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Some extensions of the Hermite-Hadamard inequalities for quasi-convex functions via weighted integral

Bahtiyar Bayraktar
Bursa Uludag University, Turkey
Juan E. Nápoles Valdés
Universidad Nacional del Nordeste, Argentina
Florencia Rabossi
Universidad Nacional del Nordeste, Argentina
and
Aylen D. Samaniego
Universidad Nacional del Nordeste, Argentina
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Abstract

In this note, starting with a lemma, we obtain several extensions of the well-known Hermite-Hadamard inequality for convex functions, using generalized weighted integral operators.

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1. Introduction

Let $\varsigma_1^*, \varsigma_2^* \in \mathbf{R}$ with $\varsigma_1^* < \varsigma_2^*$ and $I := [\varsigma_1^*, \varsigma_2^*]$, and function $\psi : I \to \mathbf{R}$.

Definition 1. *If* $\forall\ \xi, \varsigma \in I$ *and* $\kappa \in [0,1]$ *inequality* $\psi(\kappa\xi + (1-\kappa)\varsigma) \leq \kappa\psi(\xi) + (1-\kappa)\psi(\varsigma)$ *is true, then* ψ *is convex on* I. *In the case of the opposite inequality, the function concave on the interval.*

One of the most interesting and fruitful concepts in modern mathematics is the concept of a convex function. This notion has become widespread in applied and computational mathematics (an interested reader can find a fairly complete review of generalizations and extensions of the notion of a convex function in [26]).

Definition 2. *The real function* ψ *is said to be quasi-convex on* I *if inequality*

$$\psi(\kappa\xi + (1-\kappa)\varsigma) \leq \max\{\psi(\xi), \psi(\varsigma)\} \tag{1.1}$$

is fulfilled $\forall\ \xi, \varsigma \in I$ *and* $\kappa \in [0,1]$.

Remark 3. *Any convex function is a quasi-convex function. The converse is not true, that is, there exist quasi-convex functions which are not convex (see [31]).*

In recent years, the attention of many researchers working on the theory of inequalities has been drawn to the famous double Hermite-Hadamard inequality obtained for any function ψ convex on $[\varsigma_1^*, \varsigma_2^*]$.

$$\psi\left(\frac{\varsigma_1^* + \varsigma_2^*}{2}\right) \leq \frac{1}{\varsigma_2^* - \varsigma_1^*}\int_{\varsigma_1^*}^{\varsigma_2^*} \psi(\kappa)\, d\kappa \leq \frac{\psi(\varsigma_1^*) + \psi(\varsigma_2^*)}{2} \tag{1.2}$$

The peculiarity of this inequality is that it gives an estimate of the mean value of the function on the interval and, moreover, makes it possible to refine Jensen's inequality.

The study of inequality Hadamard has attracted the attention of many researchers in recent years, mainly in the following directions:
1) Using different notions of convexity.
2) Refinement of the mesh used, including more nodes.
3) Improvement of the estimates of the left and right members of Hadamard.
4) Using new generalized and fractional integral operators.

For more information and to get acquainted with various extensions of Hadamard's inequality, the reader can refer to [3, 5, 6, 7, 8, 9, 12, 14, 15, 16, 17, 19, 20, 21, 22, 23, 24, 25, 27, 35, 38] and references in them.

To make it easier to understand the subject of research, below is the definition of the fractional Riemann-Liouville integral (with $0 \leq \varsigma_1^* < \kappa < \varsigma_2^* \leq \infty$).

Definition 4. *Let $\psi \in L_1[\varsigma_1^*, \varsigma_2^*]$. Then the Riemann-Liouville fractional integrals of order $\alpha \in \mathbf{C}$, $\Re(\alpha) > 0$ are defined by (right and left respectively):*

$$
\begin{aligned}
{}^{\alpha}I_{\varsigma_1^*}\psi(x) &= \frac{1}{(\alpha)} \int_{\varsigma_1^*}^{x} (x-\kappa)^{\alpha-1}\psi(\kappa)\,d\kappa, \quad x > \varsigma_1^* \\
{}^{\alpha}I_{\varsigma_2^*}\psi(x) &= \frac{1}{(\alpha)} \int_{x}^{\varsigma_2^*} (\kappa-x)^{\alpha-1}\psi(\kappa)\,d\kappa, \quad x < \varsigma_2^*,
\end{aligned}
$$

where Euler Gamma function and $\Gamma(z) = \int_0^\infty \kappa^{z-1}e^{-\kappa}\,d\kappa, \quad \Re(z) > 0.$

Our work is based on the definition of a weighted integral operator presented below.

Definition 5. *Let $\psi \in L_1[\varsigma_1^*, \varsigma_2^*]$ and $\varpi : [0,1] \to [0,+\infty)$, with first order derivatives piecewise continuous on $[\varsigma_1^*, \varsigma_2^*]$, and $\varpi(0) = 0$. The right and left, weighted fractional integrals respectively are defined by:*

$$
{}^{\varpi}I_{\varsigma_1^*+}^{[\varsigma_1^*,\varsigma_2^*]}\psi(x) = \int_{\varsigma_1^*}^{x} \varpi'\left(\frac{x-\kappa}{\varsigma_2^*-\varsigma_1^*}\right)\psi(\kappa)\,d\kappa, \quad x > \varsigma_1^* \tag{1.3}
$$

$$
{}^{\varpi}I_{\varsigma_2^*-}^{[\varsigma_1^*,\varsigma_2^*]}\psi(x) = \int_{x}^{\varsigma_2^*} \varpi'\left(\frac{\kappa-x}{\varsigma_2^*-\varsigma_1^*}\right)\psi(\kappa)\,d\kappa, \quad x < \varsigma_2^*. \tag{1.4}
$$

and

$$
{}^{\varpi}\mathbf{I}_{\varsigma_1^*+}^{[\varsigma_1^*,\varsigma_2^*]}\psi(x) = \int_{\varsigma_1^*}^{x} \varpi''\left(\frac{x-\kappa}{\varsigma_2^*-\varsigma_1^*}\right)\psi(\kappa)\,d\kappa, \quad x > \varsigma_1^* \tag{1.5}
$$

$$
{}^{\varpi}\mathbf{I}_{\varsigma_2^*-}^{[\varsigma_1^*,\varsigma_2^*]}\psi(x) = \int_{x}^{\varsigma_2^*} \varpi''\left(\frac{\kappa-x}{\varsigma_2^*-\varsigma_1^*}\right)\psi(\kappa)\,d\kappa, \quad x < \varsigma_2^*. \tag{1.6}
$$

