$.

Case III: *If* $\xi_1 > 0, \xi_2 < 0$ *then* E_1 *is globally stable in* $R^+_{x_1,x_2,0,E,O}$.

Case IV: *If* $\xi_1 < 0, \xi_2 > 0$ *then* E_2 *is globally stable in* $R^+_{x_1,x_2,0,E,O}$.

Proof: Two competing prey, system is:

$$\begin{aligned} x_1' &= x_1\tau_1 g_1(x_1) - q_1 x_1 x_2 - r_1 O^* x_1 \\ x_2' &= x_2\tau_2 g_2(x_2) - q_2 x_1 x_2 - r_2 O^* x_2 \end{aligned}$$

and jacobian matrix is as follows:

$$M(x_1, x_2) = \begin{bmatrix} m_{11} & -q_1 x_1 \\ -q_2 x_2 & m_{22} \end{bmatrix}$$

where $m_{11} = \tau_1 g_1(x_1) - q_1 x_2 - r_1 O^* + x_1 \tau_1 g_1'(x_1)$, $m_{22} = \tau_2 g_2(x_2) - q_2 x_1 - r_2 O^* + x_2 \tau_2 g_2'(x_2)$

$M(\hat{x}_1, 0) = \begin{bmatrix} \hat{x}_1 \tau_1 g_1'(\hat{x}_1) & -q_1 \hat{x}_1 \\ 0 & \tau_2 g_2(0) - q_2 \hat{x}_1 - r_2 O^* \end{bmatrix}$,

$$M(0, \hat{x}_2) = \begin{bmatrix} \tau_1 g_1(0) - q_1 \hat{x}_2 - r_1 O^* & 0 \\ -q_2 \hat{x}_2 & \hat{x}_2 \tau_2 g_2'(\hat{x}_2) \end{bmatrix}$$

and $M(l_1, l_2) = \begin{bmatrix} l_1 \tau_1 g_1'(l_1) & -q_1 l_1 \\ -q_2 l_2 & l_2 \tau_2 g_2'(l_2) \end{bmatrix}$.

Here $tr(M(l_1, l_2)) < 0$ and $det(M(l_1, l_2)) = l_1 l_2 \tau_1 g_1'(l_1)\tau_2 g_2'(l_2) - q_1 l_1 q_2 l_2$.

Now for E_3 we have

$$\tau_1 g_1(x_1) - q_1 x_2 - r_1 O^* = 0 \tag{6.1}$$

$$\tau_2 g_2(x_2) - q_2 x_1 - r_2 O^* = 0 \tag{6.2}$$

When we draw isoclines for these two equations, (6.1) intersects positive x_1-axis at $(\hat{x}_1, 0)$ and positive x_2-axis at $(0, \frac{\tau_1 g_1(0) - r_1 O^*}{q_1})$ while (6.2) intersects positive x_2-axis at $(0, \hat{x}_2)$ and positive x_1-axis at $(\frac{\tau_2 g_2(0) - r_2 O^*}{q_2}, 0)$.

Case I: When $\frac{\tau_2 g_2(0) - r_2 O^*}{q_2} < \hat{x}_1$ and $\frac{\tau_1 g_1(0) - r_1 O^*}{q_1} < \hat{x}_2$ E_1 and E_2 are locally stable hence E_3 cannot be globally stable in place of E_3 is unstable.

We have phase plane as follows:

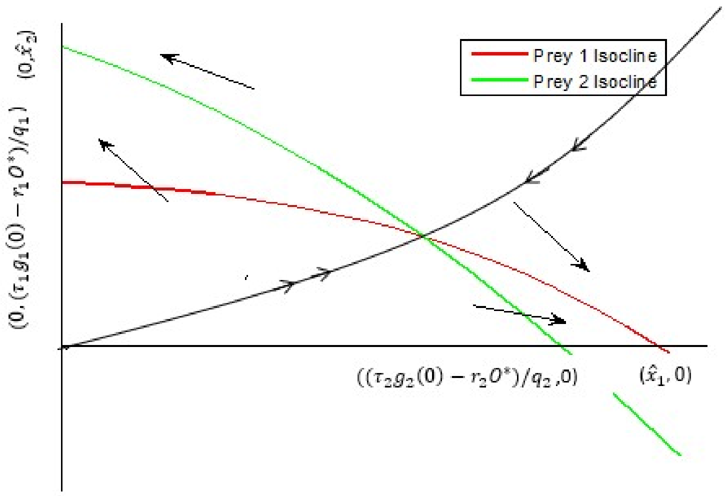

Figure 1: $\frac{\tau_2 g_2(0) - r_2 O^*}{q_2} < \hat{x}_1$ and $\frac{\tau_1 g_1(0) - r_1 O^*}{q_1} < \hat{x}_2, E_3$ (interior equilibrium) is unstable, black curve with arrow is separatrix.

Case II: When $\frac{\tau_2 g_2(0)-r_2 O^*}{q_2} > \hat{x}_1$ and $\frac{\tau_1 g_1(0)-r_1 O^*}{q_1} > \hat{x}_2$ from phase plane analysis we have E_3 is globally stable.

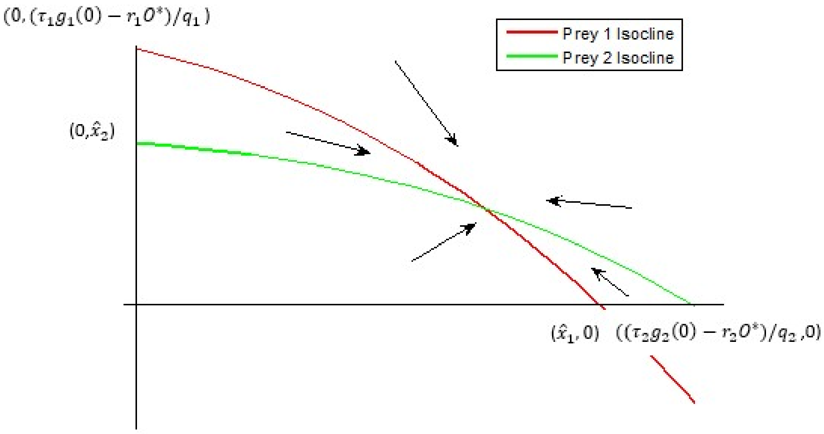

Figure 2: $\frac{\tau_2 g_2(0)-r_2 O^*}{q_2} > \hat{x}_1$ and $\frac{\tau_1 g_1(0)-r_1 O^*}{q_1} > \hat{x}_2, E_3$ (interior equilibrium) globally stable in $R^+_{x_1,x_2,0,E,O}$.

Case III: When $\frac{\tau_2 g_2(0)-r_2 O^*}{q_2} < \hat{x}_1$ and $\frac{\tau_1 g_1(0)-r_1 O^*}{q_1} > \hat{x}_2$

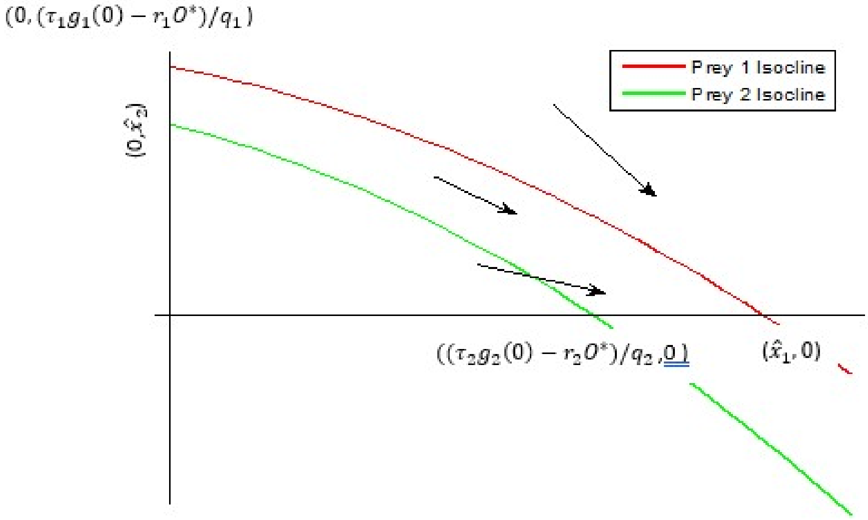

Figure 3: $\frac{\tau_2 g_2(0)-r_2 O^*}{q_2} < \hat{x}_1$ and $\frac{\tau_1 g_1(0)-r_1 O^*}{q_1} > \hat{x}_2, E_1$ is globally stable and E_3 does not exist.

