Sustainability of a system of two competing prey and a predator in polluted environment

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Abstract
In this study, a general model of interacting species consisting of two competing prey and a predator under the presence of pollution is formed. Criteria for the existence of equilibria and their (local and global) stability are derived. The conditions for persistence and bifurcation have also been derived. With the help of numerical simulation, it is shown how the change in the pollution level results in species extinction.

Keywords: Competition, Pollution, Equilibria, Stability, Bifurcation, Persistence.

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1. Introduction

The species in the ecosystem are interdependent for their survival. They interact with each other for different reasons, like food and shelter. The type of interaction depends upon their biological needs and surrounding conditions. The interacting species in a particular ecosystem form food chains or food webs. The last several decades have seen tremendous growth in industrialisation and urbanisation. The extensive use of non-biodegradable products has caused a damaging impact on our environment. The excess exploitation of nature has severely disturbed the ecological balance. It has forced the species, including humans, to be exposed to anthropogenic substances through different sources, including air, water and food. All these have resulted in the extinction of many species from the earth. The international union for the conservation of nature published a list of 160 species going extinct from 2010-2019 [15]. Keeping environment pollution free is one of the most challenging problems today. The pollutant moving upward through food chains becomes more hazardous for the species at the higher trophic level. For example, the DDT from prey breaks down into DDE in predators [17].

Mathematical models can help analyse the present situation and predict the future so that anyone can take necessary measures to control pollution levels. Some of the mathematical models formed to analyse the dynamics of two or more species are of the following types:

**Freedman and Waltman** [7] studied the system of two predators competing with each other, feeding on a single prey and a single predator feeding on two competing prey species.

**Kar and Batabyal** [13] proposed a mathematical model to analyse the dynamics of a system having two prey and one predator in the presence of a time delay due to gestation.

**Ali and Chakravarthy** [2] analysed a model with the intra-specific competition among predator populations consists of two competing prey and one predator.

**Daga et. al.** [6] proposed a predator-prey model with Holling type III functional response and analysed the dynamics of a two prey, one predator system.
The problem of approximating the effect of a toxicant on a population by mathematical models began in the early 1980s. Ma and Hallam [16] and Hauping and Ma [9] obtained a survival threshold for single and two species, respectively, under the effect of pollution. Chauhan and Mishra [3] studied a single species model under the combined impact of toxicant and infection. Sinha et al. [18] studied the predator-prey model under the influence of toxicants. Chauhan et al. [4] studied the predator-prey model under the effect of infection for prey and predator species, respectively. Huang et al. [10] formulated a toxin-dependent aquatic population model and connected the model to the experimental data via model parameterisation. Huang et al. [11] developed a toxin-dependent Predator-prey model for two species and discussed the effect of pollution on species. Lawaniya et al. [14] developed a generalised model for two species under the effect of pollution and obtained conditions regarding stability and persistence.

In this study dynamics of three species consisting of two competing prey and one predator is carried out under the effect of pollution. In next three sections the model has been formulated and has been shown to be viable. Criteria for the existence of equilibria are derived in section 5, and criteria for local and global stability have been carried out in section 6. The conditions for persistence and bifurcation have been discussed in sections 7 and 8, respectively. In section 9, the results regarding the existence of equilibria and bifurcation have been validated through the numerical examples In section 10, the results are discussed.

2. Model formation

We have Gause type model for three species consisting of two competing prey species and one predator of the form:

\[ \begin{align*}
    x'_1 &= x_1 \tau_1 g_1(x_1) - q_1 x_1 x_2 - y p_1(x_1) \\
    x'_2 &= x_2 \tau_2 g_2(x_2) - q_2 x_1 x_2 - y p_2(x_2) \\
    y' &= y(-s(y) + c_1 p_1(x_1) + c_2 p_2(x_2))
\end{align*} \]

where \( x_i \) for \( i = 1, 2 \) are density of prey species population, \( y \) denotes the predator population, \( s(y) \) is the death rate of the predator, \( g_i(x_i) \) for \( i = 1, 2 \) are growth rates of prey species, \( \tau_i \) for \( i = 1, 2 \) are growth rate coefficients, \( p_i(x_i) \) for \( i = 1, 2 \) are predation functions, constants \( q_i \) for \( i = 1, 2 \) are interspecies competition coefficients and \( c_i \) are the coefficients
for conversion of prey biomass into predator biomass.
Under the effect of environmental pollution the model is of the form;
\[
\begin{align*}
\frac{dx_1'}{dt} &= x_1 \tau_1 g_1(x_1) - q_1 x_1 x_2 - y p_1(x_1) - r_1 O x_1 \\
\frac{dx_2'}{dt} &= x_2 \tau_2 g_2(x_2) - q_2 x_1 x_2 - y p_2(x_2) - r_2 O x_2 \\
y' &= y(-s(y) + c_1 p_1(x_1) + c_2 p_2(x_2) - r_3 O) \\
E' &= -E h(E) + Q E \\
O' &= -d_1 O + p(E)
\end{align*}
\]
(2.1)
where \(E(t)\) is the concentration of toxicant in environment, \(O(t)\) is the concentration of the toxicant in the organism, \(r_i\) for \(i = 1, 2, 3\) are the rates of loss of biomass for three species \(x_1, x_2, y\) respectively due to environmental pollution, \(h(E)\) is the loss rate function of environmental pollution, \(Q E\) is the input rate of pollutant in the environment, \(d_1\) is the coefficient of depuration of organismal pollution and \(p(E)\) denotes the conversion function of environmental pollution into organismal pollution.

**Assumptions:** All the functions \(g_i, p_i, s, h(E), p(E)\) are smooth enough so that the solutions of the system exist, are unique and continuous for all \(t > 0\) Ahmad and Rao [1] and \(Q E > 0, d_1 > 0\).

**G1** Growth rate functions \(g_1(x_1)\): It is assumed that \(g_1(0) > 0, \frac{dg_1}{dx_1} < 0\) and \(\exists a K_i > 0\) for which \(g_i(K_i) = 0\) and \(0 < \tau_i\) for \(i = 1, 2\).

**P1** Predation function \(p_1(x_1)\): It is assumed that \(p_1(0) = 0, \frac{dp_1}{dx_1} > 0\) for \(i = 1, 2\).

**S1** Death rate function \(s(y)\): It is assumed that \(s(0) > 0\), and \(s'(y) > 0\). These conditions interpreted that the death rate always remains positive and is density dependent.

**Q1** Competition coefficients \(q_i\): The coefficients \(q_1, q_2\) represent competition between the prey species \(x_1, x_2\) respectively, \(q_i > 0\) for \(i = 1, 2\).

**H1** Environmental Pollution loss rate: \(h(0) > 0\) and \(h'(E) \geq 0\).

**P2** Conversion function from environmental pollution \(E\) to organismal pollution \(O\): \(p(0) > 0\) and \(p'(E) \geq 0\).