Remark 6. *If we take* $\varpi'(\kappa) = \frac{(\varsigma_2^* - \varsigma_1^*)^{1-\alpha}}{\Gamma(\alpha)} \cdot \kappa^{\alpha-1}$, *then from (1.3) and (1.4) we obtain the definition of the Riemann-Liouville fractional integral. If on the contrary* $\varpi'(\kappa) \equiv 1$, *then we obtain the classical Riemann Integral.A similar reasoning is valid in the case of the integrals of (1.5) and (1.6).*

Of course there are other known integral operators, fractional or not, that can be obtained as particular cases of the previous one, but we leave it to interested readers.

In this paper, some variants of inequality (1.2) are presented using the weighted integral operators of Definition 5 for functions with quasi-convex first and second derivatives.

2. Some results for functions first derivative is quasi-convex

Our first result establishes a variation of the Hermite-Hadamard Inequality given in (1.2).

Theorem 1. *Let* $\psi : [\varsigma_1^*, \varsigma_2^*] \longrightarrow \mathbf{R}^+$ *be a quasi-convex function on* $(\varsigma_1^*, \varsigma_2^*)$. *If* $\psi \in L[\varsigma_1^*, \varsigma_2^*]$ *and* $\varpi' \geq 0$ *then we have*

$$\varpi(1)\psi\left(\frac{\varsigma_1^* + \varsigma_2^*}{2}\right) \leq \frac{1}{\varsigma_2^* - \varsigma_1^*}\left[{}^{\varpi}I_{\varsigma_1^*+}^{[\varsigma_1^*,\varsigma_2^*]}\psi(\varsigma_2^*) + {}^{\varpi}I_{\varsigma_2^*-}^{[\varsigma_1^*,\varsigma_2^*]}\psi(\varsigma_1^*)\right]$$

$$\leq 2w(1)\max\{\psi(\varsigma_1^*), \psi(\varsigma_2^*)\}. \tag{2.1}$$

Proof. Taking into account that ψ is quasi-convex function, putting $\kappa = \frac{1}{2}$ in (1.2), we obtain $\forall u, v \in I$

$$\psi\left(\frac{u + \vartheta}{2}\right) \leq \max\{\psi(u), \psi(\vartheta)\}$$

Then, choosing $u = \kappa\varsigma_1^* + (1-\kappa)\varsigma_2^*$, $\vartheta = (1-\kappa)\varsigma_1^* + \kappa\varsigma_2^*$ and we add member to member, we obtain

$$\psi\left(\frac{\varsigma_1^* + \varsigma_2^*}{2}\right) \leq \kappa \in [0,1]\max\{\psi(\kappa\varsigma_1^* + (1-\kappa)\varsigma_2^*), \psi((1-\kappa)\varsigma_1^* + \kappa\varsigma_2^*)\},$$

by multiplying the above inequality by $\varpi'(\kappa)$ and integrating between 0 and 1 gives us

$$\varpi(1)\psi\left(\frac{\varsigma_1^* + \varsigma_2^*}{2}\right) \leq \frac{1}{\varsigma_2^* - \varsigma_1^*}\max\left\{{}^{\varpi}I_{\varsigma_1^*+}^{[\varsigma_1^*,\varsigma_2^*]}\psi(\varsigma_2^*), {}^{\varpi}I_{\varsigma_2^*-}^{[\varsigma_1^*,\varsigma_2^*]}\psi(\varsigma_1^*)\right\}$$

which allows us to obtain the first inequality of (2.1).

Now, let's prove the right inequality of (2.1). Since ψ is a quasi convex function for all $\varsigma_1^*, \varsigma_2^* \in I$, and $\kappa \in [0,1]$, we have

$$\begin{aligned}\psi(\kappa\varsigma_1^* + (1-\kappa)\varsigma_2^*) &\leq \max\{\psi(\varsigma_1^*), \psi(\varsigma_2^*)\},\\ \psi(\kappa\varsigma_2^* + (1-\kappa)\varsigma_1^*) &\leq \max\{\psi(\varsigma_2^*), \psi(\varsigma_1^*)\}.\end{aligned}$$

Multiplying both inequalities, member by member, by $\varpi'(\kappa)$, adding and integrating between 0 and 1, we obtain

$$\begin{aligned}&\varpi'(\kappa)\psi(\kappa\varsigma_1^* + (1-\kappa)\varsigma_2^*) + \varpi'(\kappa)\psi(\kappa\varsigma_2^* + (1-\kappa)\varsigma_1^*)\\ &\qquad \leq 2w'(\kappa)\max\{\psi(\varsigma_1^*), \psi(\varsigma_2^*)\}\end{aligned}$$

and

$$\begin{aligned}&\int_0^1 \varpi'(\kappa)\psi(\kappa\varsigma_1^* + (1-\kappa)\varsigma_2^*)d\kappa + \int_0^1 \varpi'(\kappa)\psi(\kappa\varsigma_2^* + (1-\kappa)\varsigma_1^*)d\kappa\\ &\qquad \leq 2w(1)\max\{\psi(\varsigma_1^*), \psi(\varsigma_2^*)\}.\end{aligned}$$

