Proyecciones Journal of Mathematics
Vol. 43, N^o 2, pp. 383-400, April 2024.
Universidad Católica del Norte
Antofagasta - Chile

10.22199/issn.0717-6279-5627

A study on derivations of inverse semirings with involution

Madhu Dadhwal
Himachal Pradesh University, India
and
Geeta Devi
Himachal Pradesh University, India
Received : October 2022. Accepted : June 2023

Abstract

In this paper, we study the influence of derivations on semirings with involution which resembles with commutativity preserving mappings. The action of derivations on Lie ideals and some differential identities regarding Lie ideals are also investigated. It is proved that for any two derivations d_1, d_2 of a prime semiring $\mathcal{S}$ with involution $\star$ such that atleast one of d_1, d_2 is nonzero and char$(\mathcal{S}) \neq 2$, hence if the identity $[d_1(a), d_1(a^\star)] + d_2(a \circ a^\star) = 0$, for all $a \in L$, then $[L, \mathcal{S}] = (0)$, where L is a Lie ideal of $\mathcal{S}$.

Key words: *Semirings; inverse semirings; Lie ideals; derivations.*

MSC 2020: *16Y60, 16W10.*

1. Introduction and Preliminaries

There has been a considerable interest on the behaviour of derivations and commutativity of rings during the last few decades (see [1],[6],[9],[13]). The notion of rings with involution was first introduced by Herstein [9] and thereafter the algebraic structure rings with involution carried much importance in ring theory (see [3],[11],[12]). In 1998, Beidar and Martindale [5] examined some functional identities in prime rings with involution. In 2020, Ali et al. proved some results concerning derivation and discussed certain differential identities of these semirings to analyse the commutativity of MA-semirings (see [4], [14],[15]). They assert an open question in [4] that is "How to control conditions of semirings which enable to induce the commutativity through Lie and other certain ideals of semirings?" This study motivated us to examine these identities for the case of Lie ideals of additively inverse semirings with involution and we settle the aforementioned question in the framework of Lie ideals.

By a semiring we mean a nonempty set $\mathcal{S}$ equipped with two binary operations $+$ and $\cdot$ (called addition and multiplication) such that $(\mathcal{S}, +)$ is a commutative monoid with identity element 0, $(\mathcal{S}, \cdot)$ is a semigroup with $0s = 0 = s0$, for all $s \in \mathcal{S}$ and multiplication distributes over addition from either side. Recall from [10] that a semiring $\mathcal{S}$ is an additively inverse semiring, if for each $a \in \mathcal{S}$ there exists a unique element $a' \in \mathcal{S}$ such that $a + a' + a = a$ and $a' + a + a' = a'$, where a' is called the pseudo inverse of a. A semiring $\mathcal{S}$ is prime, if $a\mathcal{S}b = (0)$ implies that either $a = 0$ or $b = 0$. If $L \subseteq \mathcal{S}$, then $[L, \mathcal{S}] = \{ls + s'l | l \in L, s \in \mathcal{S}\}$. Also, an additive submonoid L of a semiring $\mathcal{S}$ is called a Lie ideal, if $[L, \mathcal{S}] \subseteq L$ and it is a 2- Lie ideal, if $2ab \in L$, for all $a, b \in L$. For any $a, b \in \mathcal{S}$, $[a, b]$ (resp. $a \circ b$) symbolizes the commutator (resp. the Jordan product) $ab + b'a = ab + ba'$ (resp. $ab + ba$) and these play a vital role in the study of additively inverse semirings. Futher, an additive mapping $d : \mathcal{S} \to \mathcal{S}$ is called a derivation, if $d(ab) = d(a)b + ad(b)$, for all $a, b \in \mathcal{S}$. Moreover, an involution is an additive mapping $\star : \mathcal{S} \to \mathcal{S}$ satisfying $(a^\star)^\star = a$ and $(ab)^\star = b^\star a^\star$. An element $a \in \mathcal{S}$ is hermitian (resp. skew hermitian) if $s^\star = s$ (resp. $s^\star = s'$). The set of hermitian elements of $\mathcal{S}$ is denoted by $\mathcal{H}$ and skew hermitian elements is denoted by $\mathcal{S}_1$. In addition, an involution is of second kind if $Z(\mathcal{S})\mathcal{H}$. According to Ali et al. [15], $\mathcal{S}_1 \cap Z(\mathcal{S}) \neq (0)$ and $\mathcal{H} \cap Z(\mathcal{S}) \neq (0)$ for semiprime additively inverse semiring with second kind involution $\star$ and an ideal I of $\mathcal{S}$ is a $\star$ -ideal, if $I = I^\star$.

In this paper, we generalize some results of [4] and [15], for the case of

Lie ideals of additively inverse semirings. Throughout this paper, $\mathcal{S}$ is a prime additively inverse semiring with A_2- condition [2] i.e., for all $a \in \mathcal{S}$, $a + a' \in Z(\mathcal{S})$, where $Z(\mathcal{S})$ is the center of $\mathcal{S}$. Note that, an additively inverse semiring with A_2- condition is also known as a MA-semiring.

We collect some examples of additively inverse semiring with involution alongwith some key results which are frequently used in the sequel and we begin with

Example 1.1. *Let* $\mathcal{S} = \{0, a, b, c, d, 1\}$, *where* $0 < a < b < c < d < 1$. *Then* $(\mathcal{S}, \max, \min)$ *is an additively inverse semiring with* A_2- *condition.*

Example 1.2. *Let* $\mathcal{S}_1 = \{\mathbf{Z}^+ \cup (\infty), \oplus, \otimes\}$, *where* $\mathbf{Z}^+$ *denotes the set of all positive integers and the binary operations are defined as*

$$a \oplus b = \begin{cases} (a,b), & \textit{if } a, b \in \mathbf{Z}^+ \\ \infty, & \textit{if } a = \infty \textit{ or } b = \infty \end{cases}$$

and

$$a \otimes b = \begin{cases} gcd(a,b), & \textit{if } a, b \in \mathbf{Z}^+ \\ a, & \textit{if } b = \infty \\ b, & \textit{if } a = \infty \\ \infty, & \textit{if } a = \infty, b = \infty. \end{cases}$$

Let $R_1 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbf{Z} \right\}$ *be a non-commutative ring. Take* $\mathcal{S} = R_1 \times \mathcal{S}_1$, *where* $\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}, s \right)' = \left\{ \left(\begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}, s \right) \right.$. *Define map* $\star : \mathcal{S} \to \mathcal{S}$ *by* $\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}, s \right)^\star = \left(\begin{bmatrix} a & c \\ b & d \end{bmatrix}, s \right)$. *Clearly,* $\mathcal{S}$ *is a non-commutative additively inverse semiring with* A_2*-condition and* $\star$ *is an involution of* $\mathcal{S} = R_1 \times \mathcal{S}_1$.