As \(\lim_{E \to \infty} E h(E) > Q_E, \exists E^*\) such that \(-E^* h(E^*) + Q_E = 0\).
3. Invariant Region [14]

The solutions of (2.1) with non-negative initial conditions stay non-negative for all time $t > 0$ and the region,

$$ S = \{(x_1, x_2, y, E, O) \mid 0 \leq x_1 \leq \hat{x}_1 + \epsilon, 0 \leq x_2 \leq \hat{x}_2 + \epsilon, 0 \leq c_1 x_1 + c_2 x_2 + y \leq \frac{T}{s(0)} + \epsilon, 0 \leq E \leq \frac{Q_E}{h(0)} + \epsilon, 0 \leq O \leq O^* + \epsilon \} $$

is a positively invariant and attracting region for system (2.1) where $> 0$, $\hat{x}_i =$ carrying capacity of $i^{th}$ prey under the effect of pollution, $O^* = \frac{P(E^*)}{d_1}$ and $T = c_1 (\hat{x}_1 + \epsilon)(g_1(0) + s(0)) + c_2 (\hat{x}_2 + \epsilon)(g_2(0) + s(0))$.

4. Possible Equilibria

There are seven possible equilibria for the system

(i) $E_0(0,0,0,E^*,O^*)$

(ii) $E_1(\hat{x}_1,0,0,E^*,O^*)$

(iii) $E_2(0,\hat{x}_2,0,E^*,O^*)$

(iv) $E_3(l_1,l_2,0,E^*,O^*)$

(v) $E_4(x_4,0,y_4,E^*,O^*)$

(vi) $E_5(0,x_5,y_5,E^*,O^*)$

(vii) $E_6(x_1^*,x_2^*,y^*,E^*,O^*)$

5. Existence of Equilibria

The equilibrium $E_0(0,0,0,E^*,O^*)$ always exists and equilibria $E_1(\hat{x}_1,0,0,E^*,O^*)$ exists if $\tau_1 g_1(0) - r_1 O^* > 0$ and $E_2(0,\hat{x}_2,0,E^*,O^*)$ exists if $\tau_2 g_2(0) - r_2 O^* > 0$.

Further throughout our analysis we assume that $E_1$ and $E_2$ exist. The conditions for existence of other equilibria are as follows;

(i) $E_3(l_1,l_2,0,E^*,O^*)$ exists if

\[
\hat{x}_1 < \frac{(\tau_2 g_2(0) - r_2 O^*)}{q_2} \quad \text{and} \quad \hat{x}_2 < \frac{(\tau_1 g_1(0) - r_1 O^*)}{q_1}
\]

or

\[
\hat{x}_1 > \frac{(\tau_2 g_2(0) - r_2 O^*)}{q_2} \quad \text{and} \quad \hat{x}_2 > \frac{(\tau_1 g_1(0) - r_1 O^*)}{q_1}
\]
where,

\[ E_4(x_4, 0, y_4, E^*, O^*) \] exists if \( 0 < x_6 < \hat{x}_1 \).

\[ E_5(0, x_5, y_5, E^*, O^*) \] exists if \( 0 < x_7 < \hat{x}_2 \) such that \( s_2(0) + c_2p_2(x_7) - r_3O^* = 0 \).

\[ E_6(x_1^*, x_2^*, y^*, E^*, O^*) \] exists if our system is uniformly persistent. Conditions for uniform persistence are given in section 7.

6. Stability

6.1. Local Stability

With the help of jacobian matrix we determine the local stability of the system corresponding to equilibria Ahmad and Rao [1].

The jacobian matrix corresponding to the system (2.1) is

\[
J = \begin{bmatrix}
  j_{11} & -q_1x_1 & -p_1(x_1) & 0 & -r_1x_1 \\
  -q_2x_2 & j_{22} & -p_2(x_2) & 0 & -r_2x_2 \\
  c_1yl(x_1) & c_2yp(x_2) & j_{33} & 0 & -r_3y \\
  0 & 0 & 0 & -Eh'(E) - h(E) & 0 \\
  0 & 0 & 0 & p'(E) & -d_1
\end{bmatrix}
\]

where, \( j_{11} = x_1\tau_1g_1'(x_1) + \tau_1g_1(x_1) - yp_1'(x_1) - q_1x_2 - r_1O, j_{22} = x_2\tau_2g_2' + \tau_2g_2(x_2) - yp_2'(x_2) - q_2x_1 - r_2O \) and \( j_{33} = -s(y) - ys'(y) + c_1p_1(x_1) + c_2p_2(x_2) - r_3O \).

(i) The equilibrium \( E_0 \) is unstable.

(ii) \( E_1(\hat{x}_1, 0, 0, E^*, O^*) \) is stable if \( (\tau_2g_2(0) - \hat{x}_1q_2 - r_2O^*) < 0 \) and \( (-s(0) + c_1p_1(\hat{x}_1) - r_3O^*) < 0 \).

(iii) \( E_2(0, \hat{x}_2, 0, E^*, O^*) \) is stable if \( (\tau_1g_1(0) - \hat{x}_2q_1 - r_1O^*) < 0 \) and \( (-s(0) + c_2p_2(\hat{x}_2) - r_3O^*) < 0 \).

(iv) \( E_3(l_1, l_2, 0, E^*, O^*) \) is stable if \( \tau_1\tau_2g'_1(l_1)g'_2(l_2) - q_1q_2 > 0 \) and \( (-s(0) + c_1p_1(l_1) + c_2p_2(l_2) - r_3O^*) < 0 \).

(v) \( E_4(x_4, 0, y_4, E^*, O^*) \) is stable if \( j_{22}(4) < 0, j_{11}(4) + j_{33}(4) < 0 \) and \( j_{11}(4)j_{33}(4) + p_1(x_4)(c_1y_4p'_1(x_4)) > 0 \),

where \( j_{22}(4) = x_2\tau_2g_2(0) - y_4p_2'(0) - q_2x_4 - r_2O^*, j_{11}(4) = x_4\tau_1g_1'(x_4) + \tau_1g_1(x_4) - y_4p_1'(x_4) - r_1O^* \) and \( j_{33}(4) = -y_4s'(y_4) \).
(vi) $E_5$ is stable if $j_{11}(5) < 0, j_{22}(5) + j_{33}(5) < 0$ and $j_{22}(5).j_{33}(5) + p_2(x_5).c_2y_5p_2(x_5) > 0$,
where, $j_{11}(5) = r_1g_1(0) - q_1x_5 - r_1O^*, j_{22}(5) = x_5r_2g_2(x_5) + r_2g_2(x_5) - y_5p_2(x_5) - r_2O^*$ and $j_{33}(5) = -y_5s'(y_5)$.

**Theorem 6.1.1.** The equilibrium $E_6(x^*_1, x^*_2, y^*, E^*, O^*)$ is asymptotically stable in $R^4_{(x_1, x_2, E, O)}$ if $a_{11}, a_{22} < 0, a_{12}a_{21} < \min\{a_{11}a_{22}, a_{13}a_{31}, a_{23}a_{32}\}$, $a_{21}a_{32} > \max\{a_{31}a_{22}, a_{12}a_{23}\}$ and $a_{11}a_{32} - a_{12}a_{31} < 0$, where

$a_{11} = x_1^r1g_1(x_1^r) + r_1g_1(x_1^r) - y^*p_1(x_1^r) - q_1x_2^r - r_1O^*$, $a_{12} = -q_1x_1^r$, $a_{13} = -p_1(x_1^r)$, $a_{21} = -q_2x_2^r$, $a_{22} = x_2^r\tau_2g_2(x_2^r) + r_2g_2(x_2^r) - y^*p_2(x_2^r) - q_2x_1^r - r_2O^*$, $a_{23} = -p_2(x_2^r) + c_1y^*p_1(x_1^r)$, $a_{32} = c_2y^*p_2(x_2^r)$ and $a_{33} = -y^s'(y^*)$.