Taking into account that

$$\max\left\{{}^{\varpi}I_{\varsigma_1^*+}^{[\varsigma_1^*,\varsigma_2^*]}\psi(\varsigma_2^*), {}^{\varpi}I_{\varsigma_2^*-}^{[\varsigma_1^*,\varsigma_2^*]}\psi(\varsigma_1^*)\right\} \leq {}^{\varpi}I_{\varsigma_1^*+}^{[\varsigma_1^*,\varsigma_2^*]}\psi(\varsigma_2^*) + {}^{\varpi}I_{\varsigma_2^*-}^{[\varsigma_1^*,\varsigma_2^*]}\psi(\varsigma_1^*),$$

from above the right member of (2.1) is easily obtained. This completes the proof. □

Remark 2. *If in the previous result we take $\varpi'(\kappa) = \kappa^{\alpha-1}$, we completed the Theorem 2.1 of [31], since the authors only prove the second inequality. If we consider $\varpi'(\kappa) = 1$ then the above result is a variant of Theorem 2.2 of [13].*

The following result will be used throughout this section.

Lemma 3. *Let $\psi : [\varsigma_1^*, \varsigma_2^*] \longrightarrow \mathbf{R}$ and $\psi \in C^1(\varsigma_1^*, \varsigma_2^*)$. If $\psi' \in L[\varsigma_1^*, \varsigma_2^*]$, then the following equality*

$$\begin{aligned}&\frac{\varpi(0)-\varpi(1)}{\varsigma_2^*-\varsigma_1^*}\left(\psi(\varsigma_1^*) + \psi(\varsigma_2^*)\right) + \frac{1}{(\varsigma_2^*-\varsigma_1^*)^2}\left[{}^{\varpi}I_{\varsigma_1^*+}^{[\varsigma_1^*,\varsigma_2^*]}\psi(\varsigma_2^*) + {}^{\varpi}I_{\varsigma_2^*-}^{[\varsigma_1^*,\varsigma_2^*]}\psi(\varsigma_1^*)\right]\\ &\qquad = \int_0^1 \left[\varpi(1-\kappa) - \varpi(\kappa)\right]\psi'(\kappa\varsigma_1^* + (1-\kappa)\varsigma_2^*)d\kappa. \end{aligned} \tag{2.2}$$

holds.

Proof. Writing

$$= \int_0^1 [\varpi(1-\kappa) - \varpi(\kappa)]\, \psi'(\kappa\varsigma_1^* + (1-\kappa)\varsigma_2^*) d\kappa$$

$$= \int_0^1 \varpi(1-\kappa)\psi'(\kappa\varsigma_1^* + (1-\kappa)\varsigma_2^*) d\kappa - \int_0^1 \varpi(\kappa)\psi'(\kappa\varsigma_1^* + (1-\kappa)\varsigma_2^*) d\kappa$$

$$= A_1 - A_2.$$

From the above we have, integrating by parts we get

$$\begin{aligned} A_1 &= \int_0^1 \varpi(1-\kappa)\psi'\left(\kappa\varsigma_1^* + (1-\kappa)\varsigma_2^*\right) d\kappa \\ &= \frac{\varpi(1-\kappa)\psi\left(\kappa\varsigma_1^*+(1-\kappa)\varsigma_2^*\right)}{\varsigma_2^*-\varsigma_1^*}\Big|_0^1 + + \int_0^1 \frac{\varpi'(1-\kappa)\psi\left(\kappa\varsigma_1^*+(1-\kappa)\varsigma_2^*\right)}{\varsigma_2^*-\varsigma_1^*} d\kappa \\ &= \frac{\varpi(0)\psi\left(\varsigma_1^*\right)-\varpi(1)\psi(\varsigma_2^*)}{\varsigma_2^*-\varsigma_1^*} + \frac{1}{\left(\varsigma_2^*-\varsigma_1^*\right)}\int_0^1 \varpi'(1-\kappa)\psi\left(\kappa\varsigma_1^* + (1-\kappa)\varsigma_2^*\right) d\kappa \end{aligned}$$

Putting $z = \kappa\varsigma_1^* + (1-\kappa)\varsigma_2^*$, so $dz = (\varsigma_2^* - \varsigma_1^*)d\kappa$, with this change of variables, we obtain

$$A_1 = \frac{\varpi(0)\psi(\varsigma_1^*)}{\varsigma_2^* - \varsigma_1^*} - \frac{\varpi(1)\psi(\varsigma_2^*)}{\varsigma_2^* - \varsigma_1^*} + \frac{1}{\varsigma_2^* - \varsigma_1^*}\ {}^{\varpi}I_{\varsigma_2^*-}^{[\varsigma_1^*,\varsigma_2^*]}\psi(\varsigma_1^*).$$

Analogously,

$$A_2 = \frac{\varpi(1)\psi(\varsigma_1^*)}{\varsigma_2^* - \varsigma_1^*} - \frac{\varpi(0)\psi(\varsigma_2^*)}{\varsigma_2^* - \varsigma_1^*} - \frac{1}{\varsigma_2^* - \varsigma_1^*}\ {}^{\varpi}I_{\varsigma_1^*+}^{[\varsigma_1^*,\varsigma_2^*]}\psi(\varsigma_2^*).$$

Subtracting ς_2^* from ς_1^* and reordering, the required equality (2.2) is obtained. □

Remark 4. *Putting $\varpi(\kappa) = \kappa^\alpha$ from this result, we obtain the Lemma 2 of [36]. On the other hand, if we put $\varpi(\kappa) = \kappa$, our result contains as a particular case, Lemma 2.1 of [11].*

On the basis of this result, we can obtain the following inequality.