Example 1.3. *Let* $\mathcal{S}$ *be an additively inverse semiring as considered in Example 1.1 and* $\mathcal{S}_1 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathcal{S} \right\}$. *Then,* $\mathcal{S}_1$ *is an additively inverse semiring under the usual addition and multiplication of matrices. Define map* $\star : \mathcal{S}_1 \to \mathcal{S}_1$ *by* $\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right)^\star = \begin{bmatrix} d & b' \\ c' & a \end{bmatrix}$. *Now,* $\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} \right)^\star = \left(\begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix} \right)^\star = \begin{bmatrix} d+h & b'+f' \\ c'+g' & a+e \end{bmatrix} = \begin{bmatrix} d & b' \\ c' & a \end{bmatrix} + \begin{bmatrix} h & f' \\ g' & e \end{bmatrix} = \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right)^\star + \left(\begin{bmatrix} e & f \\ g & h \end{bmatrix} \right)^\star$, *i.e.* $\star$ *is an additive mapping* of $\mathcal{S}_1$ and $\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}^\star\right)^\star = \left(\begin{bmatrix} d & b' \\ c' & a \end{bmatrix}\right)^\star = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. *Also,*

$\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix}\right)^\star = \left(\begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}\right)^\star$

$= \begin{bmatrix} cf+dh & a'f+b'h \\ c'e+d'g & ae+bg \end{bmatrix} = \begin{bmatrix} h & f' \\ g' & e \end{bmatrix}\begin{bmatrix} d & b' \\ c' & a \end{bmatrix}$

$= \left(\begin{bmatrix} e & f \\ g & h \end{bmatrix}\right)^\star \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)^\star$. *Therefore, $\star$ is an involution of $\mathcal{S}_1$.*

Example 1.4. *If $\mathcal{S}$ is a additively inverse semiring as considered in Example 1.1, then $\mathcal{S}_1 = \left\{\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a, b, c \in \mathcal{S}\right\}$ is also an additively inverse semiring under the usual addition and multiplication of matrices. Define maps $\star, d : \mathcal{S}_1 \to \mathcal{S}_1$ by $\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right)^\star = \begin{bmatrix} c & b' \\ 0 & a \end{bmatrix}$ and $d\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$. Further, $\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} + \begin{bmatrix} d & e \\ 0 & f \end{bmatrix}\right)^\star$ $= \left(\begin{bmatrix} a+d & b+e \\ 0 & c+f \end{bmatrix}\right)^\star = \begin{bmatrix} c+f & b'+e' \\ 0 & a+d \end{bmatrix} = \begin{bmatrix} c & b' \\ 0 & a \end{bmatrix} + \begin{bmatrix} f & e' \\ 0 & d \end{bmatrix} =$ $\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right)^\star + \left(\begin{bmatrix} d & e \\ 0 & f \end{bmatrix}\right)^\star$, $\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}^\star\right)^\star = \left(\begin{bmatrix} c & b' \\ 0 & a \end{bmatrix}\right)^\star = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ and $\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\begin{bmatrix} d & e \\ 0 & f \end{bmatrix}\right)^\star = \left(\begin{bmatrix} ad & ae+bf \\ 0 & cf \end{bmatrix}\right)^\star = \begin{bmatrix} cf & a'e+b'f \\ 0 & ad \end{bmatrix}$ $= \begin{bmatrix} f & e' \\ 0 & d \end{bmatrix}\begin{bmatrix} c & b' \\ 0 & a \end{bmatrix} = \left(\begin{bmatrix} d & e \\ 0 & f \end{bmatrix}\right)^\star \left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right)^\star$. Therefore, $\star$ is an involution of $\mathcal{S}_1$. Also, $d\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\begin{bmatrix} d & e \\ 0 & f \end{bmatrix}\right) = d\left(\begin{bmatrix} ad & ae+bf \\ 0 & cf \end{bmatrix}\right)$ $= \begin{bmatrix} 0 & ae+bf \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}\begin{bmatrix} d & e \\ 0 & f \end{bmatrix} + \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\begin{bmatrix} 0 & e \\ 0 & 0 \end{bmatrix}$ $= \left[d\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right)\begin{bmatrix} d & e \\ 0 & f \end{bmatrix}\right] + \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} d\left(\begin{bmatrix} d & e \\ 0 & f \end{bmatrix}\right)$. Hence d is a derivation of $\mathcal{S}_1$.*

Lemma 1.5. *[10] If $\mathcal{S}$ is an additively inverse semiring, then the following statements hold:*
(i) $(a')' = a$; (ii) $a'b' = (a'b)' = (ab')' = ((ab)')' = ab$; (iii) $(ab)' = a'b = ab'$; (iv) $(a+b)' = a' + b'$, for all $a, b \in S$.

The proof of the upcoming lemma is quite easy, so we omit the proof.

Lemma 1.6. *If $\mathcal{S}$ is an additively inverse semiring and $a, b \in \mathcal{S}$, then $a + b = 0$ implies $a = b'$.*

Lemma 1.7. *[7] If $\mathcal{S}$ is an additively inverse semiring and $a, b, c \in \mathcal{S}$, then the following statements hold:*
(i) $[a, bc] = b[a, c]+[a, b]c$; (ii) $[ab, c] = a[b, c]+[a, c]b$; (iii) $[a, b+c] = [a, b]+[a, c]$; (iv) $[a + b, c] = [a, c] + [b, c]$; (v) $[ab, a] = a[b, a]$; (vi) $[a, ab] = a[a, b]$; (vii) $[a, ba] = [a, b]a$; (viii) $[ba, a] = [b, a]a$; (ix) $a\circ(bc) = (a\circ b)c+b'[a, c] = b(a \circ c) + [a, b]c$; (x) $(ab) \circ c = a(b \circ c) + [a, c]'b = (a \circ c)b + a[b, c]$; (ix) $f(a') = f(a)'$, for any additive mapping $f : \mathcal{S} \to \mathcal{S}$.