**Proof:** The jacobian corresponding to $E_6$ is given by:

$$J(6) = \begin{bmatrix}
    j_{11}(6) & -q_1x_1^r & -p_1(x_1^r) & 0 & -r_1x_1^r \\
    -q_2x_2^r & j_{22}(6) & -p_2(x_2^r) & 0 & -r_2x_2^r \\
    c_1y^*p_1(x_1^r) & c_2y^*p_2(x_2^r) & -y^s'(y^*) & 0 & -r_3y^s \\
    0 & 0 & 0 & -E^*h'(E^*) - h(E^*) & 0 \\
    0 & 0 & 0 & p'(E^*) & -d_1
\end{bmatrix}$$

where, $j_{11}(6) = x_1^r1g_1(x_1^r) + r_1g_1(x_1^r) - y^*p_1(x_1^r) - q_1x_2^r - r_1O^*$, $j_{22}(6) = x_2^r\tau_2g_2(x_2^r) + r_2g_2(x_2^r) - y^*p_2(x_2^r) - q_2x_1^r - r_2O^*$.

$j(6)$ can be reduced to $\begin{bmatrix}
    A & C \\
    0 & B
\end{bmatrix}$.

In $E, O$ directions the eigen values are negative.

Let $A = \begin{bmatrix}
    a_{11} & a_{12} & a_{13} \\
    a_{21} & a_{22} & a_{23} \\
    a_{31} & a_{32} & a_{33}
\end{bmatrix}$,

where $a_{11} = x_1^r1g_1(x_1^r) + r_1g_1(x_1^r) - y^*p_1(x_1^r) - q_1x_2^r - r_1O^*$, $a_{12} = -q_1x_1^r$, $a_{13} = -p_1(x_1^r)$, $a_{21} = -q_2x_2^r$, $a_{22} = x_2^r\tau_2g_2(x_2^r) + r_2g_2(x_2^r) - y^*p_2(x_2^r) - q_2x_1^r - r_2O^*$, $a_{23} = -p_2(x_2^r)$, $a_{31} = c_1y^*p_1(x_1^r)$, $a_{32} = c_2y^*p_2(x_2^r)$ and $a_{33} = -y^s'(y^*)$.

The characteristic equation of $j_6$ is $\lambda^3 + \lambda^2A_1 + \lambda A_2 + A_3 = 0$,

where $A_1 = - (a_{11} + a_{22} + a_{33})$, $A_2 = a_{22}a_{33} + a_{11}a_{22} + a_{11}a_{33} - a_{12}a_{21} - a_{23}a_{32} - a_{13}a_{31}$ and $A_3 = a_{12}a_{23}a_{31} + a_{11}a_{23}a_{32} + a_{13}a_{31}a_{22} - a_{11}a_{22}a_{33} - a_{12}a_{31}a_{23} - a_{13}a_{21}a_{32}$.  

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According to Routh-Hurwitz Criteria Ahmad and Rao [1] the equilibrium \( E_6(x^*_1, x^*_2, y^*, E^*, O^*) \) is locally asymptotically stable when \( A_1 > 0, A_3 > 0 \) and \( A_1 A_2 > A_3 \).

Hence \( E_6(x^*_1, x^*_2, y^*, E^*, O^*) \) is locally asymptotically stable.

### 6.2. Global Stability

\( E_0(0, 0, 0, E^*, O^*) \) is globally stable in \( R^+_{0,0,y,E,O} \).

**Theorem 6.2.1.** Let \( \xi_1 = (\tau_1 g_1(0) - \hat{x}_2 q_1 - r_1 O^*) \) and \( \xi_2 = (\tau_2 g_2(0) - \hat{x}_1 q_2 - r_2 O^*) \). Then

**Case I:** If \( \xi_1 < 0, \xi_2 < 0 \) then \( E_3 \) exists and is a saddle in \( R^+_{x_1,x_2,0,E,O} \).

**Case II:** If \( \xi_1 > 0, \xi_2 > 0 \) then \( E_3 \) exists and is globally stable in \( R^+_{x_1,x_2,0,E,O} \).

**Case III:** If \( \xi_1 > 0, \xi_2 < 0 \) then \( E_1 \) is globally stable in \( R^+_{x_1,x_2,0,E,O} \).

**Case IV:** If \( \xi_1 < 0, \xi_2 > 0 \) then \( E_2 \) is globally stable in \( R^+_{x_1,x_2,0,E,O} \).

**Proof:** Two competing prey, system is:

\[
\begin{align*}
  x_1' &= x_1 \tau_1 g_1(x_1) - q_1 x_1 x_2 - r_1 O^* x_1 \\
  x_2' &= x_2 \tau_2 g_2(x_2) - q_2 x_1 x_2 - r_2 O^* x_2
\end{align*}
\]

and jacobian matrix is as follows:

\[
M(x_1, x_2) = \begin{bmatrix}
m_{11} & -q_1 x_1 \\
-q_2 x_2 & m_{22}
\end{bmatrix}
\]

where \( m_{11} = \tau_1 g_1(x_1) - q_1 x_2 - r_1 O^* + x_1 \tau_1 g'_1(x_1), m_{22} = \tau_2 g_2(x_2) - q_2 x_1 - r_2 O^* + x_2 \tau_2 g'_2(x_2) \).

\[
M(\hat{x}_1, 0) = \begin{bmatrix}
  \hat{x}_1 \tau_1 g'_1(\hat{x}_1) & -q_1 \hat{x}_1 \\
  0 & \tau_2 g_2(0) - q_2 \hat{x}_1 - r_2 O^*
\end{bmatrix}
\]

\[
M(0, \hat{x}_2) = \begin{bmatrix}
  \tau_1 g_1(0) - q_1 \hat{x}_2 - r_1 O^* & 0 \\
  -q_2 \hat{x}_2 & \hat{x}_2 \tau_2 g'_2(\hat{x}_2)
\end{bmatrix}
\]

and \( M(l_1, l_2) = \begin{bmatrix}
l_1 \tau_1 g'_1(l_1) & -q_1 l_1 \\
-q_2 l_2 & l_2 \tau_2 g'_2(l_2)
\end{bmatrix} \).
Here \( tr(M(l_1, l_2)) < 0 \) and \( \text{det}(M(l_1, l_2)) = l_1 l_2 \tau_1 g'_1(l_1) \tau_2 g'_2(l_2) - q_1 l_1 q_2 l_2 \).

