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A note on fold thickness of graphs

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Abstract

A 1-fold of G is the graph G' obtained from a graph G by identifying two nonadjacent vertices in G having at least one common neighbor and reducing the resulting multiple edges to simple edges. A uniform k-folding of a graph G is a sequence of graphs $G = G_0, G_1, G_2, \ldots, G_k$, where G_{i+1} is a 1-fold of G_i for $i = 0, 1, 2, \ldots, k-1$ such that all graphs in the sequence are singular or all of them are nonsingular. The largest k for which there exists a uniform k- folding of G is called fold thickness of G and this concept was first introduced in [1]. In this paper, we determine fold thickness of corona product graph $G \odot \overline{K_m}$, $G \odot_S \overline{K_m}$ and graph join $G + \overline{K_m}$.

Key Words: *Fold thickness, Uniform folding, Singular graphs.*

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1. Introduction

The concept of graph folding was first defined by Gervacio et al. [5] The motivation for it is from the situation of folding a meter stick. Let a finite number of unit bars be joined together at ends in such a way that they are free to turn. There are some meter sticks with this structure as shown in Fig. 1. The meter stick of this structure can be treated as a physical model of the path P_n on n vertices. After a sequence of folding, it becomes a physical model of the complete graph K_2.

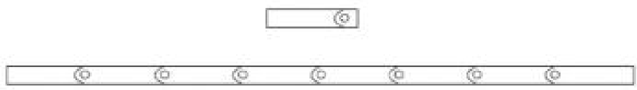

Figure 1. Meter stick-Folded and unfolded.

Let G be a graph that is not isomorphic to a complete graph. If x and y are nonadjacent vertices of G that have atleast one common neighbor, then identify x and y and reduce any resulting multiple edges to simple edges to form a new graph, G'. The graph G' is called a 1-fold of G. Consider a sequence of graphs $G = G_0, G_1, G_2, \ldots, G_k$ in which G_{i+1} is a 1-fold of G_i for $i = 0, 1, 2, \ldots, k-1$. This sequence is called a k- folding of $G = G_0$. The largest integer k for which there exists a k-folding is in the case where G_k is a complete graph. Let $\mathcal{A}(G_i)$ be the adjacency matrix corresponding to the graph G_i. A graph G_i is singular if $\mathcal{A}(G_i)$ is singular and nonsingular if $\mathcal{A}(G_i)$ is nonsingular. A graph G is said to have a uniform k - folding if there is a k- folding in which all graphs in the sequence are singular or all of them are nonsingular. The largest integer k for which there exists a uniform k - folding of G is called fold thickness of G, and is denoted by **fold** (G). If $G = G_0, G_1, G_2, \ldots, G_k$ is a k-folding of G, then the graph G_k is referred as a k-fold of G. The fold thickness of a graph was first defined by F. J. H. Campeña and S.V. Gervacio in [1] and evaluated fold thickness of some special classes of graphs such as cycle graph, wheel graph, bipartite graphs etc.

2. Preliminary results

In this paper K_n, P_n and C_n denotes the complete graph, path and cycle graph on n vertices respectively. W_n and S_n denotes the wheel graph and star graph on $n+1$ vertices respectively. $V(G)$ and $E(G)$ denotes the vertex set and edge set respectively of a graph G. $\chi(G)$ denotes the vertex chromatic number of G. For any vertex x in a graph G, $N(x)$ is the set of all vertices y in G that are adjacent to x and is called the neighbor set of x. Let $C_1, C_2, \ldots, C_n$ be the components of G. Label the vertices of G by labelling the vertices of C_1, then the vertices of C_2 and so on. The adjacency matrix of G, $\mathcal{A}(G)$ is a block diagonal matrix,

$$\mathcal{A}(G) = \begin{bmatrix} \mathcal{A}(C_1) & 0 & \cdots & 0 \\ 0 & \mathcal{A}(C_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathcal{A}(C_n) \end{bmatrix}$$

Thus, the determinant of the adjacency matrix, $det\,\mathcal{A}(G) = \prod_{i=1}^{n} det\,\mathcal{A}(C_i)$.

The null graph $\overline{K_n}$ is the graph with n vertices and zero edges. The corona product [6] $G \odot H$ of two graphs G and H is defined as the graph obtained by taking one copy of G and $|V(G)|$ copies of H and joining by an edge each vertex from the i^{th}-copy of H with the i^{th}-vertex of G. The join of two vertex disjoint graphs G and H denoted by $G+H$ is the graph consisting of G and H all edges of the form xy, where x is a vertex of G and y is a vertex of H.

Theorem 2.1. *[2] Let G be a simple connected graph. The smallest complete graph that G folds into is the complete graph with order $\chi(G)$, where $\chi(G)$ denotes the chromatic number of G.*

Thus, a maximum folding of a graph G on n vertices or simply a max fold of G is defined to be a k-folding of G, where $k = n - \chi(G)$.