Lemma 1.8. *[8, Lemma 4.6] Let $\mathcal{S}$ be a prime additively inverse semiring with char$(\mathcal{S}) \neq 2$, L be a nonzero Lie ideal of $\mathcal{S}$ and $a, b \in \mathcal{S}$ such that $aLb = 0$, then $a = 0$ or $b = 0$ or $[L, \mathcal{S}] = (0)$.*

Lemma 1.9. *[8, Proposition 3.3] If char$(\mathcal{S}) \neq 2$ and L is a Lie ideal of $\mathcal{S}$ such that $[L, L] = (0)$, then $[L, \mathcal{S}] = (0)$.*

Remark 1.10. *Let L be a 2-Lie ideal of $\mathcal{S}$ i.e., $2ab \in L$, for all $a, b \in L$. Then, we have $2s[a, b] = 2sab + 2sb'a = 2sab + 2as'b + 2asb + 2sb'a = 2[a, s']b+2[a, sb]$ for all $a, b \in L, s \in \mathcal{S}$. Since L is a 2-Lie ideal, this implies that $2\mathcal{S}[L, L] \subseteq L$. Similarly, $2[L, L]\mathcal{S} \subseteq L$.*

Lemma 1.11. *Let L be a nonzero 2-Lie ideal and d be a derivation of $\mathcal{S}$ with char$(\mathcal{S}) \neq 2$ such that $d(L) = (0)$. Then either $d = 0$ or $[L, \mathcal{S}] = (0)$.*

Proof. By hypothesis, we have $d(a) = 0$ for all $a \in L$. Since L is a 2 - Lie ideal, therefore replacing a by $2s[a, b]$, we conclude that

$$d(s)[a, b] = (0), \text{ for all } a, b \in L, s \in \mathcal{S}.$$

Now, replace s by ts, we have $d(t)\mathcal{S}[a, b] = (0)$, for all $a, b \in L, t \in \mathcal{S}$. By using the primeness of $\mathcal{S}$ and Lemma 1.9, we conclude that either $d = 0$ or $[L, \mathcal{S}] = (0)$. □

Lemma 1.12. *[8, Theorem 4.2] If L is a nonzero 2 - Lie ideal of $\mathcal{S}$ with char$(\mathcal{S}) \neq 2$, then either $[L, \mathcal{S}] = (0)$ or L contains a nonzero ideal of the form $I = 2\mathcal{S}[L, L]\mathcal{S}$ of $\mathcal{S}$.*

Lemma 1.13. *[8, Theorem 5.8] If L is a nonzero 2 - Lie ideal of $\mathcal{S}$ with char$(\mathcal{S}) \neq 2$ such that $d^2(L) = (0)$, then either $[L, \mathcal{S}] = (0)$ or $d = 0$.*

Lemma 1.14. *If d is a derivation of $\mathcal{S}$ and $a \in \mathcal{S}$ such that $[a, \mathcal{S}] = (0)$, then $[d(a), \mathcal{S}] = (0)$.*

Proof. By given hypothesis, we have $[a, s] = as + s'a = 0$, for all $s \in \mathcal{S}$. By taking derivation of $[a, s] = 0$, we conclude that $0 = d(a)s + ad(s) + d(s)'a + s'd(a) = [d(a), s] + [a, d(s)] = [d(a), s]$, for all $s \in \mathcal{S}$. Hence, $[d(a), \mathcal{S}] = (0)$. □

It is easy to conclude the following result from [7, Lemma 2.11].

Lemma 1.15. *Let d be a nonzero derivation of $\mathcal{S}$ with char$(\mathcal{S}) \neq 2$ and I be a nonzero ideal of $\mathcal{S}$ such that $[d(I), d(I)] = (0)$, then $[I, \mathcal{S}] = (0)$.*

Proposition 1.16. *If L is a 2-Lie ideal of $\mathcal{S}$ and d is a nonzero derivation of $\mathcal{S}$ with char $(\mathcal{S}) \neq 2$ such that $[d(L), L] = (0)$, then $[L, \mathcal{S}] = (0)$.*

Proof. Suppose that L is a 2-Lie ideal of $\mathcal{S}$ and d is a nonzero derivation of $\mathcal{S}$ with char$(\mathcal{S}) \neq 2$ such that $[d(a), b] = 0$, for all $a, b \in L$. Now replacing a by $2ac$, we have

$$d(a)[c, b] + [a, b]d(c) = 0, \text{ for all } a, b, c \in L. \tag{1.1}$$

Further, taking $c = 2cb$, we get $(d(a)[c, b] + [a, b]d(c))b + [a, b]cd(b) = 0$ and by (1.1), $[a, b]Ld(b) = (0)$, for all $a, b \in L$. Moreover, by Lemma 1.8, we have $[L, \mathcal{S}] = (0)$ or $[L, b] = (0)$ or $d(b) = 0$, for each $b \in L$. If $[L, \mathcal{S}] = (0)$, then we are done. Suppose that $[L, \mathcal{S}] \neq (0)$. Then we are left with the condition: For each $b \in L$, either $[L, b] = (0)$ or $d(b) = 0$. If $d(L) = 0$, then in view of Lemma 1.11, $d = 0$. But d is nonzero, so there is an element $a \in L$ such that $d(a) \neq 0$, then $[L, a] = (0)$. Now, it remain to show that $[L, b] = (0)$, for all $b \in L$. If possible, suppose that there exists an element $t \in L$ such that $[L, t] \neq (0)$ and $d(t) = 0$. Further,

$$d(a + t) = d(a) + d(t) = d(a) \neq 0$$

and $[l, t + a] = [l, t] + [l, a] = [l, t]$, for all $l \in L$. Since $[L, t] \neq (0)$, therefore the last equation infers that $[L, t + a] \neq (0)$, which is a contradiction. Thus, $[L, L] = (0)$. Moreover, by Lemma 1.9, we conclude that $[L, \mathcal{S}] = (0)$, which is a contradiction to our supposition. Hence $[L, \mathcal{S}] = (0)$. □

An immediate consequence of the above proposition is

Corollary 1.17. *If L is a 2-Lie ideal of $\mathcal{S}$ and d is a nonzero derivation of $\mathcal{S}$ with char $(\mathcal{S}) \neq 2$ such that $[d(L), \mathcal{S}] = (0)$, then $[L, \mathcal{S}] = (0)$.*

2. Lie Ideals and Derivations of Additively Inverse Semirings with Involution

This section deals with the behaviour of derivations on Lie ideals of additively inverse semirings with involution. Also, it is discussed that how derivations effect the commutativity of additivity inverse semirings with involution and some results of [4], [15] are also generalized. In this section, L is a nonzero $\star$ as well as 2-Lie ideal, $\mathcal{S}$ is a prime additively inverse semiring with involution $\star$ and char$(\mathcal{S}) \neq 2$.