In this case there is no interior equilibrium and we have E_1 globally stable.

Case IV: When $\frac{\tau_2 g_2(0)-r_2O^*}{q_2} > \hat{x}_1$ and $\frac{\tau_1 g_1(0)-r_1O^*}{q_1} < \hat{x}_2$

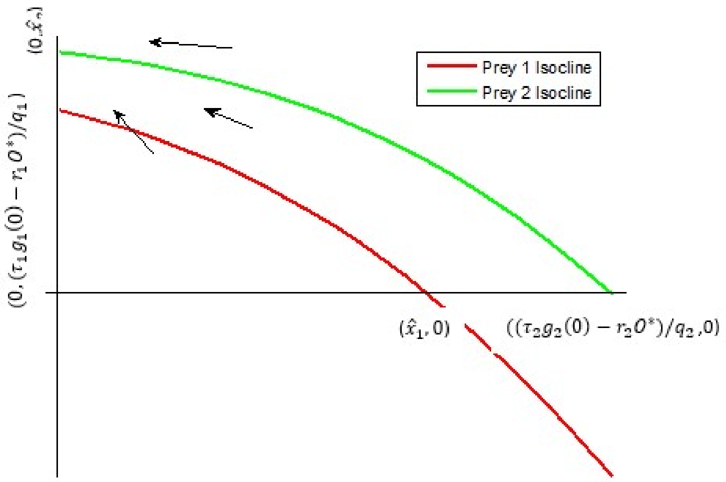

Figure 4: $\frac{\tau_2 g_2(0)-r_2O^*}{q_2} > \hat{x}_1$ and $\frac{\tau_1 g_1(0)-r_1O^*}{q_1} < \hat{x}_2, E_2$ is globally stable

In this case we also get nonexistence of interior equilibrium while E_2 is globally stable.

Theorem 6.2.2. *If E_4 is locally asymptotically stable with*

(i) $c_1p_1'(x_1)x_1 - s'(y)y > 0 \ \forall \ x_1 > 0, y > 0$

(ii) $-s(0) + c_1p_1(\hat{x}_1) - r_3O^* > 0$ *and* $E_1(\hat{x}_1, 0, 0)$ *exists*

then E_4 is globally asymptotically stable in $R^+_{x_1,0,y,E,O}$.

Proof: We have system (1) in absence of x_2

$$\begin{aligned} x_1' &= x_1\tau_1 g_1(x_1) - yp_1(x_1) - r_1O^*x_1 \\ y' &= y(-s(y) + c_1p_1(x_1) - r_3O^*) \end{aligned}$$

Then by **Theorem 3 in Cheng et.al.[5]** E_4 is globally stable when (i) and (ii) holds.
Similarly we can find criterion for global stability of E_5.

Theorem 6.2.3. *If E_5 is locally asymptotically stable with* *(i)* $c_2p_2'(x_2)x_2 - s'(y)y > 0 \; \forall \; x_2 > 0, y > 0$

(ii) $-s(0) + c_2p_2(\hat{x}_2) - r_3O^* > 0$ *and* $E_2(0, \hat{x}_2, 0)$ *exists*

then E_5 *is globally asymptotically stable in* $R^+_{0,x_2,y,E,O}$.

Proof: Similar to previous theorem.

Now we derive results for the global stability of interior equilibrium E_6 with the help of Lyapunov function.

In view of **Sinha et.al. [18]** instead of the system (1) we consider system of the following form:

$$\left.\begin{aligned} x_1' &= x_1\tau_1g_1(x_1) - q_1x_1x_2 - yp_1(x_1) - r_1O^*x_1 \\ x_2' &= x_2\tau_2g_2(x_2) - q_2x_1x_2 - yp_2(x_2) - r_2O^*x_2 \\ y' &= y(-s(y) + c_1p_1(x_1) + c_2p_2(x_2) - r_3O^*) \end{aligned}\right\} \tag{6.3}$$

Theorem 6.2.4. *Suppose* $\bar{A}$ *is positive definite matrix in the interior of* S *where* $\bar{A} = -A$, $A = \begin{bmatrix} \bar{v}_{11} & \bar{v}_{12} & \bar{v}_{13} \\ \bar{v}_{21} & \bar{v}_{22} & \bar{v}_{23} \\ \bar{v}_{31} & \bar{v}_{32} & \bar{v}_{33} \end{bmatrix}$, $\bar{v}_{ii} = v_{ii}$ *and* $\bar{v}_{ij} = \bar{v}_{ji} = (\frac{v_{ij}+v_{ji}}{2})$ *for* $j \neq i$

$v_{11} = \frac{-c_1(p_1(x_1)-p_1(x_1^*))(x_1\tau_1g_1(x_1)-q_1x_1x_2^*-y^*p_1(x_1)-r_1O^*x_1)}{p_1(x_1)(x_1-x_1^*)^2}$,
$v_{12} = \frac{q_1x_1c_1(p_1(x_1)-p_1(x_1^*))}{p_1(x_1)(x_1-x_1^*)}$,

$v_{13} = \frac{c_1(p_1(x_1)-p_1(x_1^*))}{x_1-x_1^*}$, $v_{21} = -q_2$, $v_{22} = \frac{x_2\tau_2g_2(x_2)-q_2x_2x_1^*-y^*p_2(x_2)-r_2O^*x_2}{(x_2-x_2^*)x_2}$, $v_{23} = -\frac{p_2(x_2)}{x_2}$, $v_{31} = c_1(\frac{p_1(x_1)-p_1(x_1^*)}{x_1-x_1^*})$, $v_{32} = c_2(\frac{p_2(x_2)-p_2(x_2^*)}{x_2-x_2^*})$ *and* $v_{33} = -\frac{s(y)-s(y^*}{y-y^*}$ *then* E_6 *is globally stable in* $R^+_{x_1,x_2,y,E,O}$.

Proof: Let $V(x_1, x_2, y) = V_{x_1} + V_{x_2} + V_y$ is the Lyapunov function.

Where, $V_{x_1} = \int_{x_1^*}^{x_1} \frac{-s(y^*)+c_1p_1(\xi)+c_2p_2(x_2^*)-r_3O^*}{p_1(\xi)} d\xi, V_{x_2}$

$= x_2 - x_2^* - x_2^* \ln x_2/x_2^*$ and $V_y = y - y^* - y^* \ln y/y^*$.
$\frac{dV}{dt} = \frac{-s(y^*)+c_1p_1(x_1)+c_2p_2(x_2^*)-r_3O^*}{p_1(x_1)}\frac{dx_1}{dt} + \frac{(x_2-x_2^*)}{x_2}\frac{dx_2}{dt} + \frac{(y-y^*)}{y}\frac{dy}{dt}$

$\frac{dV}{dt} = \frac{c_1(p_1(x_1)-p_1(x_1^*))}{p_1(x_1)}(x_1\tau_1 g_1(x_1) - q_1x_1x_2 - yp_1(x_1) - r_1O^*x_1)$

$+\frac{(x_2-x_2^*)}{x_2}(x_2\tau_2 g_2(x_2) - q_2x_1x_2 - yp_2(x_2) - r_2O^*x_2)$

$+(y-y^*)(-s(y) + c_1p_1(x_1) + c_2p_2(x_2) - r_3O^*)$

$\frac{dV[1]}{dt} = \frac{-c_1(p_1(x_1)-p_1(x_1^*))}{p_1(x_1)}(x_1\tau_1 g_1(x_1) - q_1x_1x_2^* - y^*p_1(x_1) - r_1O^*x_1) - \frac{c_1(p_1(x_1)-p_1(x_1^*))}{p_1(x_1)}$

$(q_1x_1x_2^* - q_1x_1x_2) - \frac{c_1(p_1(x_1)-p_1(x_1^*))}{p_1(x_1)}(y^*p_1(x_1) - yp_1(x_1))$.