Theorem 5. *Let $\psi : [\varsigma_1^*, \varsigma_2^*] \longrightarrow \mathbf{R}$ and $\psi \in C^1(\varsigma_1^*, \varsigma_2^*)$. If $\psi' \in L_1[\varsigma_1^*, \varsigma_2^*]$ and $|\psi'|$ is a quasi-convex on $[\varsigma_1^*, \varsigma_2^*]$, then following inequality holds*

$$\left|\frac{\varpi(0)-\varpi(1)}{\varsigma_2^*-\varsigma_1^*}\left(\psi(\varsigma_1^*) + \psi(\varsigma_2^*)\right) + \frac{1}{(\varsigma_2^*-\varsigma_1^*)^2}\left[{}^{\varpi}I_{\varsigma_1^*+}^{[\varsigma_1^*,\varsigma_2^*]}\psi(\varsigma_2^*) + {}^{\varpi}I_{\varsigma_2^*-}^{[\varsigma_1^*,\varsigma_2^*]}\psi(\varsigma_1^*)\right]\right|$$

$$\leq 2\max\{\psi'(\varsigma_1^*), \psi'(\varsigma_2^*)\}\int_0^1 \varpi(\kappa) d\kappa.$$

Proof. From equation (2.2) of Lemma 3, the quasi-convex of $|\psi'|$, the properties of modulus and $\varpi(\kappa)$, we have

$$\left|\frac{\varpi(0)-\varpi(1)}{\varsigma_2^*-\varsigma_1^*}\left(\psi(\varsigma_1^*)+\psi(\varsigma_2^*)\right)+\frac{1}{(\varsigma_2^*-\varsigma_1^*)^2}\left[{}^{\varpi}I_{\varsigma_1^*+}^{[\varsigma_1^*,\varsigma_2^*]}\psi(\varsigma_2^*)+{}^{\varpi}I_{\varsigma_2^*-}^{[\varsigma_1^*,\varsigma_2^*]}\psi(\varsigma_1^*)\right]\right|$$

$$\leq \int_0^1 |\varpi(1-\kappa)-\varpi(\kappa)|\,|\psi'(\kappa\varsigma_1^*+(1-\kappa)\varsigma_2^*)|\,d\kappa$$

$$\leq \int_0^1 |\varpi(1-\kappa)+\varpi(\kappa)|\,|\psi'(\kappa\varsigma_1^*+(1-\kappa)\varsigma_2^*)|\,d\kappa$$

$$\leq 2\max\left\{|\psi'(\varsigma_1^*)|,|\psi'(\varsigma_2^*)|\right\}\int_0^1 \varpi(\kappa)d\kappa.$$

using $\int_0^1 \varpi(1-\kappa)d\kappa=\int_0^1 \varpi(\kappa)d\kappa$. This completes the proof. □

Remark 6. *Considering as in the previous Remark, this result covers the Theorem 2.2 of [31]. If we put $\varpi(\kappa)=\kappa$ (see the second part of the Remark 10) we obtain Theorem 1 of [18].*

Refinements of the previous result are included in the following.

Theorem 7. *Let $\psi:[\varsigma_1^*,\varsigma_2^*]\longrightarrow \mathbf{R}$ and $\psi\in C^1(\varsigma_1^*,\varsigma_2^*)$. If $\psi'\in L_1[\varsigma_1^*,\varsigma_2^*]$ and $|\psi'|^q$ is quasi-convex on $[\varsigma_1^*,\varsigma_2^*]$, then $\forall p,q>1$ and $\frac{1}{p}+\frac{1}{q}=1$ the inequality*

$$\left|\frac{\varpi(0)-\varpi(1)}{\varsigma_2^*-\varsigma_1^*}\left(\psi(\varsigma_1^*)+\psi(\varsigma_2^*)\right)+\frac{1}{(\varsigma_2^*-\varsigma_1^*)^2}\left[{}^{\varpi}I_{\varsigma_1^*+}^{[\varsigma_1^*,\varsigma_2^*]}\psi(\varsigma_2^*)+{}^{\varpi}I_{\varsigma_2^*-}^{[\varsigma_1^*,\varsigma_2^*]}\psi(\varsigma_1^*)\right]\right|$$

$$\leq\left\{\left(\int_0^1 \varpi^p(1-\kappa)d\kappa\right)^{\frac{1}{p}}+\left(\int_0^1 \varpi^p(\kappa)d\kappa\right)^{\frac{1}{p}}\right\}\left(\max\left\{|\psi'(\varsigma_1^*)|^q,|\psi'(\varsigma_2^*)|^q\right\}\right)^{\frac{1}{q}} \tag{2.3}$$

is true.

Proof. From equation (2.2) of Lemma 3 we have

$$\left|\frac{\varpi(0)-\varpi(1)}{\varsigma_2^*-\varsigma_1^*}\left(\psi(\varsigma_1^*)+\psi(\varsigma_2^*)\right)+\frac{1}{(\varsigma_2^*-\varsigma_1^*)^2}\left[{}^{\varpi}I_{\varsigma_1^*+}^{[\varsigma_1^*,\varsigma_2^*]}\psi(\varsigma_2^*)+{}^{\varpi}I_{\varsigma_2^*-}^{[\varsigma_1^*,\varsigma_2^*]}\psi(\varsigma_1^*)\right]\right|$$

$$\leq \int_0^1 |\varpi(1-\kappa)-\varpi(\kappa)|\,|\psi'(\kappa\varsigma_1^*+(1-\kappa)\varsigma_2^*)|\,d\kappa$$

$$\leq \int_0^1 \varpi(1-\kappa)\,|\psi'(\kappa\varsigma_1^*+(1-\kappa)\varsigma_2^*)|\,d\kappa+\int_0^1 \varpi(\kappa)\,|\psi'(\kappa\varsigma_1^*+(1-\kappa)\varsigma_2^*)|\,d\kappa$$

and using well known Hölder's integral inequality, we get

$$\int_0^1 \varpi(1-\kappa)\left|\psi'(\kappa\varsigma_1^* + (1-\kappa)\varsigma_2^*)\right| d\kappa$$

$$\leq \left(\int_0^1 \varpi^p(1-\kappa)d\kappa\right)^{\frac{1}{p}} \left(\int_0^1 \left|\psi'(\kappa\varsigma_1^* + (1-\kappa)\varsigma_2^*)\right|^q d\kappa\right)^{\frac{1}{q}}$$

$$\leq \left(\int_0^1 \varpi^p(1-\kappa)d\kappa\right)^{\frac{1}{p}} \left(\max\left\{\left|\psi'(\varsigma_1^*)\right|^q, \left|\psi'(\varsigma_2^*)\right|^q\right\}\right)^{\frac{1}{q}}.$$