Lemma 2.1. *If $a \in \mathcal{S}$ such that $[a, [a, l^\star]] = 0$, for all $l \in L$, then either $[a, \mathcal{S}] = (0)$ or $[L, \mathcal{S}] = (0)$.*

Proof. The hypothesis gives that $[a, [a, l^\star]] = 0$, for all $l \in L$. As L is a $\star$-Lie ideal, so we have $[a, [a, L]] = (0)$. Moreover, by [8, Theorem 4.9], either $[a, \mathcal{S}] = (0)$ or $[L, \mathcal{S}] = (0)$. □

Lemma 2.2. *[15, Lemma 2.3] Let $\mathcal{S}$ be a semiprime inverse semiring with second kind involution $\star$. Then $\mathcal{S}_1 \cap Z(\mathcal{S}) \neq (0)$ and hence $\mathcal{H} \cap Z(\mathcal{S}) \neq (0)$.*

Proof. As involution is of second kind, therefore $Z(\mathcal{S})\mathcal{H}$. Let $z \in Z(\mathcal{S}) \backslash \mathcal{H}$. Then $zs = sz$, for all $s \in \mathcal{S}$ which gives that $(zs)^\star = (sz)^\star$. That is $s^\star z^\star = z^\star s^\star$, for all $s \in \mathcal{S}$. Further, by taking $s = s^\star$, we get $sz^\star = z^\star s$, for all $s \in \mathcal{S}$, i.e. $z^\star \in Z(\mathcal{S})$. Since $z, z' \in Z(\mathcal{S})$, therefore $z' + z^\star \in Z(\mathcal{S})$. Clearly, $z' + z^\star \in \mathcal{S}_1$. This infers that $z' + z^\star \in \mathcal{S}_1 \cap Z(\mathcal{S})$. Assume that $\mathcal{S}_1 \cap Z(\mathcal{S}) = (0)$. Thus, $z' + z^\star = 0$. Moreover, by Lemma 1.6 $z^\star = z$, i.e. $z \in \mathcal{H}$, which is a contradiction. Therefore, $\mathcal{S}_1 \cap Z(\mathcal{S}) \neq (0)$. Now, let $0 \neq s \in \mathcal{S}_1 \cap Z(\mathcal{S})$. Then $s^2 \in \mathcal{H}$ and $s^2 \in Z(\mathcal{S})$. If $s^2 = 0$, then $s\mathcal{S}s = (0)$ and by the semiprimeness of $\mathcal{S}$, $s = 0$, which is not possible. So $0 \neq s^2 \in \mathcal{H} \cap Z(\mathcal{S})$. Hence $\mathcal{H} \cap Z(\mathcal{S}) \neq (0)$. □

The upcoming result is a generalization of [15, Lemma 2.4].

Proposition 2.3. *If $\mathcal{S}$ is with second kind involution $\star$ and $[a, a^\star] = 0$, for all $a \in L$, then $[L, \mathcal{S}] = (0)$.*

Proof. By hypothesis, we have

$$[a, a^\star] = 0, \text{for all } a \in L. \tag{2.1}$$

Linearization of (2.1) gives that

$$[a, b^\star] + [b, a^\star] = 0, \text{for all } a \in L. \tag{2.2}$$

On putting $a = [a, st]$, for all $t \in \mathcal{S}, s \in \mathcal{S}_1 \cap Z\backslash\{0\}$, we have

$$[[a, st], b^\star] + [b, ([a, st])^\star] = 0. \tag{2.3}$$

As $s \in \mathcal{S}_1 \cap Z(\mathcal{S})\backslash\{0\}$, so equation (2.3) infers that $[[a, t], b^\star]s + [b, ([a, t])^\star]s' = 0$. Therefore, $([[a, t], b^\star] + [b, ([a, t])^\star]')s = 0$ and

$$([[a, t], b^\star] + [b, ([a, t])^\star]')\mathcal{S}s = (0).$$

The primeness of $\mathcal{S}$ implies that $[[a, t], b^\star] + [b, ([a, t])^\star]' = 0$ and by Lemma 1.6,

$$[[a, t], b^\star] = [b, ([a, t])^\star], \text{ for all } a, b \in L, t \in \mathcal{S}. \tag{2.4}$$

On replacing a by $[a, t]$ in (2.2) and using (2.4), we get $2[[a, t], b^\star] = 0$, for all $a, b \in L, t \in \mathcal{S}$. Since $\operatorname{char}(\mathcal{S}) \neq 2$, therefore $[[a, t], b^\star] = 0$ and putting $t = ta$, we get $[a, t][a, b^\star] = 0$, for all $a, b \in L, t \in \mathcal{S}$. Now, as L is a $\star$ - Lie ideal, so we have $[a, t][a, l] = 0$, for all $a, l \in L, t \in \mathcal{S}$. Taking $t = lt$ in the previous equation, we conclude that

$$[a, l]\mathcal{S}[a, l] = (0), \text{ for all } a, l \in L.$$

Furthermore, the primeness of $\mathcal{S}$ gives that $[L, L] = (0)$ and Lemma 1.9 follows that $[L, \mathcal{S}] = (0)$. □

The next result is an extension to [15, Lemma 2.5].

Theorem 2.4. *Let $\mathcal{S}$ with second kind involution $\star$. If d is a nonzero derivation of $\mathcal{S}$ such that $d[a, a^\star] = 0$, for all $a \in L$, then $[L, \mathcal{S}] = (0)$.*

Proof. By hypothesis, we have

$$d[a, a^\star] = 0, \text{ for all } a \in L. \tag{2.5}$$

On linearizing (2.5), we obtain that

$$d[a, b^\star] + d[b, a^\star] = 0, \text{ for all } a, b \in L. \tag{2.6}$$

By putting $b = [b, rt], r \in \mathcal{S}, t \in \mathcal{H} \cap Z(\mathcal{S})\backslash\{0\}$, (2.6) leads to,

$$\begin{aligned} 0 &= d[a, [b, rt]^\star] + d[[b, rt], a^\star] \\ &= d([a, [b, r]^\star]t) + d([[b, r], a^\star]t) \\ &= (d[a, [b, r]^\star] + d[[b, r], a^\star])t + ([a, [b, r]^\star] + [[b, r], a^\star])d(t). \end{aligned}$$