Let $\frac{dV[1]}{dt} = (x_1-x_1^*)^2 v_{11} + (x_1-x_1^*)(x_2-x_2^*)v_{12} + (x_1-x_1^*)(y-y^*)v_{13}$.

Similarly let $\frac{dV[2]}{dt} = (x_2-x_2^*)^2 v_{22} + (x_2-x_2^*)(x_1-x_1^*)v_{21} + (x_2-x_2^*)(y-y^*)v_{23}$ and $\frac{dV[3]}{dt} = (y-y^*)^2 v_{33} + (y-y^*)(x_1-x_1^*)v_{31} + (y-y^*)(x_2-x_2^*)v_{32}$

where v_{ij} for $i, j = 1, 2$ and 3 are defined as above

Let $X^T = \begin{bmatrix} (x_1-x_1^*) \\ (x_2-x_2^*) \\ (y-y^*) \end{bmatrix}$ and $A = \begin{bmatrix} \bar{v}_{11} & \bar{v}_{12} & \bar{v}_{13} \\ \bar{v}_{21} & \bar{v}_{22} & \bar{v}_{23} \\ \bar{v}_{31} & \bar{v}_{32} & \bar{v}_{33} \end{bmatrix}$

where $\bar{v}_{ii} = v_{ii}$ and $\bar{v}_{ij} = v_{ji} = (\frac{v_{ij}+v_{ji}}{2})$ for $j \neq i$.

We have $\frac{dV}{dt} = XAX^T = -X\bar{A}X^T(A = -\bar{A}$ is a symmetric matrix).

If the matrix $\bar{A}$ is positive definite in S then $\frac{dV}{dt} \leq 0$ in S and the equilibrium $E_6(x_1^*, x_2^*, y^*)$ is globally stable in

$$R^+_{x_1,x_2,y}$$

Wolkowicz and Lu [13].

As $E(t)$ and $O(t)$ tends to E^* and O^* respectively, $E_6(x^*, y^*, E^*, O^*)$ is globally stable in $R^+_{x_1,x_2,y,E,O}$.

7. Persistence

Here we obtain a result for the uniform persistence of the system (2.1). **Freedman and Waltman [7]** obtain conditions for persistence of a Kolmogorov system of two prey and one predator.
Recall a system $x' = f(x), x = (x_1, x_2., x_n)^T$ is said to be persist uniformly if $\exists\ \delta > 0$ such that for
$x_i(0) > 0, \lim inf_{t\to\infty} x_i(t) > \delta\ \forall\ i = 1, 2, \ldots, n.$

Theorem 7.1. *Let equilibria E_3, E_4, E_5 exist and be globally stable in their respective domains with*

$$-s(0) + c_1p_1(l_1) + c_2p_2(l_2) - r_3O^* > 0 \tag{7.1}$$

then system (1) will persist uniformly.

Proof: We will prove our results with the help of the Average Lyapunov function **Huston [12]**.
Let $v = x_1^\alpha x_2^\beta y^\gamma$ where α, β, γ are assumed to be positive.
Now $\frac{1}{v}\frac{dv}{dt} = \frac{\alpha}{x_1}\frac{dx_1}{dt} + \frac{\beta}{x_2}\frac{dx_2}{dt} + \frac{\gamma}{y}\frac{dy}{dt}$.
Along solutions of system (4)
$\frac{1}{v}\frac{dv}{dt} = \alpha(\tau_1 g_1(x_1) - q_1x_2 - \frac{yp_1(x_1)}{x_1} - r_1O^*) + \beta(\tau_2 g_2(x_2) - q_2x_1 - \frac{yp_2(x_2)}{x_2} - r_2O^*) + \gamma(-s(y) + c_1p_1(x_1) + c_2p_2(x_2) - r_3O^*)$
Let $\omega = \frac{1}{v}\frac{dv}{dt}$
Now we have to show that $\omega > 0$ for all boundary equilibria. Here we have six boundary equilibria E_0, E_1, E_2, E_3, E_4 and E_5. Thus $\omega > 0$ has to satisfy the following conditions corresponding to E_0, E_1, E_2, E_3, E_4 and E_5.

(i) At $E_0 : \omega = \alpha(\tau_1 g_1(0) - r_1 O^*) + \beta(\tau_2 g_2(0) - r_2 O^*) + \alpha(-s(0) - r_3 O^*) > 0$.

(ii) At $E_1 : \omega = \beta(\tau_2 g_2(0) - q_2 \hat{x}_1 - r_2 O^*) + \gamma(-s(0) + c_1 p_1(\hat{x}_1) - r_3 O^*) > 0$.

(iii) At $E_2 : \omega = \alpha(\tau_1 g_1(0) - q_1 \hat{x}_2 - r_1 O^*) + \gamma(-s(0) + c_2 p_2(\hat{x}_2) - r_3 O^*) > 0$.

(iv) At $E_3 : \omega = \gamma(-s(0) + c_1 p_1(l_1) + c_2 p_2(l_2) - r_3 O^*) > 0$.

(v) At $E_4 : \omega = \alpha(\tau_1 g_1(x_4) - \frac{y_4 p_1(x_4)}{x_4} - r_1 O^*) + \beta(\tau_2 g_2(0) - q_2 x_4 - r_2 O^*) + \gamma(-s(y_4) + c_1 p_1(x_4) - r_3 O^*) \Longrightarrow \omega = \beta(\tau_2 g_2(0) - q_2 x_4 - r_2 O^*) > 0$.

(vi) At $E_5 : \omega = \alpha(\tau_1 g_1(0) - q_1 x_5 - r_1 O^*) + \beta(\tau_2 g_2(x_5) - \frac{y_5 p_2(x_5)}{x_5} - r_2 O^*) + \gamma(-s(y_5) + c_2 p_2(x_5) - r_3 O^*) \Longrightarrow \omega = \alpha(\tau_1 g_1(0) - q_1 x_5 - r_1 O^*) > 0$.

If we choose α, β sufficiently large with $(-s(0) + c_1 p_1(l_1) + c_2 p_2(l_2) - r_3 O^*) > 0$ then conditions (i)-(vi) are satisfied and by **Lawaniya et.al.** [**14**] using the fact that for large $t, O(t) \leq O^* + \epsilon$ and standard comparison theorem we conclude that system (1) is uniformly persistent.

8. Bifurcation

As E and O tend to E^* and O^* respectively by **Lawaniya et.al.** [**14**], we consider two and three species submodel of system (1) for studying Hopf Bifurcation.

Theorem 8.1. *Suppose $E_4(x_4, 0, y_4, E^*, O^*)$ exists in an open interval containing $\tau_{1hf} > 0$, then in the absence of prey x_2 system (1) experiences Hopf Bifurcation and periodic orbit is formed around its boundary equilibrium E_4 as τ_1 passes through τ_{1hf}, whenever $y_4(s'(y_4))^2 < c_1 p_1(x_4) p_1'(x_4)$ and $\frac{d}{d\tau_1}(j_{11}(4) + j_{33}(4)) \neq 0$ at $\tau_1 = \tau_{1hf}$,*
where, $j_{11}(4) = x_4 \tau_1 g_1'(x_4) + \tau_1 g_1(x_4) - y_4 p_1'(x_4) - r_1 O^*$ and $j_{33}(4) = -y_4 s'(y_4)$.