Analogously

$$\int_0^1 \varpi(\kappa)\left|\psi'(\kappa\varsigma_1^* + (1-\kappa)\varsigma_2^*)\right| d\kappa$$

$$\leq \left(\int_0^1 \varpi^p(\kappa)d\kappa\right)^{\frac{1}{p}} \left(\max\left\{\left|\psi'(\varsigma_1^*)\right|^q, \left|\psi'(\varsigma_2^*)\right|^q\right\}\right)^{\frac{1}{q}}.$$

The last two results allow us to obtain the requested inequality (2.3). This completes the proof. □

Remark 8. *If we take $\varpi(\kappa) = \kappa^\alpha$, this result becomes the Theorem 2.3 of [31].*

Theorem 9. *Let $\psi : [\varsigma_1^*, \varsigma_2^*] \longrightarrow \mathbf{R}$ and $\psi \in C^1(\varsigma_1^*, \varsigma_2^*)$. If $\psi' \in L_1[\varsigma_1^*, \varsigma_2^*]$ and $|\psi'|^q$ is quasi-convex on $[\varsigma_1^*, \varsigma_2^*]$, then for all $q \geq 1$ inequality*

$$\left|\frac{\varpi(0)-\varpi(1)}{\varsigma_2^*-\varsigma_1^*}\left(\psi(\varsigma_1^*)+\psi(\varsigma_2^*)\right) + \frac{1}{(\varsigma_2^*-\varsigma_1^*)^2}\left[{}^{\varpi}I_{\varsigma_1^*+}^{[\varsigma_1^*,\varsigma_2^*]}\psi(\varsigma_2^*) + {}^{\varpi}I_{\varsigma_2^*-}^{[\varsigma_1^*,\varsigma_2^*]}\psi(\varsigma_1^*)\right]\right|$$
$$\leq \left(\max\left\{\left|\psi'(\varsigma_1^*)\right|^q, \left|\psi'(\varsigma_2^*)\right|^q\right\}\right)^{\frac{1}{q}} \left(\int_0^1 \varpi(1-\kappa)d\kappa + \int_0^1 \varpi(\kappa)d\kappa\right) \tag{2.4}$$

is true.

Proof. Similar to the proof of the previous theorem, we can write

$$\left|\frac{\varpi(0)-\varpi(1)}{\varsigma_2^*-\varsigma_1^*}\left(\psi(\varsigma_1^*)+\psi(\varsigma_2^*)\right) + \frac{1}{(\varsigma_2^*-\varsigma_1^*)^2}\left[{}^{\varpi}I_{\varsigma_1^*+}^{[\varsigma_1^*,\varsigma_2^*]}\psi(\varsigma_2^*) + {}^{\varpi}I_{\varsigma_2^*-}^{[\varsigma_1^*,\varsigma_2^*]}\psi(\varsigma_1^*)\right]\right|$$

$$\leq \int_0^1 \varpi(1-\kappa)\left|\psi'(\kappa\varsigma_1^* + (1-\kappa)\varsigma_2^*)\right| d\kappa + \int_0^1 \varpi(\kappa)\left|\psi'(\kappa\varsigma_1^* + (1-\kappa)\varsigma_2^*)\right| d\kappa.$$

Using the power mean inequality for the first integral, we get

$$\int_0^1 \varpi(1-\kappa)\left|\psi'(\kappa\varsigma_1^* + (1-\kappa)\varsigma_2^*)\right| d\kappa$$

$$\leq \left(\int_0^1 \varpi(1-\kappa)d\kappa\right)^{1-\frac{1}{q}} \left(\int_0^1 \varpi(1-\kappa)\left|\psi'(\kappa\varsigma_1^* + (1-\kappa)\varsigma_2^*)\right|^q d\kappa\right)^{\frac{1}{q}}$$

$$\leq \left(\int_0^1 \varpi(1-\kappa)\right) \left(\max\left\{\left|\psi'(\varsigma_1^*)\right|^q, \left|\psi'(\varsigma_2^*)\right|^q\right\} d\kappa\right)^{\frac{1}{q}}.$$

Similarly, for the second integral we can write

$$\begin{aligned}&\int_0^1 \varpi(\kappa)\left|\psi'(\kappa\varsigma_1^*+(1-\kappa)\varsigma_2^*)\right|d\kappa\\ &\leq \left(\int_0^1 \varpi(\kappa)d\kappa\right)^{1-\frac{1}{q}}\left(\int_0^1 \varpi(\kappa)\left|\psi'(\kappa\varsigma_1^*+(1-\kappa)\varsigma_2^*)\right|^q d\kappa\right)^{\frac{1}{q}}\\ &\leq \left(\int_0^1 \varpi(\kappa)\right)\left(\max\left\{\left|\psi'(\varsigma_1^*)\right|^q,\left|\psi'(\varsigma_2^*)\right|^q\right\}d\kappa\right)^{\frac{1}{q}}.\end{aligned}$$

So we have

$$\begin{aligned}&\left|\frac{\varpi(1)}{\varsigma_2^*-\varsigma_1^*}\left(\psi(\varsigma_1^*)+\psi(\varsigma_2^*)\right)-\frac{1}{(\varsigma_2^*-\varsigma_1^*)^2}\left[{}^{\varpi}I_{\varsigma_1^*+}^{[\varsigma_1^*,\varsigma_2^*]}\psi(\varsigma_2^*)+{}^{\varpi}I_{\varsigma_2^*-}^{[\varsigma_1^*,\varsigma_2^*]}\psi(\varsigma_1^*)\right]\right|\\ &\leq \left(\max\left\{\left|\psi'(\varsigma_1^*)\right|^q,\left|\psi'(\varsigma_2^*)\right|^q\right\}\right)^{\frac{1}{q}}\left(\int_0^1 \varpi(1-\kappa)d\kappa+\int_0^1 \varpi(\kappa)d\kappa\right)\end{aligned}$$

which is the inequality (2.4) sought. In this way we complete the proof. □

Remark 10. *It is easy to see that Theorem 2.4 of [31] is obtained from the previous one, putting $\varpi(\kappa)=\kappa^{\alpha}$, we also obtain Theorem 2 of [18], if we consider $\varpi(\kappa)=\kappa$.*

3. Some results for functions whose second derivative is quasi-convex

We will use the following result throughout this section.