Now for \( E_3 \) we have

\[
(6.1) \quad \tau_1 g_1(x_1) - q_1 x_2 - r_1 O^* = 0
\]

\[
(6.2) \quad \tau_2 g_2(x_2) - q_2 x_1 - r_2 O^* = 0
\]

When we draw isoclines for these two equations, (6.1) intersects positive \( x_1 \)-axis at \((\hat{x}_1, 0)\) and positive \( x_2 \)-axis at \((0, \frac{\tau_1 g_1(0) - r_1 O^*}{q_1})\) while (6.2) intersects positive \( x_2 \)-axis at \((0, \hat{x}_2)\) and positive \( x_1 \)-axis at \((\frac{\tau_2 g_2(0) - r_2 O^*}{q_2}, 0)\).

**Case I:** When \( \frac{\tau_2 g_2(0) - r_2 O^*}{q_2} < \hat{x}_1 \) and \( \frac{\tau_1 g_1(0) - r_1 O^*}{q_1} < \hat{x}_2 \), \( E_1 \) and \( E_2 \) are locally stable hence \( E_3 \) cannot be globally stable in place of \( E_3 \) is unstable.

We have phase plane as follows:

**Figure 1:** \( \frac{\tau_2 g_2(0) - r_2 O^*}{q_2} < \hat{x}_1 \) and \( \frac{\tau_1 g_1(0) - r_1 O^*}{q_1} < \hat{x}_2 \), \( E_3 \) (interior equilibrium) is unstable, black curve with arrow is separatrix.
Case II: When $\frac{\tau g_2(0) - r_2 O^*}{q_2} > \hat{x}_1$ and $\frac{\tau g_1(0) - r_1 O^*}{q_1} > \hat{x}_2$ from phase plane analysis we have $E_3$ is globally stable.

![Figure 2:](image)

Case III: When $\frac{\tau g_2(0) - r_2 O^*}{q_2} < \hat{x}_1$ and $\frac{\tau g_1(0) - r_1 O^*}{q_1} > \hat{x}_2$, $E_1$ is globally stable and $E_3$ does not exist.

![Figure 3:](image)
In this case there is no interior equilibrium and we have $E_1$ globally stable.

**Case IV:** When $\frac{r_2 q_2}{q_2} > \hat{x}_1$ and $\frac{r_1 q_1}{q_1} < \hat{x}_2$

In this case we also get nonexistence of interior equilibrium while $E_2$ is globally stable.

**Theorem 6.2.2.** If $E_4$ is locally asymptotically stable with

(i) $c_1 p_1(x_1) x_1 - s(y) y > 0 \forall x_1 > 0, y > 0$

(ii) $-s(0) + c_1 p_1(\hat{x}_1) - r_3 O^* > 0$ and $E_1(\hat{x}_1, 0, 0)$ exists

then $E_4$ is globally asymptotically stable in $\mathbb{R}^+ x_1, 0, y, E, O$.

**Proof:** We have system (1) in absence of $x_2$

$$x_1' = x_1 r_1 q_1(x_1) - y p_1(x_1) - r_3 O^* x_1$$

$$y' = y(-s(y) + c_1 p_1(x_1) - r_3 O^*)$$

Then by **Theorem 3 in Cheng et.al.**[5] $E_4$ is globally stable when (i) and (ii) holds.

Similarly we can find criterion for global stability of $E_5$.

**Theorem 6.2.3.** If $E_5$ is locally asymptotically stable with
\((i)\) \(c_2p_2'(x_2)x_2 - s'(y)y > 0 \ \forall \ x_2 > 0, y > 0\)

\((ii)\) \(-s(0) + c_2p_2(\dot{x}_2) - r_3O^* > 0\) and \(E_2(0, \dot{x}_2, 0)\) exists

then \(E_5\) is globally asymptotically stable in \(R^+_0, x_2, y, E, O\).

**Proof:** Similar to previous theorem.

Now we derive results for the global stability of interior equilibrium \(E_6\) with the help of Lyapunov function.

In view of Sinha et.al. [18] instead of the system (1) we consider system of the following form:

\[
\begin{align*}
  x_1' &= x_1 \tau_1 g_1(x_1) - q_1x_1x_2 - yp_1(x_1) - r_1O^*x_1 \\
  x_2' &= x_2 \tau_2 g_2(x_2) - q_2x_1x_2 - yp_2(x_2) - r_2O^*x_2 \\
  y' &= y(-s(y) + c_1p_1(x_1) + c_2p_2(x_2) - r_3O^*)
\end{align*}
\]

(6.3)

**Theorem 6.2.4.** Suppose \(\bar{A}\) is positive definite matrix in the interior of \(S\) where \(\bar{A} = -A, A = \begin{bmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{bmatrix}\), \(v_{ii} = \bar{v}_{ii}\) and \(\bar{v}_{ij} = \bar{v}_{ji} = \frac{v_{ij} + v_{ji}}{2}\)

for \(j \neq i\)

\[
\begin{align*}
v_{11} &= -\frac{c_1(p_1(x_1) - p_1(x_1^*))}{p_1(x_1)(x_1 - x_1^*)}, \\
v_{12} &= \frac{q_1x_1c_1^1(p_1(x_1) - p_1(x_1^*))}{p_1(x_1)(x_1 - x_1^*)}, \\
v_{13} &= \frac{c_1^1(p_1(x_1) - p_1(x_1^*))}{x_1 - x_1^*}, \\
v_{21} &= -q_2, \\
v_{22} &= \frac{x_2 \tau_2 g_2(x_2) - q_2x_1x_2 - yp_2(x_2) - r_2O^*x_2}{(x_2 - x_2^*)x_2}, \\
v_{23} &= -\frac{p_2(x_2)}{x_2}, \\
v_{31} &= c_1^1(p_1(x_1) - p_1(x_1^*)), \\
v_{32} &= \frac{c_2^1(p_2(x_2) - p_2(x_2^*))}{x_2 - x_2^*}
\end{align*}
\]

\(-s(y) - s(y^*)\) then \(E_6\) is globally stable in \(R^+_0, x_2, y, E, O\).

**Proof:** Let \(V(x_1, x_2, y) = V_{x_1} + V_{x_2} + V_y\) is the Lyapunov function.

Where, \(V_{x_1} = \int_{x_1}^{x_1^*} \frac{-s(y^*) + c_1p_1(\xi) + c_2p_2(x_2^*) - r_3O^*}{p_1(\xi)} d\xi, V_{x_2}\)
\[
\frac{dV}{dt} = \frac{c_1(p_1(x_1) - p_1(x_1^*))}{p_1(x_1)}(x_1 r_1 g_1(x_1) - q_1 x_1 x_2 - y p_1(x_1) - r_1 O^* x_1) - \frac{c_1(p_1(x_1) - p_1(x_1^*))}{c_1(p_1(x_1) - p_1(x_1^*))}(y^* p_1(x_1) - y p_1(x_1)).
\]

Let \( \frac{dV}{dt} = (x_1 - x_1^*)^2 v_{11} + (x_1 - x_1^*)(x_2 - x_2^*) v_{12} + (x_1 - x_1^*)(y - y^*) v_{13}. \)

Similarly let \( \frac{dV}{dt} = (x_2 - x_2^*)^2 v_{22} + (x_2 - x_2^*)(x_1 - x_1^*) v_{21} + (x_2 - x_2^*)(y - y^*) v_{23} \) and \( \frac{dV}{dt} = (y - y^*)^2 v_{33} + (y - y^*)(x_1 - x_1^*) v_{31} + (y - y^*)(x_2 - x_2^*) v_{32} \)

where \( v_{ij} \) for \( i, j = 1, 2 \) and \( 3 \) are defined as above.