Proof: For jacobian we have submatrix

$$A = \begin{bmatrix} j_{11}(4) & -p_1(x_4) \\ c_1 y_4 p_1'(x_4) & j_{33}(4) \end{bmatrix}$$

The characteristic equation for A is $Q(\lambda) \equiv \lambda^2 + a_1 \lambda + a_2 = 0$,
where $a_1 = -(j_{11}(4) + j_{33}(4))$ and $a_2 = j_{11}(4).j_{33}(4) + c_1 p_1(x_4) y_4 p_1'(x_4)$

at $\tau_1 = \tau_{1hf}$ if A possesses purely imaginary eigen values then we have $j_{11}(4) + j_{33}(4) = 0$ and $j_{11}(4).j_{33}(4) + c_1 p_1(x_4) y_4 p_1'(x_4) > 0$.
$\Leftrightarrow a_1 = 0$ and $a_2 = -(j_{33}(4))^2 + c_1 p_1(x_4) y_4 p_1'(x_4)$
$\Rightarrow y_4(s'(y_4))^2 < c_1 p_1(x_4) p_1'(x_4)$.
Let two eigenvalues be $\lambda_{1,2} = \chi \pm i\psi$
Putting $\lambda = \chi + i\psi$ in characteristic equation of A and differentiating it with respect to τ_1
$(2\chi + a_1)\frac{d\chi}{d\tau_1} + (-2\psi)\frac{d\psi}{d\tau_1} = -\chi\frac{da_1}{d\tau_1} - \frac{da_2}{d\tau_1}$ and $(2\chi + a_1)\frac{d\psi}{d\tau_1} + 2\psi\frac{d\chi}{d\tau_1} = -\psi\frac{da_1}{d\tau_1}$.
At $\tau_1 = \tau_{1hf}, \psi^2 = a_2$ and $a_1 = 0$.
After simplifying we get
$\frac{d\chi}{d\tau_1}\Big|_{\tau_{1hf}} = -\frac{1}{2}(\frac{da_1}{d\tau_1})\Big|_{\tau_{1hf}} = \frac{1}{2}(\frac{d}{d\tau_1}(j_{11}(4) + j_{33}(4)))$.
If $\frac{d}{d\tau_1}(j_{11}(4) + j_{33}(4))\Big|_{\tau_{1hf}} \neq 0$ (transversality condition) hence this provides our result, system experienced Hopf Bifurcation and periodic orbits are formed by **Hassard et.al. [8]**.

Theorem 8.2. *Assume that E_5 exists in an open interval containing $\tau_{2hf} > 0$, then in the absence of prey x_1, the system (1) experiences Hopf Bifurcation and periodic orbit is formed around its boundary equilibrium E_5 as τ_2 passes through τ_{2hf} whenever $y_5(s'(y_5))^2 < c_2 p_2(x_5) p_2'(x_5)$ and $\frac{d}{d\tau_2}(j_{22}(5) + j_{33}(5)) \neq 0$ at $\tau_2 = \tau_{2hf}$,*
where, $j_{22}(5) = x_5 \tau_2 g_2'(x_5) + \tau_2 g_2(x_5) - y_5 p_2'(x_5) - r_2 O^*$ and $j_{33}(5) = -y_5 s'(y_5)$.

Proof: Proceed as Theorem 8.1.

For interior Equilibrium
Assume that in an open interval containing $\tau_2 = \tau_{2HF} > 0$ interior equilibrium exists.

Theorem 8.3. *Suppose there exists $\tau_2 = \tau_{2HF} > 0$ such that $n_i(\tau_{2HF}) > 0, 1 \leq i \leq 3, \Delta_2(\tau_{2HF}) = 0$ and $(\frac{dn_3}{d\tau_2} - (n_2\frac{dn_1}{d\tau_2} + n_1\frac{dn_2}{d\tau_2}))\Big|_{\tau_{2HF}} \neq 0$*
where, $\Delta_2 = det\begin{bmatrix} n_1 & 1 \\ n_3 & n_2 \end{bmatrix}$ and n_i for all $i = 1, 2, 3$ are defined as above and evaluated at E_6
Then system experiences Hopf Bifurcation and periodic orbits are formed.

Proof: The jacobian for system is as follows:

$$J = \begin{bmatrix} j_{11} & -q_1x_1 & -p_1(x_1) \\ -q_2x_2 & j_{22} & -p_2(x_2) \\ c_1yp_1'(x_1) & c_2yp_2'(x_2) & j_{33} \end{bmatrix}$$

The characteristic polynomial is

$$P(\lambda) = \lambda^3 + n_1\lambda^2 + n_2\lambda + n_3 \tag{8.1}$$

where, $n_1 = -(j_{11} + j_{22} + j_{33}), n_2 = j_{11}j_{22} + j_{11}j_{33} + j_{22}j_{33} - q_1q_2x_1x_2 + c_2yp_2'(x_2)p_2(x_2) + c_1yp_1'(x_1)p_1(x_1), n_3 = -j_{11}j_{22}j_{33} - j_{11}c_2yp_2'(x_2)p_2(x_2) + q_1q_2x_1x_2j_{33} - q_1x_1c_1yp_1'(x_1)p_2(x_2) - q_2x_2c_2yp_2'(x_2)p_1(x_1) - j_{22}c_1yp_1'(x_1)p_1(x_1), j_{11} = x_1\tau_1g_1'(x_1) + \tau_1g_1(x_1) - yp_1'(x_1) - q_1x_2 - r_1O^*, j_{22} = x_2\tau_2g_2'(x_2) + \tau_2g_2(x_2) - yp_2'(x_2) - q_2x_1 - r_2O^*, j_{33} = -s(y) - ys'(y) + c_1p_1(x_1) + c_2p_2(x_2) - r_3O^*$, and n_i's are evaluated at E_6 and are functions of τ_2.
We claim that $\Delta_2(\tau_{2HF}) = 0$ if and only if $P(\lambda)$ has a pair of purely imaginary roots. First let $\Delta_2(\tau_{2HF}) = 0$, then by Orlando formula $P(\lambda)$ has a pair of roots with opposite signs.
Suppose the eigen values are real, λ_1 and $-\lambda_1$ then by characteristic equation

$$\begin{aligned} & P(\lambda_1) \equiv \lambda_1^3 + n_1\lambda_1^2 + n_2\lambda_1 + n_3 = 0 \\ \text{and}\quad & P(-\lambda_1) \equiv -\lambda_1^3 + n_1\lambda_1^2 - n_2\lambda_1 + n_3 = 0 \\ & \Rightarrow n_1\lambda_1^2 + n_3 = 0 \end{aligned}$$

But this contradicts the fact since $n_i > 0$.
Hence there will be pair of purely imaginary roots of $P(\lambda) = 0$ at τ_{2HF}.
Conversely let $P(\lambda)$ has imaginary roots $i\lambda_1$ and $-i\lambda_1$ $(\lambda_1 \in R)$ for $\tau_2 = \tau_{2HF}$.