Lemma 1. *Let $\psi:[\varsigma_1^*,\varsigma_2^*]\longrightarrow \mathbf{R}$ and $\psi\in C^2(\varsigma_1^*,\varsigma_2^*)$. If $\psi''\in L[\varsigma_1^*,\varsigma_2^*]$, then the following equality*

$$\begin{aligned}&\frac{(\varpi(0)-\varpi(1))\left(\psi'(\varsigma_1^*)+\psi'(\varsigma_2^*)\right)}{\varsigma_2^*-\varsigma_1^*}-\frac{(\varpi'(0)+\varpi'(1))\left(\psi(\varsigma_2^*)-\psi(\varsigma_1^*)\right)}{(\varsigma_2^*-\varsigma_1^*)^2}\\ &+\frac{1}{(\varsigma_2^*-\varsigma_1^*)^2}\left({}^{\varpi}\mathbf{I}_{\varsigma_1^*}^{[\varsigma_1^*,\varsigma_2^*]}\psi(\varsigma_2^*)+{}^{\varpi}\mathbf{I}_{\varsigma_2^*}^{[\varsigma_1^*,\varsigma_2^*]}\psi(\varsigma_1^*)\right)\\ &=\int_0^1\left[\varpi(1-\kappa)-\varpi(\kappa)\right]\psi''(\kappa\varsigma_1^*+(1-\kappa)\varsigma_2^*)d\kappa\end{aligned}\tag{3.1}$$

is true.

Proof. Writing

$$\begin{aligned}B&=\int_0^1\left[\varpi(1-\kappa)-\varpi(\kappa)\right]\psi''(\kappa\varsigma_1^*+(1-\kappa)\varsigma_2^*)d\kappa\\ &=\int_0^1 \varpi(1-\kappa)\psi''(\kappa\varsigma_1^*+(1-\kappa)\varsigma_2^*)d\kappa-\int_0^1 \varpi(\kappa)\psi''(\kappa\varsigma_1^*+(1-\kappa)\varsigma_2^*)d\kappa\\ &=B_1-B_2.\end{aligned}$$

From the above, integrating by parts twice and making a change of variables, we have

$$\int_0^1 \varpi(1-\kappa)\psi''\left(\kappa\varsigma_1^*+(1-\kappa)\varsigma_2^*\right)d\kappa = \frac{\varpi(0)\psi'\left(\varsigma_1^*\right)-\varpi(1)\psi'(\varsigma_2^*)}{\varsigma_2^*-\varsigma_1^*} + \frac{\varpi'(0)\psi\left(\varsigma_1^*\right)-\varpi'(1)\psi(\varsigma_2^*)}{(\varsigma_2^*-\varsigma_1^*)^2} + \int_0^1 \frac{\varpi''(1-\kappa)\psi\left(\kappa\varsigma_1^*+(1-\kappa)\varsigma_2^*\right)}{(\varsigma_2^*-\varsigma_1^*)^2}d\kappa$$

Putting $z=\kappa\varsigma_1^*+(1-\kappa)\varsigma_2^*$, so $dz=(\varsigma_2^*-\varsigma_1^*)d\kappa$, with this change of variables, we obtain

$$B_1=\frac{\varpi(0)\psi'\left(\varsigma_1^*\right)-\varpi(1)\psi'(\varsigma_2^*)}{\varsigma_2^*-\varsigma_1^*}+\frac{\varpi'(0)\psi\left(\varsigma_1^*\right)-\varpi'(1)\psi(\varsigma_2^*)}{(\varsigma_2^*-\varsigma_1^*)^2}+\frac{{}^{\varpi}\mathbf{I}_{\varsigma_2^*}^{[\varsigma_1^*,\varsigma_2^*]}\psi(\varsigma_1^*)}{(\varsigma_2^*-\varsigma_1^*)^2}$$

Analogously

$$\int_0^1 \varpi(\kappa)\psi''\left(\kappa\varsigma_1^*+(1-\kappa)\varsigma_2^*\right)d\kappa = \frac{\varpi(1)\psi'\left(\varsigma_1^*\right)-\varpi(0)\psi'(\varsigma_2^*)}{\varsigma_2^*-\varsigma_1^*} + \frac{\varpi'(0)\psi\left(\varsigma_2^*\right)-\varpi'(1)\psi(\varsigma_1^*)}{(\varsigma_2^*-\varsigma_1^*)^2} + \int_0^1 \frac{\varpi''(\kappa)\psi\left(\kappa\varsigma_1^*+(1-\kappa)\varsigma_2^*\right)}{(\varsigma_2^*-\varsigma_1^*)^2}d\kappa$$

Putting $z=\kappa\varsigma_1^*+(1-\kappa)\varsigma_2^*$, so $dz=(\varsigma_2^*-\varsigma_1^*)d\kappa$, with this change of variables, we obtain

$$B_2=\frac{\varpi(1)\psi'\left(\varsigma_1^*\right)-\varpi(0)\psi'(\varsigma_2^*)}{\varsigma_2^*-\varsigma_1^*}+\frac{\varpi'(0)\psi\left(\varsigma_2^*\right)-\varpi'(1)\psi(\varsigma_1^*)}{(\varsigma_2^*-\varsigma_1^*)^2}+\frac{{}^{\varpi}\mathbf{I}_{\varsigma_1^*}^{[\varsigma_1^*,\varsigma_2^*]}\psi(\varsigma_2^*)}{(\varsigma_2^*-\varsigma_1^*)^2}.$$