Let \( X^T = \begin{bmatrix} (x_1 - x_1^*) \\ (x_2 - x_2^*) \\ (y - y^*) \end{bmatrix} \) and \( A = \begin{bmatrix} \bar{v}_{11} & \bar{v}_{12} & \bar{v}_{13} \\ \bar{v}_{21} & \bar{v}_{22} & \bar{v}_{23} \\ \bar{v}_{31} & \bar{v}_{32} & \bar{v}_{33} \end{bmatrix} \)

where \( \bar{v}_{ii} = v_{ii} \) and \( \bar{v}_{ij} = v_{ji} = \frac{(v_{ij} + v_{ji})}{2} \) for \( j \neq i \).

We have \( \frac{dV}{dt} = XAX^T = -X\bar{A}X^T (A = -\bar{A} \text{ is a symmetric matrix}) \).
If the matrix $\bar{A}$ is positive definite in $S$ then $\frac{dV}{dt} \leq 0$ in $S$ and the equilibrium $E_6(x_1^*, x_2^*, y^*)$ is globally stable in $R_{x_1, x_2, y}^+$. 

Wolkowicz and Lu [13].

As $E(t)$ and $O(t)$ tends to $E^*$ and $O^*$ respectively, $E_6(x^*, y^*, E^*, O^*)$ is globally stable in $R_{x_1, x_2, y, E, O}^+$. 

7. Persistence

Here we obtain a result for the uniform persistence of the system (2.1). Freedman and Waltman [7] obtain conditions for persistence of a Kolmogorov system of two prey and one predator. Recall a system $x' = f(x)$, $x = (x_1, x_2, x_3)^T$ is said to be persist uniformly if $\exists \delta > 0$ such that for $x_i(0) > 0, \liminf_{t \to \infty} x_i(t) > \delta \forall i = 1, 2, \ldots, n$.

**Theorem 7.1.** Let equilibria $E_3, E_4, E_5$ exist and be globally stable in their respective domains with

\begin{equation}
-s(0) + c_1p_1(l_1) + c_2p_2(l_2) - r_3O^* > 0
\end{equation}

then system (1) will persist uniformly.

**Proof:** We will prove our results with the help of the Average Lyapunov function Huston [12]. Let $v = x_1^2 x_2^2 y^2$ where $\alpha, \beta, \gamma$ are assumed to be positive. Now $\frac{dv}{dt} = \frac{\alpha}{x_1} \frac{dx_1}{dt} + \frac{\beta}{x_2} \frac{dx_2}{dt} + \frac{\gamma}{y} \frac{dy}{dt}$.

Along solutions of system (4)

\begin{equation}
\frac{dv}{dt} = \alpha(\tau_1g_1(x_1) - q_1 x_2 - \frac{wp_1(x_1)}{x_1} - \tau_1 O^*) + \beta(\tau_2g_2(x_2) - q_2 x_1 - \frac{wp_2(x_2)}{x_2} - \tau_2 O^*) + \gamma(-s(y) + c_1p_1(x_1) + c_2p_2(x_2) - r_3O^*)
\end{equation}

Let $\omega = \frac{1}{v} \frac{dv}{dt}$

Now we have to show that $\omega > 0$ for all boundary equilibria. Here we have six boundary equilibria $E_0, E_1, E_2, E_3, E_4$ and $E_5$. Thus $\omega > 0$ has to satisfy the following conditions corresponding to $E_0, E_1, E_2, E_3, E_4$ and $E_5$. 
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(i) At $E_0: \omega = \alpha(\tau_1 g_1(0) - r_1 O^*) + \beta(\tau_2 g_2(0) - r_2 O^*) + \alpha(-s(0) - r_3 O^*) > 0$.

(ii) At $E_1: \omega = \beta(\tau_2 g_2(0) - q_2 x_1 - r_2 O^*) + \gamma(-s(0) + c_1 p_1(x_1) - r_3 O^*) > 0$.

(iii) At $E_2: \omega = \alpha(\tau_1 g_1(0) - q_1 x_2 - r_1 O^*) + \gamma(-s(0) + c_2 p_2(x_2) - r_3 O^*) > 0$.

(iv) At $E_3: \omega = \gamma(-s(0) + c_1 p_1(l_1) + c_2 p_2(l_2) - r_3 O^*) > 0$.

(v) At $E_4: \omega = \alpha(\tau_1 g_1(x_4) - \frac{\mu p_1(x_4)}{x_4} - r_1 O^*) + \beta(\tau_2 g_2(0) - q_2 x_4 - r_2 O^*) + \gamma(-s(y_4) + c_1 p_1(x_4) - r_3 O^*) \implies \omega = \beta(\tau_2 g_2(0) - q_2 x_4 - r_2 O^*) > 0$.

(vi) At $E_5: \omega = \alpha(\tau_1 g_1(0) - q_1 x_5 - r_1 O^*) + \beta(\tau_2 g_2(x_5) - \frac{q_2 p_2(x_5)}{x_5} - r_2 O^*) + \gamma(-s(y_5) + c_2 p_2(x_5) - r_3 O^*) \implies \omega = \alpha(\tau_1 g_1(0) - q_1 x_5 - r_1 O^*) > 0$.

If we choose $\alpha, \beta$ sufficiently large with $(-s(0) + c_1 p_1(l_1) + c_2 p_2(l_2) - r_3 O^*) > 0$ then conditions (i)-(vi) are satisfied and by Lawaniya et al. [14] using the fact that for large $t, O(t) \leq O^* + \epsilon$ and standard comparison theorem we conclude that system (1) is uniformly persistent.

8. Bifurcation

As $E$ and $O$ tend to $E^*$ and $O^*$ respectively by Lawaniya et al. [14], we consider two and three species submodel of system (1) for studying Hopf Bifurcation.

Theorem 8.1. Suppose $E_4(x_4, 0, y_4, E^*, O^*)$ exists in an open interval containing $\tau_{1hf} > 0$, then in the absence of prey $x_2$ system (1) experiences Hopf Bifurcation and periodic orbit is formed around its boundary equilibrium $E_4$ as $\tau_1$ passes through $\tau_{1hf}$, whenever $y_4(s'(y_4))^2 < c_1 p_1(x_4) p_1'(x_4)$ and $\frac{d}{d\tau_1}(j_{11}(4) + j_{33}(4)) \neq 0$ at $\tau_1 = \tau_{1hf}$, where, $j_{11}(4) = x_4 \tau_1 g_1'(x_4) + \tau_1 g_1(x_4) - y_4 p_1'(x_4) - r_1 O^*$ and $j_{33}(4) = -y_4 s'(y_4)$.

Proof: For jacobian we have submatrix

$$A = \begin{bmatrix} j_{11}(4) & -p_1(x_4) \\ c_1 y_4 p_1'(x_4) & j_{33}(4) \end{bmatrix}$$

The characteristic equation for $A$ is $Q(\lambda) \equiv \lambda^2 + a_1 \lambda + a_2 = 0$, where $a_1 = -(j_{11}(4) + j_{33}(4))$ and $a_2 = j_{11}(4) j_{33}(4) + c_1 p_1(x_4) y_4 p_1'(x_4)$.
at \( \tau_1 = \tau_{1hf} \) if \( A \) possesses purely imaginary eigen values then we have
\( j_{11}(4) + j_{33}(4) = 0 \) and \( j_{11}(4), j_{33}(4) + c_1p_1(x_4)y_4p'_1(x_4) > 0 \).