Then $P(i\lambda_1) = 0$,
i.e. $-i\lambda_1^3 - n_1\lambda_1^2 + in_2\lambda_1 + n_3 = 0$
or $\lambda_1^3 - n_2\lambda_1 = 0$ and $n_1\lambda_1^2 - n_3 = 0 \Rightarrow \lambda_1^2 = n_2$
thus $n_1n_2 - n_3 = 0$ i.e. $\Delta_2(\tau_{2HF}) = 0$
Also the third root is negative.
Let the one complex root be $\lambda_1 = \alpha + i\beta$
then we are to prove that $Re(\frac{d\lambda_1}{d\tau_2})\Big|_{\tau_{2HF}} \neq 0$.
Setting $\lambda = \alpha + i\beta$ in (6) and differentiating with respect to τ_2,

$$\frac{d\alpha}{d\tau_2}(3(\alpha^2-\beta^2)+2\alpha n_1+n_2)+\frac{d\beta}{d\tau_2}(-6\alpha\beta-2\beta n_1) = \frac{dn_1}{d\tau_2}(\beta^2-\alpha^2)-\alpha\frac{dn_2}{d\tau_2}-\frac{dn_3}{d\tau_2} \tag{8.2}$$

and$\frac{d\alpha}{d\tau_2}(6\alpha\beta + 2\beta n_1) + \frac{d\beta}{d\tau_2}(3(\alpha^2 - \beta^2) + 2\alpha n_1 + n_2) = -2\alpha\beta\frac{dn_1}{d\tau_2} - \beta\frac{dn_2}{d\tau_2}$

(8.3)

From (8.2) and (8.3) we get
$\frac{d\alpha}{d\tau_2}(3(\alpha^2 - \beta^2) + 2\alpha n_1 + n_2)^2 + (6\alpha\beta + 2\beta n_1)^2 = (\frac{dn_1}{d\tau_2}(\beta^2 - \alpha^2) - \alpha\frac{dn_2}{d\tau_2} - \frac{dn_3}{d\tau_2})(3(\alpha^2 - \beta^2) + 2\alpha n_1 + n_2) - (6\alpha\beta + 2\beta n_1)(2\alpha\beta\frac{dn_1}{d\tau_2} + \beta\frac{dn_2}{d\tau_2})$
At $\tau_2 = \tau_{2HF}$, $\beta^2 = n_2$ and $\alpha = 0 \Rightarrow \frac{d\alpha}{d\tau_2}\Big|_{\tau_{2HF}} = \frac{[\frac{dn_3}{d\tau_2} - (n_2\frac{dn_1}{d\tau_2} + n_1\frac{dn_2}{d\tau_2})]}{2(n_2+n_1^2)}$.
Hence if $(\frac{dn_3}{d\tau_2} - (n_2\frac{dn_1}{d\tau_2} + n_1\frac{dn_2}{d\tau_2}))\Big|_{\tau_{2HF}} \neq 0$ then by **Hassard et.al.[8]** system experiences Hopf Bifurcation and periodic orbits are formed.

9. Numerical Simulation

The numerical simulation has been carried out with MATLAB 2010a.

$$x_1' = x_1\tau_1 g_1(x_1) - q_1x_1x_2 - yp_1(x_1) - r_1Ox_1 \tag{9.1}$$
$$x_2' = x_2\tau_2 g_2(x_2) - q_2x_1x_2 - yp_2(x_2) - r_2Ox_2 \tag{9.2}$$
$$y' = y(-s(y) + c_1p_1(x_1) + c_2p_2(x_2) - r_3O) \tag{9.3}$$
$$E' = -Eh(E) + Q_E \tag{9.4}$$
$$O' = -d_1O + p(E) \tag{9.5}$$

Taking following set of functions:
$g_i(x_i) = (1 - (x_i/K_i)^3), p_i(x_i) = \frac{x_i}{x_i+m_i}\forall i = 1, 2, s(y) = c(d + \frac{y}{w}), h(E) = a$ and $p(E) = l + mE$,
and the following values of parameters: $\tau_1 = \tau_2 = 0.5, K_1 = 900, K_2 = 950, m_1 = 800, m_2 = 750, q_1 = 0.00002, q_2 = 0.00001, c = 0.003, d = 75, w = 75, r_1 = 0.004, r_2 = 0.005, r_3 = 0.006, c_1 = 0.6, c_2 = 0.7$

Stage I: Pollution parameters $Q_E = 0.4, a = 1, d_1 = 0.5, l = 0.0234, m = 0.04$

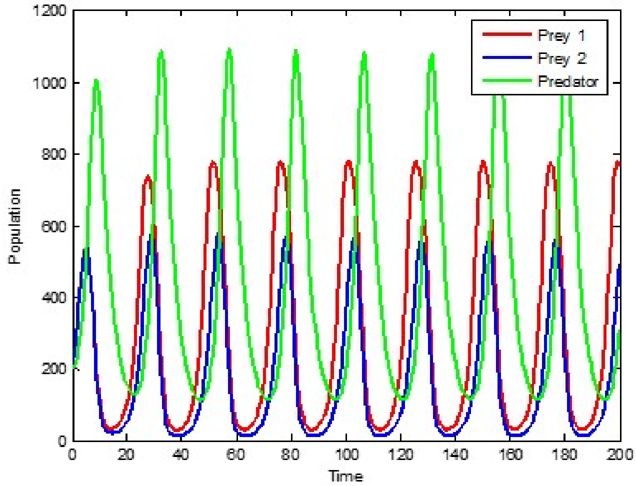

Figure 5: System at very low level of Pollution

We get $\tau_1 g_1(0) - r_1 O^* > 0, \tau_2 g_2(0) - r_2 O^* > 0, 0 < x_1^* < x_4 < l_1 < \hat{x}_1, 0 < x_2^* < x_5 < l_2 < \hat{x}_2$. Here all conditions needed for existence of equilibria are satisfied.

Stage II: Pollution parameters, $Q_E = 10, a = 0.35, d_1 = 0.5, l = 0.6, m = 0.5$

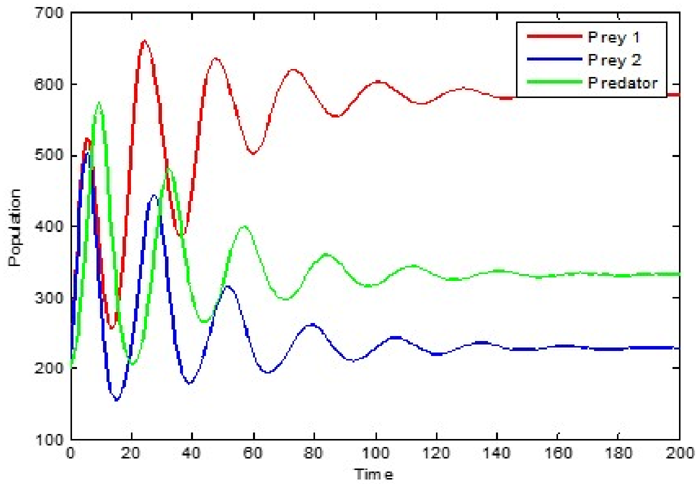

Figure 6: System after a little increase in Pollution level

We also get $\tau_1 g_1(0) - r_1 O^* > 0, \tau_2 g_2(0) - r_2 O^* > 0, 0 < x_1^* < x_4 < l_1 < \hat{x}_1, 0 < x_2^* < x_5 < l_2 < \hat{x}_2$, hence all equilibria exist.

Stage III: Pollution parameters, $Q_E = 30, a = 0.35, d_1 = 0.5, l = 0.6, m = 0.5$

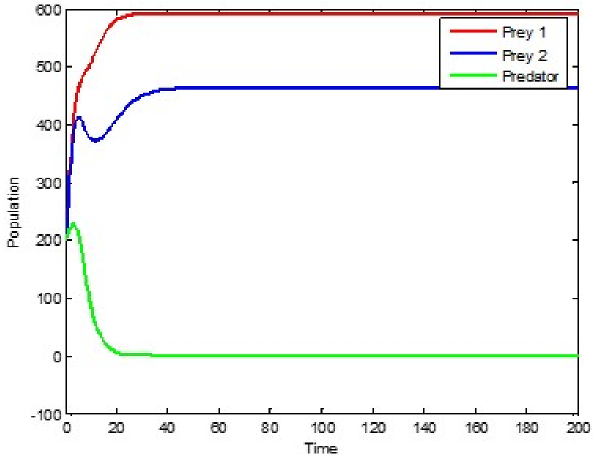

Figure 7: System after a little increase in Pollution level

We also get $\tau_1 g_1(0) - r_1 O^* > 0, \tau_2 g_2(0) - r_2 O^* > 0, 0 < x_1^* < x_4 = l_1 < \hat{x}_1, 0 < x_2^* < x_5 = l_2 < \hat{x}_2$, hence E_1, E_2, E_3 exist but E_4, E_5, E_6 do not exist.

