Proyecciones Journal of Mathematics
Vol. 42, N$^o$ 4, pp. 931-956, August 2023.
Universidad Católica del Norte
Antofagasta - Chile

10.22199/issn.0717-6279-5608

# Maximal matching cover pebbling number for variants of hypercube

*S. Sarah Surya*
*Stella Maris College(Autonomous), India*
*and*
*Lian Mathew*
*CHRIST (Deemed to be University), India*
*Received : August 2022. Accepted : March 2023*

**Abstract**

*An edge pebbling move is defined as the removal of two pebbles from one edge and placing one on the adjacent edge. The maximal matching cover pebbling number, $f_{mmcp}(G)$, of a graph $G$, is the minimum number of pebbles that must be placed on $E(G)$, such that after a sequence of pebbling moves the set of edges with pebbles forms a maximal matching regardless of the initial configuration. In this paper, we find the maximal matching cover pebbling number for variants of hypercube.*

**Keywords:** *Graph pebbling, Maximal matching, Cover pebbling number, Maximal matching cover pebbling number.*

**2010 Subject Mathematics Classification:** *05C69, 05C70, 54A25*

## 1. Introduction

A pebbling move is defined as the removal of two pebbles from one vertex and placing one on the adjacent vertex whereas an edge pebbling move is the removal of two pebbles from one edge and placing one on the adjacent edge. The pebbling number of a graph $G$ denoted by $f(G)$, is the least $n$ such that however $n$ pebbles are placed on the vertices of $G$, we can move a pebble to any vertex by a sequence of pebbling moves[3] and an edge pebbling number of a graph $G$, denoted by, $f_e(G)$, is the least $n$ such that however $n$ pebbles are placed on the edges of $G$, we can move a pebble to any edge by a sequence of pebbling moves[7]. For a survey of additional results, refer [6]. Also, the variants of graph pebbling can be found in [4, 5, 10, 11]. This graph variant has wide applications in the architecture field, finding the minimum quantity of drugs required for injecting drugs in medical science, placing stones in jewels, networking, plumbing work and so on.

## 2. Preliminaries

**Definition 1.** [1] Given a graph $G = (V, E)$, a **matching** $M$ in $G$ is a set of pairwise non-adjacent edges, none of which are loops; i.e., no edges share common vertices.

**Definition 2.** [1] A matching $M$ of a graph $G$ is **maximal** if every edge in $G$ has a non-empty intersection with at least one edge in $M$ and a **maximum matching** is a matching that contains the largest possible number of edges in $M$.

**Definition 3.** [4]The **cover pebbling number** denoted by $\gamma(G)$ of a graph $G$ is the minimum number of pebbles required to place a pebble on every vertex simultaneously under any initial configuration.

**Definition 4.** [9] The **maximal matching cover pebbling number** denoted by, $f_{mmcp}(G)$ is defined as the minimum number of pebbles that must be placed on $E(G)$, such that after a sequence of pebbling moves the set of edges with pebbles forms a maximal matching regardless of the initial configuration.

**Definition 5.** The distance between the vertices in the corresponding line graph is called the **edge distance** of a graph $G$.

**Definition 6.** [2] The **hypercube** $Q_n$ is defined recursively in terms of the cartesian product of two graphs as $Q_1 = K_2$ and $Q_n = K_2 \times Q_{n-1}$.

**Definition 7.** [2] The $n$**- dimensional folded hypercube**, denoted by $FQ_n$, is an undirected graph obtained f rom hypercube $Q_n$ by adding all the complementary edges.

**Definition 8.** [2] The **crossed cube** $CQ_n$ is obtained by interchanging a pair of edges of the ordinary hypercube.

**Definition 9.** [2] The **folded crossed cube** $FCQ_n$ is constructed by connecting each node to a node farthest from it.

## 3. Main Results

**Theorem 1.** *The maximal matching cover pebbling number for hypercube $Q_2$ is, $f_{mmcp}(Q_2) = 4$.*

**Proof.** Let the edges of hypercube $Q_2$ be denoted by $e_1, e_2, e_3$ and $e_4$ in clockwise direction as seen in Figure 1. Then, either $\{e_1, e_3\}$ or $\{e_2, e_4\}$ forms the maximal matching for $Q_2$.

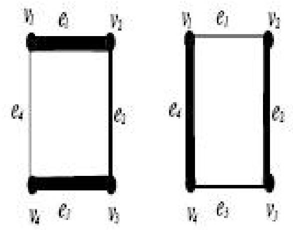

**Figure 1: Possible maximal matching edge sets for hypercube $Q_2$**

Let the proof be divided into the following cases based on the number of pebbles distributed to each edge of $Q_2$.

**Case 1:** Four pebbles are distributed to any edge of $Q_2$
Let the pebbled edge be denoted by $e_1$. Then, by a pebbling move, it is possible to place a pebble on the adjacent edges of $e_1$, which produces a maximal matching cover solution.

In order to show that the worst case configuration is by placing all the pebbles on a single edge, let us prove that it is always possible to place a pebble on the target edge set with three pebbles if we distribute all the pebbles to more than one edge.