\( \Rightarrow a_1 = 0 \) and \( a_2 = -(j_{33}(4))^2 + c_1p_1(x_4)y_4p'_1(x_4) \)

\( \Rightarrow y_4(s'(y_4))^2 < c_1p_1(x_4)p'_1(x_4) \).

Let two eigenvalues be \( \lambda = \chi \pm i\psi \)

Putting \( \lambda = \chi + iv \) in characteristic equation of \( A \) and differentiating it with respect to \( \tau_1 \)

\( (2\chi + a_1)\frac{d\chi}{d\tau_1} + (-2\psi)\frac{d\psi}{d\tau_1} = -\chi \frac{d\psi}{d\tau_1} - \frac{d\chi}{d\tau_1} \) and \( (2\chi + a_1)\frac{d\psi}{d\tau_1} + 2\psi \frac{d\chi}{d\tau_1} = -\psi \frac{d\chi}{d\tau_1} \).

At \( \tau_1 = \tau_{1hf} \), \( \psi^2 = a_2 \) and \( a_1 = 0 \).

After simplifying we get

\( \left. \frac{d\chi}{d\tau_1} \right|_{\tau_{1hf}} = \left. -\frac{1}{2}\left( \frac{d\psi}{d\tau_1} \right) \right|_{\tau_{1hf}} = \frac{1}{2}\left( \frac{d}{d\tau_1}(j_{11}(4) + j_{33}(4)) \right) \).

If \( \left. \frac{d}{d\tau_1}(j_{11}(4) + j_{33}(4)) \right|_{\tau_{1hf}} \neq 0 \) (transversality condition) hence this provides our result, system experienced Hopf Bifurcation and periodic orbits are formed by Hassard et al. [8].

**Theorem 8.2.** Assume that \( E_5 \) exists in an open interval containing \( \tau_{2hf} > 0 \), then in the absence of prey \( x_1 \), the system (1) experiences Hopf Bifurcation and periodic orbit is formed around its boundary equilibrium \( E_5 \) as \( \tau_2 \) passes through \( \tau_{2hf} \) whenever \( y_5(s'(y_5))^2 < c_2p_2(x_5)p'_2(x_5) \) and

\( \left. \frac{d}{d\tau_2}(j_{22}(5) + j_{33}(5)) \right|_{\tau_{2hf}} \neq 0 \) at \( \tau_2 = \tau_{2hf} \),

where, \( j_{22}(5) = x_5\tau_2g'_2(x_5) + \tau_2g_2(x_5) - y_5p'_2(x_5) - r_2O^* \) and \( j_{33}(5) = -y_5s'(y_5) \).

**Proof:** Proceed as Theorem 8.1.

**For interior Equilibrium**

Assume that in an open interval containing \( \tau_2 = \tau_{2HF} > 0 \) interior equilibrium exists.

**Theorem 8.3.** Suppose there exists \( \tau_2 = \tau_{2HF} > 0 \) such that \( n_i(\tau_{2HF}) > 0, 1 \leq i \leq 3, \Delta_2(\tau_{2HF}) = 0 \) and

\( \left. \left( \frac{dn_1}{d\tau_2} - (n_2 \frac{dn_1}{d\tau_2} + n_3 \frac{dn_2}{d\tau_2}) \right) \right|_{\tau_{2HF}} \neq 0 \)

where, \( \Delta_2 = \det \begin{bmatrix} n_1 & 1 & 1 \\ n_2 & n_3 & n_2 \end{bmatrix} \) and \( n_i \) for all \( i = 1, 2, 3 \) are defined as above and evaluated at \( E_6 \).

Then system experiences Hopf Bifurcation and periodic orbits are formed.
Proof: The jacobian for system is as follows:

\[ J = \begin{bmatrix} j_{11} & -q_1x_1 & -p_1(x_1) \\ -q_2x_2 & j_{22} & -p_2(x_2) \\ c_{13}y'^1(x_1) & c_{23}y'^2(x_2) & j_{33} \end{bmatrix} \]

The characteristic polynomial is

\[ P(\lambda) = \lambda^3 + n_1\lambda^2 + n_2\lambda + n_3 \]

where, \(n_1 = -(j_{11} + j_{22} + j_{33}), n_2 = j_{11}j_{22} + j_{11}j_{33} + j_{22}j_{33} - q_1q_2x_1x_2 + c_{23}y'^2(x_2)p_2(x_2) + c_{13}y'^1(x_1)p_1(x_1), n_3 = -j_{11}j_{22}j_{33} - j_{11}c_{23}y'^2(x_2)p_2(x_2) + q_1q_2x_1j_{33} - q_1x_1c_{13}y'^1(x_1)p_2(x_2) - q_2x_2c_{23}y'^2(x_2)p_1(x_1) - j_{22}c_{13}y'^1(x_1)p_1(x_1), j_{11} = x_1\tau g'_1(x_1) + \tau g_1(x_1) - y'^1(x_1) + q_1x_2 - r_1O^*, j_{22} = x_2\tau g'_2(x_2) + \tau g_2(x_2) - y'^2(x_2) - q_2x_1 - r_2O^*, j_{33} = -s(y) - ys'(y) + c_1p_1(x_1) + c_2p_2(x_2) - r_3O^* \]

and \(n_i's\) are evaluated at \(E_0\) and are functions of \(\tau_2\).

We claim that \(\Delta_2(\tau_{2HF}) = 0\) if and only if \(P(\lambda)\) has a pair of purely imaginary roots. First let \(\Delta_2(\tau_{2HF}) = 0\), then by Orlando formula \(P(\lambda)\) has a pair of roots with opposite signs.

Suppose the eigen values are real, \(\lambda_1\) and \(-\lambda_1\) then by characteristic equation

\[
P(\lambda_1) \equiv \lambda^3_1 + n_1\lambda^2_1 + n_2\lambda_1 + n_3 = 0 \]

and \(P(-\lambda_1) \equiv -\lambda^3_1 + n_1\lambda^2_1 - n_2\lambda_1 + n_3 = 0 \)

\[
\Rightarrow n_1\lambda^2_1 + n_3 = 0
\]

But this contradicts the fact since \(n_i > 0\).

Hence there will be pair of purely imaginary roots of \(P(\lambda) = 0\) at \(\tau_{2HF}\).

Conversely let \(P(\lambda)\) has imaginary roots \(i\lambda_1\) and \(-i\lambda_1\) \((\lambda_1 \in R)\) for \(\tau_2 = \tau_{2HF}\).

Then \(P(i\lambda_1) = 0, \)

i.e. \(-i\lambda^3_1 - n_1\lambda^2_1 + in_2\lambda_1 + n_3 = 0 \)

or \(\lambda^3_1 - n_2\lambda_1 = 0\) and \(n_1\lambda^2_1 - n_3 = 0 \Rightarrow \lambda^2_1 = n_2 \)

thus \(n_1n_2 = n_3 = 0\) i.e. \(\Delta_2(\tau_{2HF}) = 0\)

Also the third root is negative.