Stage IV: Pollution parameters, $Q_E = 35, a = 0.35, d_1 = 0.5, l = 0.7, m = 0.5$

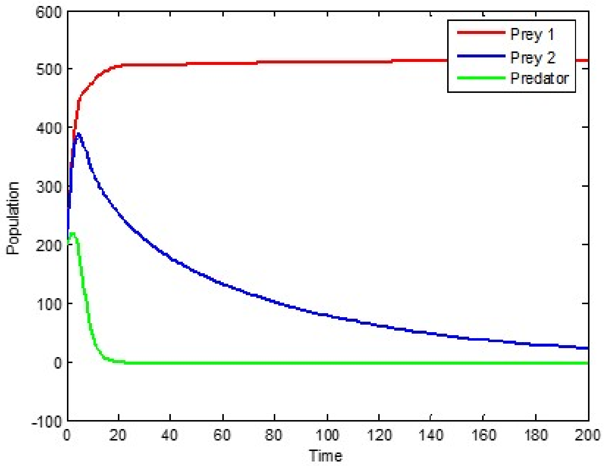

Figure 8: System at very large increase in level of Pollution

Here we also get $\tau_1 g_1(0) - r_1 O^* > 0, \tau_2 g_2(0) - r_2 O^* > 0, 0 < x_1^* < x_4 = l_1 < \hat{x}_1, 0 < x_2^* < x_5 = l_2 < \hat{x}_2$, hence E_1, E_2, E_3 exist but E_4, E_5, E_6 do not exist.

Stage V: Pollution parameters, $Q_E = 45, c = 0.35, d_1 = 0.5, l = 0.7, m = 0.5$

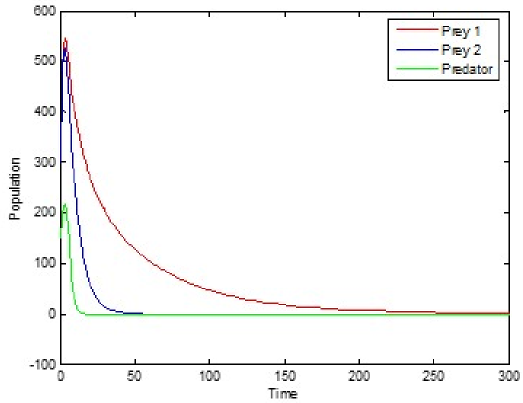

Figure 9: System at hazardous level of Pollution

Here none of the conditions for existence of equilibria is satisfied, hence no equilibrium except E_0 exists.

For Bifurcation: (i) In absence of Prey 2
Taking τ_1 as Bifurcation parameter and keeping other as follows
$K_1 = 1000, m_1 = 700, q_1 = 0.00002, c = 0.003, d = 70, w = 50, r_1 = 0.05, r_3 = 0.08, c_1 = 0.64, a = 0.4, b = 1, d_1 = 0.5, l = 0.0234, m = 0.04.$

We get the graphical structure as follows:

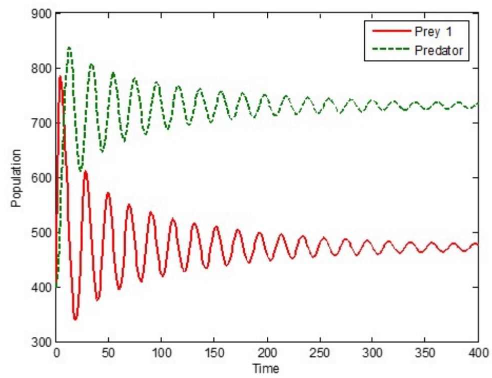

Figure 10: The Predator and Prey 1 population with $\tau_1 = 0.7$

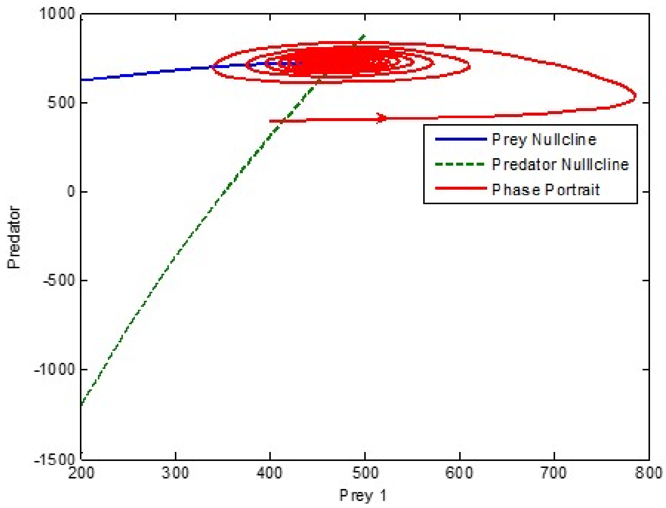

Figure 11: The Predator and Prey 1 system with $\tau_1 = 0.7$

Here we have $E_4 = (472.3877, 0, 731.2307, 0.04, 0.05)$ and eigenvalues of jacobian matrix are $-0.0070 + i0.3077, -0.0070 - i0.3077$ and E_4 is stable.

At $\tau_1 = 0.5$ we get $E_4 = (431.1909, 0, 499.2866)$ and eigenvalues of jacobian matrix are $0.0850 + i0.2311, 0.0850 - i0.2311$ and E_4 is unstable.

When $\tau_1 = 0.640625$, the behavior of system is followed by Fig. 12 and Fig. 13.

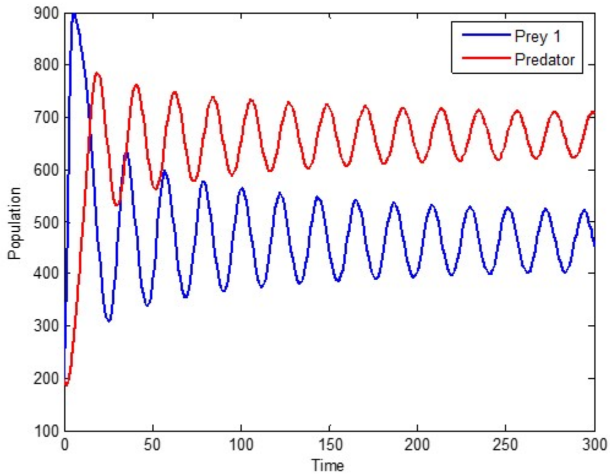

Figure 12: Predator and Prey 1 population system at $\tau_1 = 0.640625$

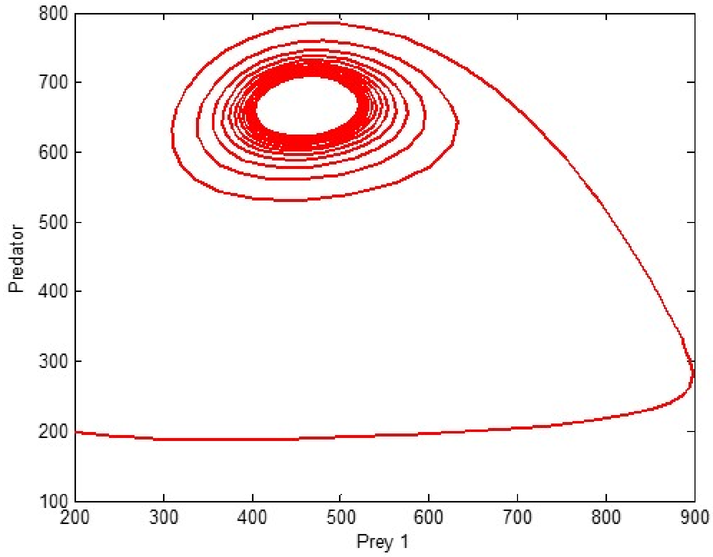

Figure 13: Predator and Prey 1 system at $\tau_1 = 0.640625$

Here the periodic orbit arises and we get eigenvalues of jacobian matrix are $0.0001 + i8.2083, 0.0001 - i8.2083$ with $\frac{d}{d\tau_1}(j_{11}(4) + j_{33}(4)) = 0.006084 \neq 0$ and $y_4(s'(y_4))^2 < c_1 p_1(x_4) p_1'(x_4)$.
Hence $\tau_{1hf} \approx 0.640625$ and when $\tau_1 > \tau_{1hf}$ the system shows a stable equilibrium point and below the critical value equilibrium point it is unstable.