Theorem 2.2. *[4] If x and y are vertices in a graph G such that $N(x) = N(y)$, then G is singular.*

Theorem 2.3. *[4] For each $n \geq 1$, $det\,\mathcal{A}(K_n) = (-1)^{n-1}(n-1)$.*

Theorem 2.4. *[4] Let x and y be vertices in a graph G such that $N(x) \subseteq N(y)$. If G' is the graph obtained from G by deleting all the edges of the form yz, where z is a neighbor of x, then $det\,\mathcal{A}(G) = det\,\mathcal{A}(G')$.*

The following theorem gives an upper bound for the fold thickness of graphs.

Theorem 2.5. *[1] For any connected graph G of order n,*

$$\mathbf{fold}(G) \leq \begin{cases} n - \chi(G) & if\ G\ is\ \ nonsingular, \\ n - \chi(G) - 1 & if\ G\ is\ \ singular\ . \end{cases}$$

Remark 2.1. *In view of the above theorem, if there exists a uniform k-folding of a connected graph G where k is equal to the upper bound in the theorem, then k must be the fold thickness of the graph. This observation will be used to obtain the fold thickness of most of the graphs.*

Theorem 2.6. *[1] For each integer $n \geq 1$,*

$$det\,\mathcal{A}(P_n) = \begin{cases} 0 & if\ \text{n}\ is\ odd, \\ (-1)^{n/2} & if\ \text{n}\ is\ even. \end{cases}$$

Theorem 2.7. *[1] For each integer $n \geq 3$,*

$$det\,\mathcal{A}(C_n) = \begin{cases} 0 & if\ n \equiv 0\ (mod\ 4), \\ 2 & if\ n \equiv 1\, or\, 3\ (mod\ 4), \\ -4 & if\ n \equiv 2\ (mod\ 4). \end{cases}$$

Theorem 2.8. *[1] The path P_n has fold thickness given by,*

$$\mathbf{fold}\,(P_n) = \begin{cases} 0 & if\ n\ is\ even \\ max\{0, n-3\} & if\ n\ is\ odd. \end{cases}$$

Theorem 2.9. *[1] The cycle C_n, has fold thickness given by*

$$\mathbf{fold}\,(C_n) = \begin{cases} 0 & if\ n \equiv 2\ (mod\ 4), \\ n-3 & otherwise. \end{cases}$$

3. Main Results

3.1. Fold Thickness of $G \odot \overline{K_m}$

In this section we evaluate the fold thickness of corona product, $G \odot \overline{K_m}$ of a connected graph G and a null graph $\overline{K_m}$, $m \geq 2$. The vertices of the graph $G \odot \overline{K_m}$ is labelled as follows : let $v_1, v_2, \ldots v_n$ be the vertices of G and let $u_{i1}, u_{i2}, \ldots u_{im}$ be the pendant vertices adjacent to the i^{th} vertex v_i of G for $i = 1, 2, \ldots n$.

Theorem 3.1. *If $m \geq 2$, then the fold thickness of $G \odot \overline{K_m}$ is given by,*

$$\textbf{fold}\,(G \odot \overline{K_m}) = \begin{cases} (m+1)n - \chi(G) - 1 & \text{if } \chi(G) = 2, \\ (m+1)n - \chi(G) - 2 & \text{otherwise.} \end{cases}$$

noindent where n is the number of vertices of G.

Proof. The graph $G \odot \overline{K_m}$, $m \geq 2$ is singular, since the vertices u_{ij} and u_{ik}, where $i \in \{1, 2, \dots n\}$, $j, k \in \{1, 2, \dots m\}$ has common neighbor v_i. Therefore, by Theorem 2.5, $\textbf{fold}\,(G \odot \overline{K_m}) \leq (m+1)n - \chi(G \odot \overline{K_m}) - 1 = (m+1)n - \chi(G) - 1$. For $i = 1, 2, \dots n-1$, first identify the pendant vertices $u_{i1}, u_{i2}, \dots u_{im}$ to a single vertex and then identify it with an eligible vertex of G. Thus, a uniform $m(n-1)$-folding $G_0 = G, G_1, \dots G_{m(n-1)}$ is obtained in which every graph in the sequence is singular and $G_{m(n-1)}$ is the graph G plus m pendant vertices $u_{n1}, u_{n2}, \dots u_{nm}$ adjacent to the vertex v_n.

The maximum folding of G is $n - \chi(G)$. So, identifying repeatedly every pairs of eligible vertices of G, after $n - \chi(G)$ steps a complete graph with $\chi(G)$ vertices is obtained. Hence, a new graph G' is obtained from $G_{m(n-1)}$ which is the complete graph $K_{\chi(G)}$ plus m pendant vertices $u_{n1}, u_{n2}, \dots u_{nm}$ adjacent to one of its vertices v_n.