On using (2.6), the above equation infers that

$$([a, [b, r]^\star] + [[b, r], a^\star])d(t) = 0, \tag{2.7}$$

for all $a, b \in L, r \in \mathcal{S}, t \in \mathcal{H} \cap Z(\mathcal{S})\backslash\{0\}$. Replacing r by rs with $s \in \mathcal{S}_1 \cap Z(\mathcal{S})\backslash\{0\}$, we are left with

$$([a, [b, r]^\star]' + [[b, r], a^\star])d(t)s = 0.$$

This gives that $([a, [b, r]^\star]' + [[b, r], a^\star])d(t)\mathcal{S}s = (0)$, for all $a, b \in L, r \in \mathcal{S}, t \in \mathcal{H} \cap Z(\mathcal{S})\backslash\{0\}$, $s \in \mathcal{S}_1 \cap Z(\mathcal{S})\backslash\{0\}$. The primeness of $\mathcal{S}$ deduces that $([a, [b, r]^\star]' + [[b, r], a^\star])d(t) = 0$ and by Lemma 1.6, we conclude that

$$[a, [b, r]^\star]d(t) = [[b, r], a^\star]d(t), \text{ for all } a, b \in L, r \in \mathcal{S}, t \in \mathcal{H} \cap Z(\mathcal{S})\backslash\{0\}.$$

On using the above equation in (2.7), we have $2[[b, r], a^\star]d(t) = 0$, for all $a, b \in L, r \in \mathcal{S}, t \in \mathcal{H} \cap Z(\mathcal{S})\backslash\{0\}$. As $\text{char}(S) \neq 2$, so $[[b, r], a^\star]d(t) = 0$. By using the fact L is $\star$ - Lie ideal the previous equation concludes that $[[b, r], a])d(t) = 0$, for all $a, b \in L, r \in \mathcal{S}, t \in \mathcal{H} \cap Z(\mathcal{S})\backslash\{0\}$. Now, taking $2la$ in place of a, we find that

$$[[b, r], l]Ld(t) = (0), \text{ for all } b, l \in L, r \in \mathcal{S}, t \in \mathcal{H} \cap Z(\mathcal{S})\backslash\{0\}. \tag{2.8}$$

In view of Lemma 1.8, (2.8) gives that $[[b, r], L] = (0)$, for all $b \in L, r \in \mathcal{S}$ or $d(t) = (0)$, $t \in \mathcal{H} \cap Z(\mathcal{S})\backslash\{0\}$ or $[L, \mathcal{S}] = (0)$. If $[L, \mathcal{S}] = (0)$, then we are done. If

$$[[b, r], l] = 0, \text{ for all } b, l \in L, r \in \mathcal{S}$$

then taking $l = b, r = a^\star$, for any $a \in L$ and using Lemma 2.1, $[L, \mathcal{S}] = (0)$. On the other side, if

$$d(t) = 0, t \in \mathcal{H} \cap Z(\mathcal{S}) \backslash \{0\}. \tag{2.9}$$

Since for each $s \in Z(\mathcal{S}), s + s^\star \in \mathcal{H} \cap Z(\mathcal{S})$, therefore

$$d(s) + d(s^\star) = 0. \tag{2.10}$$

As $s' + s^\star \in \mathcal{S}_1 \cap Z(\mathcal{S})$, for each $s \in Z(\mathcal{S})$, we get from (2.9)

$$\begin{aligned} 0 &= d(s') + d(s^\star) \\ &= d(s)' + d(s^\star). \end{aligned}$$

In view of Lemma 1.6, $d(s) = d(s^\star)$ and using this in equation (2.10), we conclude that

$$d(s) = 0, \text{ for all } s \in Z(\mathcal{S}). \tag{2.11}$$

On replacing b by $[b, st], s \in Z(\mathcal{S})$ in (2.6), we have

$$\begin{aligned} 0 &= d[a, [b, st]^\star] + d[[b, st], a^\star] \\ &= d([a, [b, t]^\star])s^\star + [a, [b, t]^\star]d(s^\star) + d([[b, t], a^\star])s + [[b, t], a^\star]d(s). \end{aligned}$$

Now, using (2.11), the above equation infers that

$$d([a, [b, t]^\star])s^\star + d([[b, t], a^\star])s = 0, \text{ for all } a, b \in L, t \in \mathcal{S}, s \in Z(\mathcal{S}). \tag{2.12}$$

By applying Lemma 1.6, equation (2.6) deduces that $d[a, [b, t]^\star] = d[[b, t], a^\star]'$ and using this in (2.12), we have $d([[b, t], a^\star])(s + (s^\star)\prime) = 0$. As $s + (s^\star)' \in Z(\mathcal{S})$, so

$$d([[b, t], a^\star])\mathcal{S}(s + (s^\star)') = (0).$$

The primeness of $\mathcal{S}$ concludes that either $d([[b, t], a^\star]) = 0$, for all $a, b \in L, t \in \mathcal{S}$ or $s + (s^\star)' = 0$. By Lemma 1.6, the latter case gives that $s = s^\star$, which implies that $Z(\mathcal{S}) \subset \mathcal{H}$, a contradiction. Therefore

$$d([[b, t], a^\star]) = 0, \text{ for all } a, b \in L, t \in \mathcal{S}. \tag{2.13}$$

As L is a $\star$ - Lie ideal, so (2.13) gives that $d([[b,t],a]) = 0$.

On taking $a = 2al$, we are left with $([[b,t],a])d(l) + d(a)[[b,t],l] = 0$, for all $a, b, l \in L, t \in \mathcal{S}$. Further, putting $[[b,t],a^\star]$ in place of a and using (2.13), we get $([[b,t],[[b,t],a^\star]])d(l) = 0$. Replacing l by $2lk$ and using the fact that char $(\mathcal{S}) \neq 2$, we obtain that

$$([[b,t],[[b,t],a^\star]])Ld(k) = (0), \text{ for all } a, b, k \in L, t \in \mathcal{S}. \tag{2.14}$$

By Lemma 1.8, $[[b,t],[[b,t],a^\star]] = 0$ or $d(L) = (0)$ or $[L,\mathcal{S}] = (0)$. If

$$[[b,t],[[b,t],a^\star]] = 0, \text{ for all } a, b \in L, t \in \mathcal{S}$$

by Lemma 2.1, $[[b,t],s] = 0$, for all $b \in L, s, t \in \mathcal{S}$ or $[L,\mathcal{S}] = (0)$. Now, putting $s = b$ and $t = l^\star$ with $l \in L$ and again using Lemma 2.1, we find that $[L,\mathcal{S}] = (0)$.
By Lemma 1.11, $d(L) = (0)$ also infers that $[L,\mathcal{S}] = (0)$. This completes the proof. □

The upcoming corollaries are immediate consequences of Theorem 2.3.