(ii) In absence of Prey 1
Now taking τ_2 as bifurcation parameter and values of other parameters as follows:
$K_2 = 950, m_2 = 750, q_2 = 0.00001, c = 0.003, d = 70, w = 50, r_2 = 0.06, r_3 = 0.08, c_2 = 0.7, a = 0.4, b = 1, d_1 = 0.5, l = 0.0234, m = 0.04,$
At $\tau_2 = 0.65$

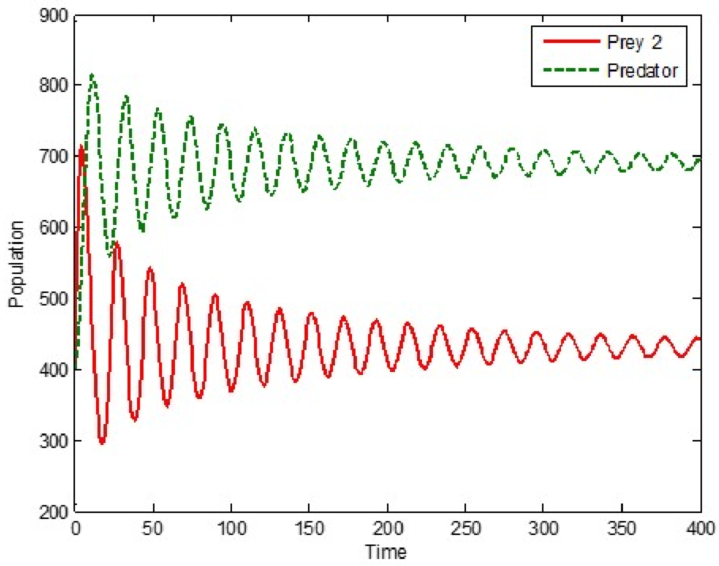

Figure 14: Predator and Prey 2 population at $\tau_2 = 0.65$

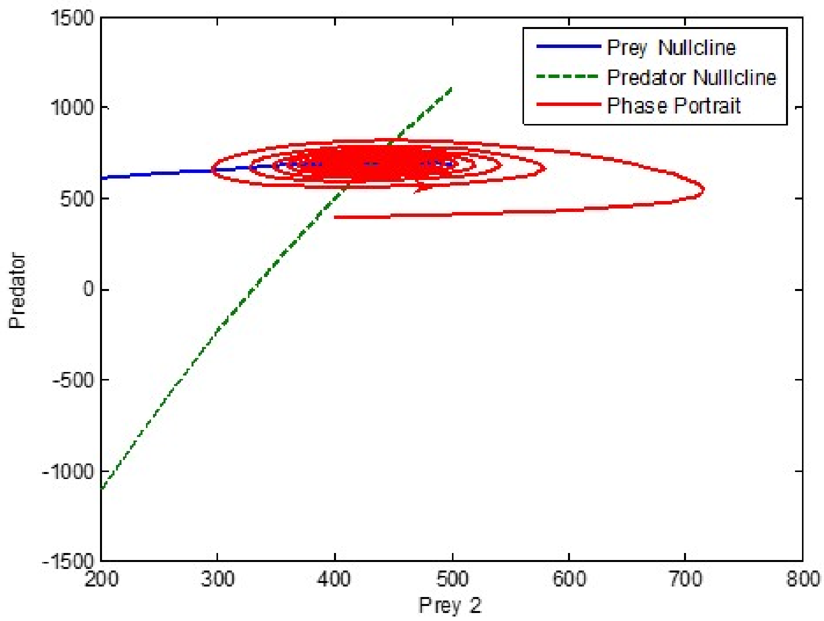

Figure 15: Predator and Prey 2 system at $\tau_2 = 0.65$

We get equilibrium is $E_5 = (0, 431.2260, 692.4423, 0.04, 0.05)$ and eigenvalues of jacobian matrix are $-0.0060 + i0.3063, -0.0060 - i0.3063$. Hence E_5 is stable.
At $\tau_2 = 0.5$ we get $E_5 = (0, 405.7851, 529.3887, 0.04, 0.05)$ and eigenvalues of jacobian matrix are $0.0061 + i0.2676, 0.0061 - i0.2676$. Hence E_5 is unstable.
At $\tau_2 = 0.59375$, behavior of system is followed by Fig 16 and Fig 17.

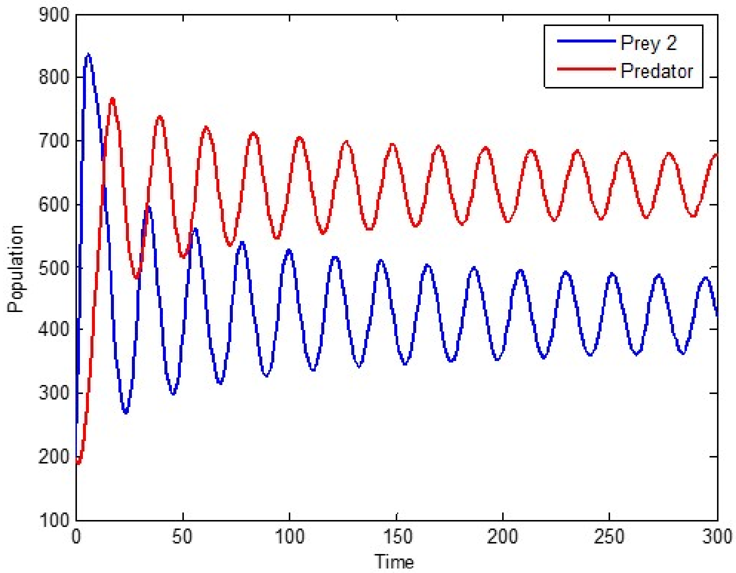

Figure 16: Predator and Prey 2 system at $\tau_2 = 0.59375$

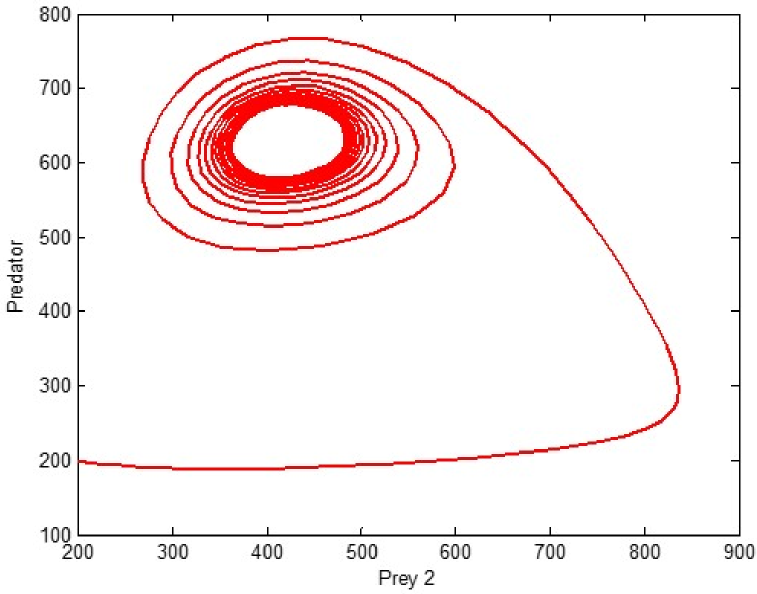

Figure 17: Predator and Prey 2 system at $\tau_2 = 0.59375$

Here we get eigenvalues of jacobian matrix are $0.00002 + i7.5642$ and $0.00002 - i7.5642$ also we have $\frac{d}{d\tau_2}(j_{22}(5) + j_{33}(5)) = 0.006505 \neq 0$ and $y_5(s'(y_5))^2 < c_2p_2(x_5)p_2'(x_5)$.
Hence $\tau_{2hf} \approx 0.59375$. Hence as $\tau_2 > \tau_{2hf}$ the system shows a stable equilibrium point and at the threshold value it has a periodic orbit enclosing an equilibrium point.