If $\chi(G) = 2$, the graph G' will be the star graph $K_{1,m+1}$, which is singular. Next, identify the vertices $u_{n2}, u_{n3}, \dots u_{nm}$ of $K_{1,m+1}$ one by one to obtain the graph $K_{1,3}$ which can be folded to another singular graph $K_{1,2}$. If the non- adjacent vertices of $K_{1,2}$ is identified, the non-singular graph K_2 is obtained. In this case, the sequence of graphs $G \odot \overline{K_m} = G_0, G_1, \dots, G_{m(n-1)}, \dots G' = K_{1,n+1} \dots K_{1,3}, K_{1,2}$ is a uniform k- folding with $k = m(n-1) + n - \chi(G) + m - 2 + 1 = (m+1)n - \chi(G) - 1$. So, in this case $\textbf{fold}(G \odot \overline{K_m}) = (m+1)n - \chi(G) - 1$.

If $\chi(G) \neq 2$, identify the vertices $u_{n2}, u_{n3}, \dots u_{nm}$ of G' one by one to obtain the graph G'' which is the complete graph $K_{\chi(G)}$ plus a pair of pendant vertices adjacent to one of its vertices. If G'' is again folded by identifying a pair of its non adjacent vertices, then we obtain a non singular graph. Hence the sequence of graphs $G \odot \overline{K_m} = G_0, G_1, \dots, G_{m(n-1)}, \dots G' \dots G''$ forms a uniform k-folding of G with $k = m(n-1) + n - \chi(G) + m - 2 = (m+1)n - \chi(G) - 2$. So, in this case $\textbf{fold}(G \odot \overline{K_m}) = (m+1)n - \chi(G) - 2$. □

Corollary 3.2. *If C_n is a cycle graph on n vertices,*

$$\textbf{fold}\,(C_n \odot \overline{K_m}) = \begin{cases} (m+1)n - 3 & \text{if } n \text{ is even,} \\ (m+1)n - 5 & \text{if } n \text{ is odd.} \end{cases}$$

Proof. $\chi(C_n) = 2$, if n is even and $\chi(C_n) = 3$, if n is odd. Hence, the result follows by Theorem 3.1. □

Corollary 3.3. *If S_n is the star graph $K_{1,n}$,* $\textbf{fold}\,(S_n \odot \overline{K_m}) = m(n+1) + n - 2$.

Proof. $S_n = K_{1,n}$ is a bipartite graph, that is $\chi(S_n) = 2$. So, the result follows by Theorem 3.1. □

Corollary 3.4. *If W_n is the wheel graph $C_{n-1} + K_1$,*

$$\textbf{fold}\,(W_n \odot \overline{K_m}) = \begin{cases} (m+1)n - 5 & \text{if } n \text{ is odd,} \\ (m+1)n - 6 & \text{if } n \text{ is even.} \end{cases}$$

.

Proof. $\chi(W_n) = 3$, if n is odd and $\chi(W_n) = 4$, if n is even. Thus the result follows by Theorem 3.1. □

Definition 3.5. *The corona product of a graph G and H with respect to a subset of vertices in G say $S \subset V(G)$ denoted by $G \odot_S H$ is defined to be the graph obtained by joining every vertex in H to the vertex v in S.*

Corollary 3.6. *Let $S \subset V(G)$ such that $|S| = p$, and $m \geq 2$, then the fold thickness of $G \odot_S \overline{K_m}$ is given by*

$$\textbf{fold}\,(G \odot_S \overline{K_m}) = \begin{cases} mp + n - \chi(G) - 1 & \text{if } \chi(G) = 2, \\ mp + n - \chi(G) - 2 & \text{otherwise.} \end{cases}$$

where n is the number of vertices of G.

3.2. Fold Thickness of $G + \overline{K_m}$

In this section, we evaluate the fold thickness of graph join $G + \overline{K_m}$, where G is any connected graph and $\overline{K_m}$ is the null graph on m vertices.

Theorem 3.7. *If $m \geq 2$, then the fold thickness of $G + \overline{K_m}$ is given by,*

$$\textbf{fold}\,(G + \overline{K_m}) = m + n - \chi(G) - 2$$

Proof. The graph $G+\overline{K_m}$, $m \geq 2$ is singular since, for any two vertices x and y in $V(\overline{K_m})$, $N(x) = N(y) = V(G)$. Note that $\chi(G+\overline{K_m}) = \chi(G)+1$. By Theorem 2.5, **fold** $(G+\overline{K_m}) \leq m+n-\chi(G+\overline{K_m})-1 = m+n-\chi(G)-2$. The maximum folding of G is $n - \chi(G)$.

Identify repeatedly every pairs of eligible vertices of G, $n - \chi(G)$ times to obtain a complete graph on $\chi(G)$ vertices. Thus, a uniform $(n - \chi(G))$-folding $G_0 = G + \overline{K_m}, G_1, \ldots G_{n-\chi(G)}$ is obtained in which all graphs are singular. Then, fold $G_{n-\chi(G)}$ $m - 2$ times by identifying pairs of eligible vertices of $V(\overline{K_m})$ to obtain the graph $K_{\chi(G)}+\overline{K_2}$. If the two vertices of $\overline{K_2}$ are identified, then we get a complete graph on $\chi(G) + 1$ vertices which is nonsingular. Hence, **fold**$(G+\overline{K_m}) = n-\chi(G)+m-2 = m+n-\chi(G)-2$. □

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