Corollary 2.5. *If $\mathcal{S}$ is with second kind involution $\star$ and d is derivation a of $\mathcal{S}$ such that $d(t) = 0, t \in \mathcal{H} \cap Z(\mathcal{S}) \backslash \{0\}$. Then, $d(s) = 0$, for all $s \in Z(\mathcal{S})$.*

Corollary 2.6. *Let I be a nonzero $\star$-ideal of $\mathcal{S}$ with second kind involution $\star$. If d is a nonzero derivation of $\mathcal{S}$ such that $d[a,a^\star] = 0$, for all $a \in I$, then $[I,\mathcal{S}] = (0)$. Moreover, $\mathcal{S}$ is commutative.*

By applying a similar technique as in the above theorem with suitable changes, the following theorem can be proved.

Theorem 2.7. *Let $\mathcal{S}$ with second kind involution $\star$. If d is a nonzero derivation of $\mathcal{S}$ such that $d(a \circ a^\star) = 0$, for all $a \in L$, then $[L,\mathcal{S}] = (0)$.*

The next result is a generalization of [4, Lemma 5].

Proposition 2.8. *If $a, b \in \mathcal{S}$ such that $alb + bla = 0$, for all $l \in L$, then $a = 0$ or $b = 0$ or $[L,\mathcal{S}] = (0)$.*

Proof. Suppose that $a, b \in \mathcal{S}$ with

$$alb + bla = 0, \text{ for all } l \in L. \tag{2.15}$$

As L is a 2 -Lie ideal, so replacing l by $2lbs[u,v]$ in (2.15), we get

$$2(albs[u,v]b + blbs[u,v]a) = 0.$$

Since $\mathrm{char}(\mathcal{S}) \neq 2$, therefore we have

$$albs[u,v]b + blbs[u,v]a = 0.$$

On applying Lemma 1.6 on equation (2.15), it concludes that $alb = bla'$ and using this in the previous equation we find that

$$bL(a's[u,v]b + bs[u,v]a) = (0), \text{ for all } u, v \in L, s \in \mathcal{S}. \tag{2.16}$$

By applying Lemma 1.8 to equation (2.16), we obtain that $b = 0$ or $a's[u,v]b + bs[u,v]a = 0$ or $[L, \mathcal{S}] = (0)$. If $a's[u,v]b + bs[u,v]a = 0$, then by Lemma 1.6, we are left with

$$as[u,v]b = bs[u,v]a, \text{ for all } u, v \in L, s \in \mathcal{S}. \tag{2.17}$$

Now, replacing l by $2s[u,v]$ in (2.15), then by the fact $\mathrm{char}(\mathcal{S}) \neq 2$ and (2.17), we get $a\mathcal{S}[u,v]b = (0)$, for all $u, v \in L$. The primeness of $\mathcal{S}$ infers that either $a = 0$ or $[u,v]b = 0$. By taking $v = 2vl$, we conclude that

$$[u,v]Lb = (0), \text{ for all } u, v \in L.$$

In view of Lemma 1.8 and 1.9, the above equation infers that $[L, \mathcal{S}] = (0)$ or $b = 0$. $\square$

Theorem 2.9. *Let d be a nonzero derivation of $\mathcal{S}$ such that $[d(L), d(L)] = (0)$, then $[L, \mathcal{S}] = (0)$. Moreover, $L \subseteq Z(\mathcal{S})$.*

Proof. By hypothesis, we have

$$[d(a), d(b)] = 0, \text{ for all } a, b \in L.$$

Then, by Lemma 1.12, the above equation gives that $[d(a), d(b)] = 0$, for all $a, b \in I = 2\mathcal{S}[L, L]\mathcal{S}$.

By using Lemma 1.15, we conclude that $[2s[a,b]t,u]=0$, for all $a,b\in L,t,s,u\in\mathcal{S}$. On using the fact that $\mathrm{char}(\mathcal{S})\neq 2$ and then replacing t by tv we conclude that

$$s[a,b]\mathcal{S}[v,u]=(0),\ \text{for all } a,b,v\in L,u,s\in\mathcal{S}.$$

The primeness of $\mathcal{S}$ concludes that $[L,L]=(0)$ and by Lemma 1.9, $[L,\mathcal{S}]=(0)$. Moreover, by Lemma 1.6, $L\subseteq\mathcal{S}$. Hence the proof. □

Corollary 2.10. *Let d be a nonzero derivation of $\mathcal{S}$ such that $[d(\mathcal{S}),d(\mathcal{S})]=(0)$, then $[\mathcal{S},\mathcal{S}]=(0)$. Moreover, $\mathcal{S}$ is commutative.*

The upcoming theorem is a generalized version of [4, Theorem 1].

Theorem 2.11. *If d_1 and d_2 are derivations of $\mathcal{S}$ with second kind involution $\star$ such that atleast one of d_1, d_2 is nonzero with $[d_1(a),d_1(a^\star)]+d_2(a\circ a^\star)=0$, for all $a\in L$, then $[L,\mathcal{S}]=(0)$.*

Proof. The hypothesis implies that

$$[d_1(a),d_1(a^\star)]+d_2(a\circ a^\star)=0,\ \text{for all } a\in L \tag{2.18}$$

and atleast one of d_1 and d_2 is nonzero. Here, we have three possible cases:
(i) $d_1=0$ and $d_2\neq 0$;
(ii) $d_1\neq 0$ and $d_2=0$;
(iii) $d_1\neq 0$ and $d_2\neq 0$.

Now, we will discuss the aforementioned possible cases in detail:
(i) If $d_1=0$ and $d_2\neq 0$, then (2.18) infers that $d_2(a\circ a^\star)=0$, for all $a\in L$. Further, Theorem 2.7 concludes that $[L,\mathcal{S}]=(0)$.