Let the one complex root be \(\lambda_1 = \alpha + i\beta \)

then we are to prove that \(\text{Re}(\frac{d\lambda_1}{d\tau_2})|_{\tau_{2HF}} \neq 0\).

Setting \(\lambda = \alpha + i\beta\) in (6) and differentiating with respect to \(\tau_2\),

\[
\frac{d\alpha}{d\tau_2}(3(\alpha^2 - \beta^2) + 2\alpha n_1 + n_2) + \frac{d\beta}{d\tau_2}(-6\alpha\beta - 2\beta n_1) = \frac{dn_1}{d\tau_2}(\beta^2 - \alpha^2) - \alpha \frac{dn_2}{d\tau_2} - \frac{dn_3}{d\tau_2}
\]

(8.2)
\[
\frac{d\alpha}{d\tau^2}(6\alpha\beta + 2\beta n_1) + \frac{d\beta}{d\tau^2}(3(\alpha^2 - \beta^2) + 2\alpha n_1 + n_2) = -2\alpha\beta \frac{dn_1}{d\tau^2} - \beta \frac{dn_2}{d\tau^2}
\]

(8.3)

From (8.2) and (8.3) we get
\[
\frac{d\alpha}{d\tau^2}(3(\alpha^2 - \beta^2) + 2\alpha n_1 + n_2) + (6\alpha\beta + 2\beta n_1)^2
\]
\[
-(6\alpha\beta + 2\beta n_1)(2\alpha\beta \frac{dn_1}{d\tau^2} + \beta \frac{dn_2}{d\tau^2})
\]

At \(\tau = \tau_{HF}\), \(\beta = n_2\) and \(\alpha = 0\) \(\Rightarrow \frac{d\alpha}{d\tau^2}\bigg|_{\tau_{HF}} = \left[\frac{\frac{dn_3}{d\tau^2} - (n_2 \frac{dn_1}{d\tau^2} + n_1 \frac{dn_2}{d\tau^2})}{2(n_2 + n_1)}\right]^{\tau_{HF}}\).

Hence if \(\left(\frac{dn_3}{d\tau^2} - (n_2 \frac{dn_1}{d\tau^2} + n_1 \frac{dn_2}{d\tau^2})\right)\bigg|_{\tau_{HF}} \neq 0\) then by Hassard et.al.\[8\] system experiences Hopf Bifurcation and periodic orbits are formed.

9. Numerical Simulation

The numerical simulation has been carried out with MATLAB 2010a.

(9.1) \(x' = x_1 \tau_1 g_1(x_1) - q_1 x_1 x_2 - y p_1(x_1) - r_1 O x_1\)
(9.2) \(x'_2 = x_2 \tau_2 g_2(x_2) - q_2 x_1 x_2 - y p_2(x_2) - r_2 O x_2\)
(9.3) \(y' = y(-s(y) + c_1 p_1(x_1) + c_2 p_2(x_2) - r_3 O)\)
(9.4) \(E' = -E h(E) + Q E\)
(9.5) \(O' = -d_1 O + p(E)\)

Taking following set of functions:
\(g_i(x_i) = (1 - (x_i/K_i)^3), p_i(x_i) = \frac{x_i}{x_i + m_i} \forall i = 1, 2, s(y) = c(d + \frac{y}{w}), h(E) = a\)
and \(p(E) = l + m E,\)
and the following values of parameters: \(\tau_1 = \tau_2 = 0.5, K_1 = 900, K_2 = 950, m_1 = 800, m_2 = 750, q_1 = 0.00002, q_2 = 0.00001, c = 0.003, d = 75, w = 75, r_1 = 0.004, r_2 = 0.005, r_3 = 0.006, c_1 = 0.6, c_2 = 0.7\)
Stage I: Pollution parameters $Q_E = 0.4, a = 1, d_1 = 0.5, l = 0.0234, m = 0.04$

![Figure 5: System at very low level of Pollution](image)

We get $\tau_1 g_1(0) - r_1 O^* > 0, \tau_2 g_2(0) - r_2 O^* > 0, 0 < x_4^* < l_1 < \hat{x}_1, 0 < x_5^* < l_2 < \hat{x}_2$. Here all conditions needed for existence of equilibria are satisfied.

Stage II: Pollution parameters, $Q_E = 10, a = 0.35, d_1 = 0.5, l = 0.6, m = 0.5$

![Figure 6: System after a little increase in Pollution level](image)
We also get \( \tau_1 g_1(0) - r_1 O^* > 0, \tau_2 g_2(0) - r_2 O^* > 0, 0 < x_1^* < x_4 < l_1 < \hat{x}_1, 0 < x_2^* < x_5 < l_2 < \hat{x}_2 \), hence all equilibria exist.

**Stage III:** Pollution parameters, \( Q_E = 30, a = 0.35, d_1 = 0.5, l = 0.6, m = 0.5 \)

![Figure 7: System after a little increase in Pollution level](image1)

We also get \( \tau_1 g_1(0) - r_1 O^* > 0, \tau_2 g_2(0) - r_2 O^* > 0, 0 < x_1^* < x_4 = l_1 < \hat{x}_1, 0 < x_2^* < x_5 = l_2 < \hat{x}_2 \), hence \( E_1, E_2, E_3 \) exist but \( E_4, E_5, E_6 \) do not exist.

**Stage IV:** Pollution parameters, \( Q_E = 35, a = 0.35, d_1 = 0.5, l = 0.7, m = 0.5 \)

![Figure 8: System at very large increase in level of Pollution](image2)
Here we also get \( \tau_1 g_1(0) - r_1 O^* > 0, \tau_2 g_2(0) - r_2 O^* > 0, 0 < x_1^* < x_4 = l_1 < \hat{x}_1, 0 < x_2^* < x_5 = l_2 < \hat{x}_2 \), hence \( E_1, E_2, E_3 \) exist but \( E_4, E_5, E_6 \) do not exist.

**Stage V:** Pollution parameters, \( Q_E = 45, c = 0.35, d_1 = 0.5, l = 0.7, m = 0.5 \)

![Figure 9: System at hazardous level of Pollution](image)

Here none of the conditions for existence of equilibria is satisfied, hence no equilibrium except \( E_0 \) exists.

**For Bifurcation: (i) In absence of Prey 2**

Taking \( \tau_1 \) as Bifurcation parameter and keeping other as follows
\( K_1 = 1000, m_1 = 700, q_1 = 0.00002, c = 0.003, d = 70, w = 50, r_1 = 0.05, r_3 = 0.08, c_1 = 0.64, a = 0.4, b = 1, d_1 = 0.5, l = 0.0234, m = 0.04 \).

We get the graphical structure as follows:

![Figure 10: The Predator and Prey 1 population with \( \tau_1 = 0.7 \)](image)
Here we have $E_4 = (472.3877, 0, 731.2307, 0.04, 0.05)$ and eigenvalues of jacobian matrix are $-0.0070 + 0.3077i, -0.0070 - 0.3077i$ and $E_4$ is stable.