(iii) For interior equilibrium:
Taking τ_2 as Bifurcation parameter and keeping others fixed and following set of parameters;
$\tau_1 = 0.5, K_1 = 1000, K_2 = 950, m_1 = 800, m_2 = 750, q_1 = 0.0000015, q_2 = 0.0000025, c = 0.0035, d = 30, w = 50, r_1 = 0.04, r_2 = 0.06, r_3 = 0.08, c_1 = 0.3, c_2 = 0.2$ and $O^* = 0.05$.
We get at $\tau_2 = 0.6$ interior equilibrium is $E_6 = (163.9249, 621.3034, 466.1853)$ and eigenvalues of jacobian matrix are
$-0.0141+0.1359i, -0.0141-0.1359i, -0.3083$ which shows that E_6 is stable as we can see in fig. 18.

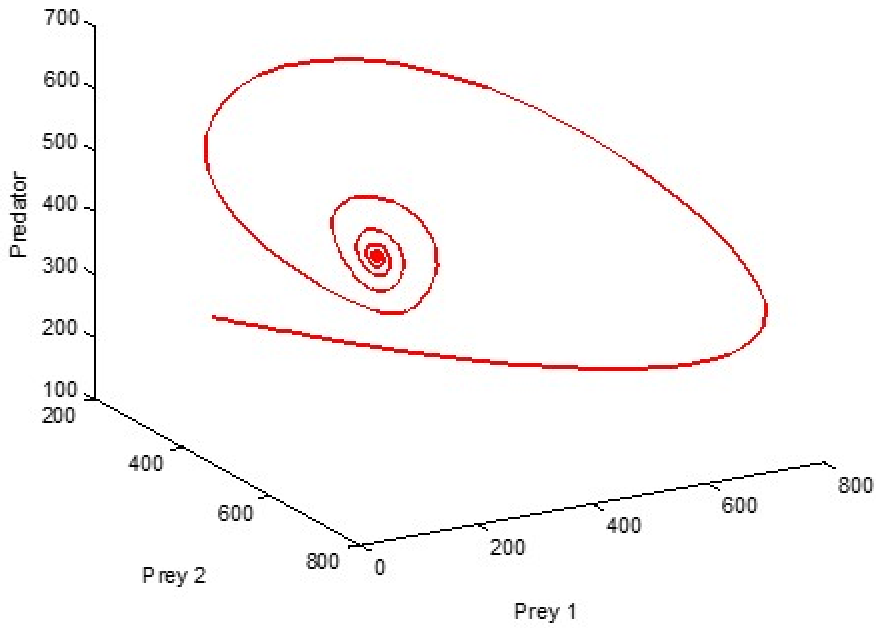

Figure 18: Predator, Prey 1 and Prey 2 system at $\tau_2 = 0.6$

At $\tau_2 = 0.9$ interior equilibrium is $E_6 = (118.2906, 776.8789, 450.2551)$ and the eigenvalues of jacobian matrix are $0.0298+i0.1246, 0.0298-i0.1246, -1.0957$. Hence E_6 is unstable.

Here with $\tau_2 = 0.7935$,

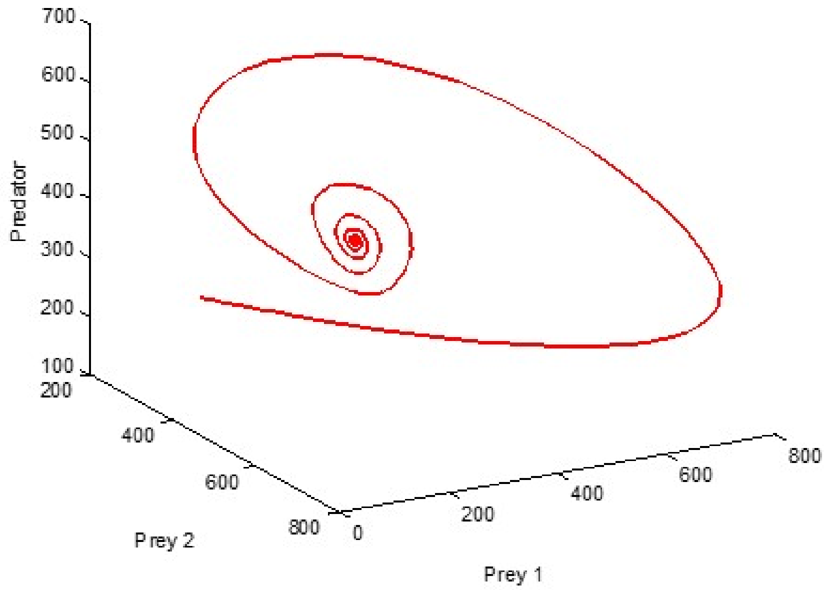

Figure 19: Predator, Prey 1 and Prey 2 system at $\tau_2 = 0.7935$

We get interior equilibrium is $E_6 = (126.8799, 744.7612, 453.0900)$ and eigenvalues of jacobian matrix are $-0.000011 + i0.122911, -0.000011 - i0.122911, -0.805206$ with

$$n_1 = 0.805228, n_2 = 0.0151, n_3 = 0.0122, (n_1.n_2 - n_3) = -0.000041 \approx 0$$

and $\left(\frac{dn_3}{d\tau_2} - (n_2\frac{dn_1}{d\tau_2} + n_1\frac{dn_2}{d\tau_2})\right)\Big|_{\tau_{2HF}} = 0.0016.$
Hence $\tau_{2HF} \approx 0.7935$.
Here we conclude from Theorem 8.3 that when τ_2 crosses its critical value the system has periodic solution. This illustrated in Fig 19.

Persistence:
From the previous example we have $l_1 = 996.5735, l_2 = 947.0973, x_4 = 672.5703, y_4 = 400.2809, x_5 = 938.6522, y_5 = 31.0251$ and inequality (5) holds as

$$-s(0) + c_1p_1(l_1) + c_2p_2(l_2) - r_3O^* = 0.1690 > 0$$

hence the system persists uniformly.

10. Results and Conclusion

In this study mathematical criteria are developed for the stability and persistence of a system of three species (two competing prey and one predator) in the presence of pollution. With the help of numerical example it is shown how the sustainability of the system gets affected as the pollution level increases. It was observed that when the pollution is very low all the species can survive in a stable equilibrium, but as the level of pollution increases, initially the value of predator equilibrium decreases and later on it results in the extinction of species one by one from top to bottom in trophic level:

Stage	Q_E **Input rate of pollutant in the environment**	**a Loss rate of environmental pollution**	$p(E) = l + mE$ **Conversion function of environmental pollution into organismal pollution.**		**Surviving species**
			l	**m**	
Stage I	0.4	1	0.0234	0.04	All three species surviving
Stage II	10	0.35	0.6	0.5	All three surviving and equilibrium value of predator has decreased.
Stage III	30	0.35	0.6	0.5	Both prey survive and predator goes for extinction.
Stage IV	35	0.35	0.7	0.5	Second prey and predator go for extinction and only first prey survives.
Stage V	45	0.35	0.7	0.5	All three species go for extinction

In the following three cases, conditions are derived under which system experiences Bifurcation.
The pollution level is assumed to be low.

(i) In the absence of second prey

(ii) In the absence of first prey

(iii) With all three species.

In each of the above cases as the bifurcation parameter crosses the threshold value, the system moving towards stable equilibrium point shows change in behavior and moves towards a periodic orbit.

Declaration

There is no relevant financial or non-financial competing interest.

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