(ii) If $d_1\neq 0$ and $d_2=0$, then (2.18) leads to

$$[d_1(a),d_1(a^\star)]=0,\ \text{for all } a\in L. \tag{2.19}$$

On linearizing (2.19), we

$$[d_1(a),d_1(b^\star)]+[d_1(b),d_1(a^\star)]=0,\ \text{for all } a,b\in L. \tag{2.20}$$

Taking $a=2r[a,st],t\in\mathcal{H}\cap Z(\mathcal{S})\backslash\{0\}$ in (2.20), we have

$2(([d_1(r[a,s]),d_1(b^\star)]+[d_1(b),d_1(r[a,s])^\star])t+[r[a,s]d_1(t),d_1(b^\star)]+[d_1(b),$

$$(r[a,s])^{\star}d_1(t)]) = 0$$

for all $a, b, r \in L, s \in \mathcal{S}$. By using char$(\mathcal{S}) \neq 2$ and (2.19) in the above equation, we get

$$[r[a,s]d_1(t), d_1(b^{\star})] + [d_1(b), r[a,s]^{\star}d_1(t)] = 0. \tag{2.21}$$

Putting $s = su, u \in \mathcal{S}_1 \cap Z(\mathcal{S})\backslash\{0\}$, we are left with $([r[a,s]d_1(t), d_1(b^{\star})] + [d_1(b), r[a,s]^{\star}d_1(t)]')u = 0$. Therefore,

$$([r[a,s]d_1(t), d_1(b^{\star})] + [d_1(b), r[a,s]^{\star}d_1(t)]')\mathcal{S}u = (0)$$

for all $a, b, r \in L, s \in \mathcal{S}, u \in \mathcal{S}_1 \cap Z(\mathcal{S})\backslash\{0\}$. The primeness of $\mathcal{S}$ gives that

$$[r[a,s]d_1(t), d_1(b^{\star})] + [d_1(b), r[a,s]^{\star}d_1(t)]' = 0.$$

Moreover, by lemma 1.6, we have

$$[r[a,s]d_1(t), d_1(b^{\star})] = [d_1(b), (r[a,s])^{\star}d_1(t)]$$

and using this in equation (2.21), we get that $2[r[a,s]d_1(t), d_1(b^{\star})] = 0$. This implies that

$$\begin{aligned} 0 &= [r[a,s]d_1(t), d_1(b^{\star})] \\ &= r[a,s][d_1(t), d_1(b^{\star})] + [r[a,s], d_1(b^{\star})]d_1(t). \end{aligned}$$

Putting $r = 2rc$ we get that

$$[r, d_1(b^{\star})]c[a,s]d_1(t) = 0 \text{ for all } a, b, c, r \in L, s \in \mathcal{S}, t \in \mathcal{H} \cap Z(\mathcal{S})\backslash\{0\}.$$

By Lemma 1.8, $[r, d_1(b^{\star})] = 0$ or $[a,s]d_1(t) = 0$ or $[L, \mathcal{S}] = (0)$. If $[r, d_1(b^{\star})] = 0$, for all $b, r \in L$. As L is a $\star$ - Lie ideal, then we find that $[r, d_1(b)] = 0$ and by Proposition 1.16, $[L, \mathcal{S}] = (0)$. Now, consider the case $[a,s]d_1(t) = 0$. By replacing s by rs, we get that

$$[a,r]\mathcal{S}d_1(t) = 0, \text{ for all } a \in L, r \in \mathcal{S}, t \in \mathcal{H} \cap Z(\mathcal{S})\backslash\{0\}.$$

The primeness of $\mathcal{S}$ infers that either $[L, \mathcal{S}] = 0$ and $d_1(t) = 0$, $t \in \mathcal{H} \cap Z(\mathcal{S})\backslash\{0\}$. Then, by Corollary 2.5

$$d_1(s) = 0, \text{ for all } s \in Z(\mathcal{S}).$$

Now, putting $b = 2r[b, ts]$ in (2.20) and using the above equation, we find that $([d_1(a), d_1((r[b, s])^\star)]' + [d_1(r[b, s]), d_1(a^\star)])t = 0$, for all $a, b, s \in L, r \in \mathcal{S}, t \in \mathcal{S}_1 \cap Z(\mathcal{S})\backslash\{0\}$. This implies that

$$([d_1(a), d_1((r[b, s])^\star)]' + [d_1(r[b, s]), d_1(a^\star)])\mathcal{S}t = (0).$$

The primeness of $\mathcal{S}$ gives that

$$[d_1(a), d_1((r[b, s])^\star)]' + [d_1(r[b, s]), d_1(a^\star)] = 0. \tag{2.22}$$

By Lemma 1.6, the above equation infers that

$$[d_1(a), d_1((r[b, s])^\star)] = [d_1(r[b, s]), d_1(a^\star)]$$

for all $a, b, s \in L, r \in \mathcal{S}$. Taking $b = 2r[b, s]$ in (2.20), then using char $(\mathcal{S}) \neq 2$ and the above equation, we are left with

$$[d_1(r[b, s]), d_1(a^\star)] = 0$$

for all $a, b, s \in L, r \in \mathcal{S}$. Since L is a $\star$ - Lie ideal, therefore $[d_1(r[b, s]), d_1(a)] = 0$. Further, putting $r = d_1(a)r$, we get

$$\begin{aligned} 0 &= [d_1^2(a)r[b, s], d_1(a)] + d_1(a)[d_1(r[b, s]), d_1(a)] \\ &= [d_1^2(a)r[b, s], d_1(a)] = d_1^2(a)[r[b, s], d_1(a)] + [[d_1^2(a), d_1(a)]r[b, s] \end{aligned}$$

$a, b, s \in L, r \in \mathcal{S}$. Replacing s by sb, we have

$$d_1^2(a)\mathcal{S}[b, s][b, d_1(a)] = (0).$$

Since $\mathcal{S}$ is a prime, therefore the primeness of $\mathcal{S}$ infers that for each $a \in L$, either $d_1^2(a) = 0$ or $[b, s][b, d_1(a)] = 0$, for all $b, s \in L$. Assume that $[L, \mathcal{S}] \neq (0)$. If $d_1^2(L) = 0$, then by Lemma 1.13, $d_1 = 0$, which is not possible since $d_1 \neq 0$. So there exists some $l \in L$ such that $d_1^2(l) \neq 0$, then $[b, s][b, d_1(l)] = 0$, for all $b, s \in L$. We claim that $[b, s][b, d_1(a)] = 0$, for all $a, b, s \in L$. If possible, let $m(\neq l) \in L$ such that $[b, s][b, d_1(m)] \neq 0$. Then $d_1^2(m) = 0$. Now,