Subtracting B_2 from B_1 and reordering, the required equality (3.1) is obtained. □

Remark 2. *Putting $\varpi(\kappa)=\kappa(1-\kappa)$ from this result, we obtain the Lemma 1 of [4]. On the other hand, if we put $\varpi(\kappa)=k(\kappa)$, our result contains as a particular case, Lemma 2 of [29], where $k(\kappa)$ is the function defined in this paper.*

For brevity of expressions, we introduce the notation

$$L(HH)=\frac{(\varpi(0)-\varpi(1))\left(\psi'\left(\varsigma_1^*\right)+\psi'(\varsigma_2^*)\right)}{\varsigma_2^*-\varsigma_1^*}-\frac{(\varpi'(0)+\varpi'(1))\left(\psi\left(\varsigma_2^*\right)-\psi(\varsigma_1^*)\right)}{(\varsigma_2^*-\varsigma_1^*)^2} + \frac{1}{(\varsigma_2^*-\varsigma_1^*)^2}\left({}^{\varpi}\mathbf{I}_{\varsigma_1^*}^{[\varsigma_1^*,\varsigma_2^*]}\psi(\varsigma_2^*)+{}^{\varpi}\,\mathbf{I}_{\varsigma_2^*}^{[\varsigma_1^*,\varsigma_2^*]}\psi(\varsigma_1^*)\right).$$

On the basis of the Lemma 1, we can obtain the following inequality.

Theorem 3. *Let $\psi : [\varsigma_1^*, \varsigma_2^*] \longrightarrow \mathbf{R}$ and $\psi \in C^2(\varsigma_1^*, \varsigma_2^*)$. If $\psi'' \in L[\varsigma_1^*, \varsigma_2^*]$ and ψ'' is a quasi-convex function on $[\varsigma_1^*, \varsigma_2^*]$, then the following inequality holds:*

$$|L(HH)| \leq 2\max\{\psi''(\varsigma_1^*), \psi''(\varsigma_2^*)\} \int_0^1 \varpi(\kappa)d\kappa.$$

Proof. From Lemma 1, property of the modulus and using the hypothesis that ψ'' is quasi-convex, we have:

$$\begin{aligned} |L(HH)| &\leq \int_0^1 \varpi(1-\kappa) - \varpi(\kappa) \left|\psi''(\kappa\varsigma_1^* + (1-\kappa)\varsigma_2^*)\right| d\kappa \\ &\leq \int_0^1 \varpi(1-\kappa) + \varpi(\kappa) \left|\psi''(\kappa\varsigma_1^* + (1-\kappa)\varsigma_2^*)\right| d\kappa \\ &\leq 2\max\{\psi''(\varsigma_1^*), \psi''(\varsigma_2^*)\} \int_0^1 \varpi(\kappa)d\kappa. \end{aligned}$$

using $\int_0^1 \varpi(1-\kappa)d\kappa = \int_0^1 \varpi(\kappa)d\kappa$. □

Remark 4. *Considering the function $\varpi(\kappa) = k(\kappa)$ as the previous Remark, we obtain the Theorem 2 of [30]. We also obtain Theorem 3 of [4] if we put $\varpi(\kappa) = \kappa(1-\kappa)$.*

The next result includes refinements of the previous result.

Theorem 5. *Let $\psi : [\varsigma_1^*, \varsigma_2^*] \longrightarrow \mathbf{R}$ and $\psi \in C^2(\varsigma_1^*, \varsigma_2^*)$. If $\psi'' \in L[\varsigma_1^*, \varsigma_2^*]$ and $|\psi''|^q$ is a quasi-convex function on $[\varsigma_1^*, \varsigma_2^*]$, then $\forall p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$ the following inequality holds:*

$$|L(HH)| \leq \left\{ \left(\int_0^1 \varpi^p(1-\kappa)d\kappa \right)^{\frac{1}{p}} \left(\int_0^1 \varpi^p(\kappa)d\kappa \right)^{\frac{1}{p}} \right\} \left(\max\left\{|\psi''(\varsigma_1^*)|^q, |\psi''(\varsigma_2^*)|^q\right\}\right)^{\frac{1}{q}}. \tag{3.2}$$

Proof. From Lemma 1 and property of the modulus we have

$$\begin{aligned} |L(HH)| &\leq \int_0^1 |\varpi(1-\kappa) - \varpi(\kappa)| \left|\psi''(\kappa\varsigma_1^* + (1-\kappa)\varsigma_2^*)\right| d\kappa \\ &\leq \int_0^1 \varpi(1-\kappa) \left|\psi''(\kappa\varsigma_1^* + (1-\kappa)\varsigma_2^*\right) \kappa + \int_0^1 \varpi(\kappa) \left|\psi''(\kappa\varsigma_1^* + (1-\kappa)\varsigma_2^*)\right| d\kappa \end{aligned}$$

and using well known Hölder's integral inequality and the quasi convex of $|\psi''|^q$, we get

$$\int_0^1 \varpi(1-\kappa)\left|\psi''(\kappa\varsigma_1^*+(1-\kappa)\varsigma_2^*)\right| d\kappa$$

$$\leq \left(\int_0^1 \varpi^p(1-\kappa)d\kappa\right)^{\frac{1}{p}} \left(\int_0^1 \left|\psi''(\kappa\varsigma_1^*+(1-\kappa)\varsigma_2^*)\right|^q d\kappa\right)^{\frac{1}{q}}$$

$$\leq \left(\int_0^1 \varpi^p(1-\kappa)d\kappa\right)^{\frac{1}{p}} \left(\max\left\{\left|\psi''(\varsigma_1^*)\right|^q, \left|\psi''(\varsigma_2^*)\right|^q\right\}\right)^{\frac{1}{q}}.$$