At $\tau_1 = 0.5$ we get $E_4 = (431.1909, 0, 499.2866)$ and eigenvalues of jacobian matrix are $0.0850 + 0.2311i, 0.0850 - 0.2311i$ and $E_4$ is unstable.

When $\tau_1 = 0.640625$, the behavior of system is followed by Fig. 12 and Fig. 13.
Here the periodic orbit arises and we get eigenvalues of jacobian matrix are $0.0001 + i8.2083$, $0.0001 - i8.2083$ with $rac{d}{d\tau_1}(j_{11}(4) + j_{33}(4)) = 0.006084 \neq 0$ and $y_1(s'(y_4))^2 < c_1 p_1(x_4)p_1'(x_4)$.

Hence $\tau_{hf} \approx 0.640625$ and when $\tau_1 > \tau_{hf}$ the system shows a stable equilibrium point and below the critical value equilibrium point it is unstable.

(ii) In absence of Prey 1

Now taking $\tau_2$ as bifurcation parameter and values of other parameters as follows:

$K_2 = 950, m_2 = 750, q_2 = 0.00001, c = 0.003, d = 70, w = 50, r_2 = 0.06, r_3 = 0.08, c_2 = 0.7, a = 0.4, b = 1, d_1 = 0.5, l = 0.0234, m = 0.04$,

At $\tau_2 = 0.65$
Figure 14: Predator and Prey 2 population at $\tau_2 = 0.65$

Figure 15: Predator and Prey 2 system at $\tau_2 = 0.65$
We get equilibrium is $E_5 = (0, 431.2260, 692.4423, 0.04, 0.05)$ and eigenvalues of jacobian matrix are $-0.0060 + i0.3063, -0.0060 - i0.3063$. Hence $E_5$ is stable.

At $\tau_2 = 0.5$ we get $E_5 = (0, 405.7851, 529.3887, 0.04, 0.05)$ and eigenvalues of jacobian matrix are $0.0061 + i0.2676, 0.0061 - i0.2676$. Hence $E_5$ is unstable.

At $\tau_2 = 0.59375$, behavior of system is followed by Fig 16 and Fig 17.

![Figure 16: Predator and Prey 2 system at $\tau_2 = 0.59375$](image-url)
Here we get eigenvalues of jacobian matrix are $0.00002 + i7.5642$ and $0.00002 - i7.5642$ also we have $\frac{d}{d\tau_2}(j_{22}(5) + j_{33}(5)) = 0.006505 \neq 0$ and $y_5(s'(y_5))^2 < c_2p_2(x_5)p'_5(x_5)$. Hence $\tau_{2h} \approx 0.59375$. Hence as $\tau_2 > \tau_{2h}$ the system shows a stable equilibrium point and at the threshold value it has a periodic orbit enclosing an equilibrium point.

(iii) For interior equilibrium:
Taking $\tau_2$ as Bifurcation parameter and keeping others fixed and following set of parameters:
$\tau_1 = 0.5, K_1 = 1000, K_2 = 950, m_1 = 800, m_2 = 750, q_1 = 0.000015, q_2 = 0.0000025, c = 0.0035, d = 30, w = 50, r_1 = 0.04, r_2 = 0.06, r_3 = 0.08, c_1 = 0.3, c_2 = 0.2$ and $O^* = 0.05$.
We get at $\tau_2 = 0.6$ interior equilibrium is $E_6 = (163.9249, 621.3034, 466.1853)$ and eigenvalues of jacobian matrix are $-0.0141 + 0.1359i, -0.0141 - 0.1359i, -0.3083$ which shows that $E_6$ is stable as we can see in fig. 18.
Figure 18: Predator, Prey 1 and Prey 2 system at $\tau_2 = 0.6$

At $\tau_2 = 0.9$ interior equilibrium is $E_6 = (118.2906, 776.8789, 450.2551)$ and the eigenvalues of jacobian matrix are $0.0298 + i0.1246, 0.0298 - i0.1246, -1.0957$. Hence $E_6$ is unstable.
Here with $\tau_2 = 0.7935$,

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure19.png}
\caption{Predator, Prey 1 and Prey 2 system at $\tau_2 = 0.7935$}
\end{figure}

We get interior equilibrium is $E_6 = (126.8799, 744.7612, 453.0900)$ and eigenvalues of jacobian matrix are $-0.000011 + i0.122911, -0.000011 - i0.122911, -0.805206$ with

$$n_1 = 0.805228, n_2 = 0.0151, n_3 = 0.0122, (n_1.n_2 - n_3) = -0.000041 \approx 0$$

and \( \left. \left( \frac{dn_3}{d\tau_2} - (n_2 \frac{dn_1}{d\tau_2} + n_1 \frac{dn_2}{d\tau_2}) \right) \right|_{\tau_2=HF} = 0.0016 \).

Hence $\tau_{2HF} \approx 0.7935$.

Here we conclude from Theorem 8.3 that when $\tau_2$ crosses its critical value the system has periodic solution. This illustrated in Fig 19.

**Persistence:**

From the previous example we have $l_1 = 996.5735, l_2 = 947.0973, x_4 = 672.5703, y_4 = 400.2809, x_5 = 938.6522, y_5 = 31.0251$ and inequality (5) holds as

$$-s(0) + c_1p_1(l_1) + c_2p_2(l_2) - r_3O^* = 0.1690 > 0$$

hence the system persists uniformly.
10. Results and Conclusion

In this study mathematical criteria are developed for the stability and persistence of a system of three species (two competing prey and one predator) in the presence of pollution. With the help of numerical example it is shown how the sustainability of the system gets affected as the pollution level increases. It was observed that when the pollution is very low all the species can survive in a stable equilibrium, but as the level of pollution increases, initially the value of predator equilibrium decreases and later on it results in the extinction of species one by one from top to bottom in trophic level:

<table>
<thead>
<tr>
<th>Stage</th>
<th>Q &amp; Input rate of pollutant in the environment</th>
<th>$a$ &amp; Loss rate of environmental pollution</th>
<th>$p(E) = 1 + mE$ &amp; Conversion function of environmental pollution into organismal pollution.</th>
<th>Surviving species</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>0.4</td>
<td>1</td>
<td>0.0234</td>
<td>0.04</td>
</tr>
<tr>
<td>II</td>
<td>10</td>
<td>0.35</td>
<td>0.6</td>
<td>0.5</td>
</tr>
<tr>
<td>III</td>
<td>30</td>
<td>0.35</td>
<td>0.6</td>
<td>0.5</td>
</tr>
<tr>
<td>IV</td>
<td>45</td>
<td>0.35</td>
<td>0.7</td>
<td>0.5</td>
</tr>
<tr>
<td>V</td>
<td>45</td>
<td>0.35</td>
<td>0.7</td>
<td>0.5</td>
</tr>
</tbody>
</table>

In the following three cases, conditions are derived under which system experiences Bifurcation.
The pollution level is assumed to be low.

(i) In the absence of second prey
(ii) In the absence of first prey
(iii) With all three species.

In each of the above cases as the bifurcation parameter crosses the threshold value, the system moving towards stable equilibrium point shows change in behavior and moves towards a periodic orbit.
Declaration

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References


Sustainability of a system of two competing prey and a predator ...


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