$$d_1^2(l + m) = d_1^2(l) + d_1^2(m) = d_1^2(l).$$

But $d_1^2(l) \neq 0$, therefore

$$0 = [b, s][b, d_1(m + l)] = [b, s][b, d_1(m)] + [b, s][b, d_1(l)] = [b, s][b, d_1(m)]$$

which is a contradiction. So,

$$[b,s][b,d_1(a)]=0, \text{ for all } a,b,s\in L. \tag{2.23}$$

On taking $s=2d_1(a)s[u,v]$, we have

$$0=d_1(a)[b,2s[u,v]][b,d_1(a)]+2[b,d_1(a)]s[u,v][b,d_1(a)]=0$$

for all $a,b,s,u,v\in L$. Since $2s[u,v]\in L$, therefore by (2.23), the preceding equation leads to

$$[b,d_1(a)]s[u,v][b,d_1(a)]=0$$

for all $a,b,s,u,v\in L$. This gives that $[u,v][b,d_1(a)]L[u,v][b,d_1(a)]=(0)$, for all $a,b,u,v\in L$. In view of Lemma 1.8, $[u,v][b,d_1(a)]=0$, for all $a,b,u,v\in L$. This infers that $[u,c]v[b,d_1(a)]=0$, for all $a,b,c,u,v\in L$, by taking $v=2cv$. Again, by using Lemma 1.8, we get that $[L,L]=(0)$ or $[L,d_1(L)]=(0)$. By Lemma 1.9, $[L,L]=(0)$ implies that $[L,\mathcal{S}]=(0)$, which is a contradiction. In view of Proposition 1.16, $[L,d_1(L)]=(0)$ gives that $[L,\mathcal{S}]=(0)$, again a contradiction. In each case, we get a contradiction to our assumption. Hence, $[L,\mathcal{S}]=(0)$.

(iii) Let $d_1\neq 0, d_2\neq 0$. Since L is $\star$ Lie ideal, therefore replacing a by $a^\star$ in (2.18), we conclude that

$$\begin{aligned} 0 &= [d_1(a^\star),d_1(a)]+d_2(a\circ a^\star)\\ &= [d_1(a),d_1(a^\star)]'+d_2(a\circ a^\star). \end{aligned}$$

By Lemma 1.6, the above equation leads to $[d_1(a),d_1(a^\star)]=d_2(a\circ a^\star)$ and by using this in (2.18), we get that

$$2d_2(a\circ a^\star)=0, \text{ for all } a\in L.$$

This gives that $d_2(a\circ a^\star)=0$. Moreover, by Theorem 2.7, $[L,\mathcal{S}]=(0)$. This completes the proof. □

The upshot of the above theorem is

Corollary 2.12. *If d_1 and d_2 are derivations of $\mathcal{S}$ with second kind involution $\star$ such that atleast one of d_1, d_2 is nonzero with $[d_1(a),d_1(a^\star)]+d_2(a\circ a^\star)=(0)$, for all $a\in\mathcal{S}$, then $\mathcal{S}$ is commutative.*

Acknowledgement

The second author gratefully acknowledges the financial assistance by UGC.

References

[1] M. A. Javed, M. Aslam and M. Hussain, "On condition (A2) of Bandlet and Petrich for inverse semirings", *International Mathematical Forum,* Vol. 7, No. 59, pp. 2903-2914, 2012.

[2] H. J. Bandlet, M. Petrich, "Subdirect products of rings and distributive lattices", *Proceeding of the Edinburgh Mathematical Society,* Vol. 25, No. 2, pp. 135-171, 1982, doi: 10.1017/S0013091500016643

[3] L. Ali, M. Aslam and Y. A. Khan, "Commutativity of semirings with involution", *Asian-European journal of mathematics*, Art ID. 2050153, 2019, doi: 10.1142/S1793557120501533.

[4] L. Ali, Y. A. Khan, A. A. Mousa, S. Abdel-Khalek and G. Farid, "Some differential identities of MA-semirings with involution", *AIMS Mathematics*, Vol. 6, No. 3, pp. 2304-2314, doi:10.3934/math.2021.2021139

[5] K. I. Beidar and W. S. Martindale, "On functional identities in prime rings with involution", *Journal of Algebra*, Vol. 203, No. 2, pp. 491-532, 1998, doi:10.1006/jabr.1997.7285

[6] J. Berger, I. N. Herstein and J. W. Kerr, "Lie ideals and derivations of prime rings", *Journal of Algebra*, Vol. 71, No. 1, pp. 259-267, 1981, doi.org/10.1016/0021-8693(81)90120-4

[7] M. Dadhwal and G. Devi, "On generalized derivations of semirings", *Georgian Mathematical Journal*, doi.org/10.1515/gmj-2022-2178

[8] M. Dadhwal and Neelam, "On derivations and Lie structure of semirings", *Advances and Applications in Mathematical Sciences*, Accepted

[9] I. N. Herstein, "Rings with involution", Chicago, IL: University of Chicago, 1976.

[10] P. H. Karvellas, Inversive semiring, *Cambridge University Press*, Vol. 18, No. 3, pp. 277-288, 1974, doi.org/10.1017/S1446788700022850

[11] C. Lanski, "Commutation with skew elements in rings with involution", *Pacific Journal of Mathematics,* Vol. 83, No. 2, pp. 393-399, 1979

[12] T. K Lee, "On derivations of prime rings with involution", *Chinese Journal Mathematics*, Vol. 20, No. 2, 191-203, 1992. Available: https://bit.ly/2Zn14oP

[13] E. C. Posner, "Derivations in prime rings", *Proceedings of American Mathematical Society*, Vol. 8, No. 6, pp.1093-1100, 1957, doi.org/10.2307/2032686

[14] L. Ali, M. Aslam and Y. A. Khan, "On additive maps of MA-semirings with involution", *Proyecciones (Antofagasta)*, Vol. 39, No. 4, pp. 1097-1112, 2020, doi:10.22.199/issn.0717-6279-2020-04-0067

[15] L. Ali, M. Aslam and Y. A. Khan, "Some results on commutativity of MA-semirings", *Indian Journal Science and Technology*, Vol. 13, No. 31, 3198-3203, 2020, doi:10.17485/IJST/v13i31.1022

Madhu Dadhwal
Department of Mathematics and Statistics,
Himachal Pradesh University,
Summer Hill, Shimla-171005,
India
e-mail: mpatial.math@gmail.com
Corresponding author

and

Geeta Devi
Department of Mathematics and Statistics,
Himachal Pradesh University,
Summer Hill, Shimla-171005,
India
e-mail: geetasharmamath@gmail.com