Analogously

$$\int_0^1 \varpi(\kappa)\left|\psi''(\kappa\varsigma_1^*+(1-\kappa)\varsigma_2^*)\right| d\kappa$$
$$\leq \left(\int_0^1 \varpi^p(\kappa)d\kappa\right)^{\frac{1}{p}} \left(\max\left\{\left|\psi''(\varsigma_1^*)\right|^q, \left|\psi''(\varsigma_2^*)\right|^q\right\}\right)^{\frac{1}{q}}.$$

The last two results allow us to obtain the requested inequality (3.2). This completes the proof. □

Remark 6. *Considering as in the previous Remark, this result covers the Theorem 3 of [30], if we take $\varpi(\kappa) = k(\kappa)$, the Theorem 1 of [28] and Theorem 4 of [4], if we take $\varpi(\kappa) = \kappa(1-\kappa)$.*

The following theorem gives us another form of the previous result.

Theorem 7. *Let $\psi : [\varsigma_1^*, \varsigma_2^*] \longrightarrow \mathbf{R}$ and $\psi \in C^2(\varsigma_1^*, \varsigma_2^*)$. If $\psi'' \in L[\varsigma_1^*, \varsigma_2^*]$ and $|\psi''|^q$ is a quasi-convex function on $[\varsigma_1^*, \varsigma_2^*]$, then for all $q \geq 1$ the following inequality holds:*

$$|L(HH)| \leq \left(\max\left\{\left|\psi''(\varsigma_1^*)\right|^q, \left|\psi''(\varsigma_2^*)\right|^q\right\}\right)^{\frac{1}{q}} \left(\int_0^1 \varpi(1-\kappa)d\kappa + \int_0^1 \varpi(\kappa)d\kappa\right) \tag{3.3}$$

is true.

Proof. Proceeding as in the previous Theorem's proof, we have

$$|L(HH)| \leq \int_0^1 \varpi(1-\kappa)\left|\psi''(\kappa\varsigma_1^*+(1-\kappa)\varsigma_2^*)\right| d\kappa + \int_0^1 \varpi(\kappa)\left|\psi''(\kappa\varsigma_1^*+(1-\kappa)\varsigma_2^*)\right| d\kappa$$

and using the well known power mean inequality, we get

$$\int_0^1 \varpi(1-\kappa)\left|\psi''(\kappa\varsigma_1^* + (1-\kappa)\varsigma_2^*)\right| d\kappa$$

$$\leq \left(\int_0^1 \varpi(1-\kappa)d\kappa\right)^{1-\frac{1}{q}} \left(\int_0^1 \varpi(1-\kappa)\left|\psi''(\kappa\varsigma_1^* + (1-\kappa)\varsigma_2^*)\right|^q d\kappa\right)^{\frac{1}{q}}$$

$$\leq \left(\int_0^1 \varpi(1-\kappa)d\kappa\right)\left(\max\left\{|\psi''(\varsigma_1^*)|^q, |\psi''(\varsigma_2^*)|^q\right\}\right)^{\frac{1}{q}}.$$

Similarly

$$\int_0^1 \varpi(\kappa)\left|\psi''(\kappa\varsigma_1^* + (1-\kappa)\varsigma_2^*)\right| d\kappa$$

$$\leq \left(\int_0^1 \varpi(\kappa)d\kappa\right)^{1-\frac{1}{q}} \left(\int_0^1 \varpi(\kappa)\left|\psi''(\kappa\varsigma_1^* + (1-\kappa)\varsigma_2^*)\right|^q d\kappa\right)^{\frac{1}{q}}$$

$$\leq \left(\int_0^1 \varpi(\kappa)d\kappa\right)\left(\max\left\{|\psi''(\varsigma_1^*)|^q, |\psi''(\varsigma_2^*)|^q\right\}\right)^{\frac{1}{q}}.$$

So, we can write

$$|L(HH)| \leq \left(\max\left\{|\psi''(\varsigma_1^*)|^q, |\psi''(\varsigma_2^*)|^q\right\}\right)^{\frac{1}{q}} \left(\int_0^1 \varpi(1-\kappa)d\kappa + \int_0^1 \varpi(\kappa)d\kappa\right)$$

which is the inequality (3.3) sought. In this way we complete the proof. □

Remark 8. *It is easy to see that Theorem 2 of [28] is obtained from the previous one, putting $\varpi(\kappa) = \kappa(1-\kappa)$. We also obtain Theorem 5 of [4] if we considerig the same function.*

4. Conclusions

In this article, we have obtained new integral inequalities related to the Hermite-Hadamard inequality, using quasi-convex functions, under the weighted operators of the Definition 5. We point out that several known results from the literature are obtained as particular cases of those presented here. Finally, we want to point out that the working method used can be extrapolated to other notions of convexity, for example, to the case of harmonically-convex functions and can even be used in obtain new inequalities of the Hermite-Hadamard-Fejer type.

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Bahtiyar Bayraktar
Bursa Uludag University,
Faculty of Education,
Gorukle Campus,
Bursa,
Turkey
e-mail: bbayraktar@uludag.edu.tr
Corresponding author

Juan E. Nápoles Valdés
Universidad Nacional del Nordeste,
Facultad de Ciencias Exactas,
Ave. Libertad 5450, Corrientes 3400,
Argentina
e-mail: jnapoles@exa.unne.edu.ar

Florencia Rabossi
Universidad Nacional del Nordeste,
Facultad de Ciencias Exactas,
Ave. Libertad 5450, Corrientes 3400,
Argentina
e-mail: florenciarabossi@exa.unne.edu.ar

and

Aylen D. Samaniego
Universidad Nacional del Nordeste,
Facultad de Ciencias Exactas,
Ave. Libertad 5450, Corrientes 3400,
Argentina
e-mail: aylen.samaniego@comunidad.unne.edu.ar