**Case 2:** Three pebbles are distributed to any two of the edges of $Q_2$
If the pebbles are distributed to independent edges of $Q_2$, then there is nothing to prove. So, consider the case where the pebbles are distributed to the adjacent edges of $Q_2$. Let the pebbled edge be denoted by $e_1$ and $e_2$. Consider the distribution of a single pebble to $e_1$ and two pebbles to the edge $e_2$. Then, it is always possible to place a pebble on $e_3$ by a pebbling move and the result follows. □

Here, we note that the worst-case configuration is by placing all the pebbles on a single edge.

### 3.1. Maximal matching cover pebbling number for $Q_3$

**Theorem 2.** *The maximal matching cover pebbling number of a hypercube $Q_3$ is, $f_{mmcp}(Q_3) = 12$.*

**Proof.** Denote the vertices on the face $f_1$ of $Q_3$ by $v_1, v_2, v_3, v_4$ and the vertices on the opposite face $f_2$ as $v_5, v_6, v_7, v_8$. See Figure 2. The faces $f_3, f_4, f_5, f_6$ are formed by connecting the vertices $\{v_1, v_2, v_5, v_6\}$, $\{v_3, v_4, v_7, v_8\}$, $\{v_1, v_4, v_5, v_8\}$, $\{v_2, v_3, v_6, v_7\}$ respectively. Let $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11}, e_{12}$ be the edges which joins the vertices $(v_1, v_2)$,$(v_2, v_3)$,$(v_3, v_4)$, $(v_4, v_1)$, $(v_5, v_6)$, $(v_6, v_7)$, $(v_7, v_8)$, $(v_8, v_1)$, $(v_1, v_5)$, $(v_2, v_6)$, $(v_3, v_7)$, $(v_4, v_8)$ respectively.

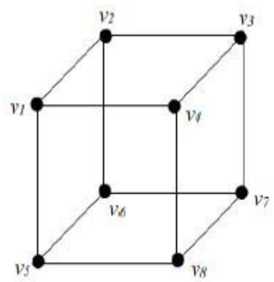


**Figure 2: Hypercube $Q_3$**

To prove the lemma, the proof is divided into the following cases based on the distribution of pebbles to the edges of $Q_3$.

**Case 1:** Twelve pebbles are distributed on any edge of $Q_3$

Let the pebbled edge be $e_1$. Place a pebble on the adjacent edges $e_2$ and $e_4$ by a pebbling move. With the remaining pebbles on $e_1$, it is also possible to place a pebble on $e_6$ and $e_8$ by a sequence of pebbling moves and hence the result follows.

In order to show that the worst case is by placing all the pebbles on a single edge, let us prove that it is always possible to place a pebble on the target edge set with eleven pebbles if we distribute all the pebbles to more than one edge.

**Case 2:** Eleven pebbles are distributed to any two edges of $Q_3$

Denote the pebbled edge by $e^{'}$ and $e^{''}$. The proof is divided into the following cases based on the distance between the pebbled edges.

Denote the number of pebbles distributed to the edges $e_1$ and $e_2$ as $S_1$ and $S_2$ where $S_1 \geq S_2$. Then, clearly $S_1 \geq 6$.

**Case 2.1:** $e^{'}$ and $e^{''}$ are adjacent

Let us assume that both the edges $e^{'}$ and $e^{''}$ are on the face $f^{'}$.

Let $S_1 = 6$. Place a pebble on the adjacent edges $e^{'}$ which is not on the face $f^{'}$ by a sequence of pebbling moves. Let the newly pebbled edge be on the face $f^{*}$. Transfer the remaining pebbles to the edge $e^{''}$ by a pebbling move. Now $e^{''}$ has $S_2 + [\frac{S_1-4}{2}] \geq 6$ pebbles as $S_2 \geq 1$ and hence the remaining pebbles are sufficient to place a pebble on the corresponding edges of the opposite face $f^{*}$.

Let $S_1 = 7$. Then, the edge $e^{''}$ has four pebbles on it. Place a pebble on the adjacent edges of $e^{''}$ which is not on the face $f^{'}$ by a pebbling move.

Let the newly pebbled edge $e^{'''}$ and $e^{''''}$ be on the face $f^{*}$. Now, it is always possible to place a pebble on the corresponding edges of $e^{'''}$ and $e^{''''}$ on the opposite face of $f^{*}$ from $S_1$ by a pebbling move.

Let $8 \leq S_1 \leq 9$. Place a pebble on the adjacent edges of $e^{'}$ which is not on the face $f^{'}$ by a pebbling move. Let the newly pebbled edge $e^{'''}$ and $e^{''''}$ be on the face $f^{*}$. With the remaining pebbles from $S_1$, place a pebble on the corresponding edge of $e^{'''}$ and $e^{''''}$ which is on the opposite face of $f^{*}$ but is not adjacent to $e^{''}$ by a pebbling move. Use the pebbles from $S_1$ to place a pebble on the remaining corresponding edges of $e^{'''}$ or $e^{''''}$ on the opposite face $f^{*}$ by a pebbling move and we are done.

Let $S_1 = 10$. Place a pebble on the opposite edge of $e^{''}$ which is on the same face $f^{'}$ by a pebbling move. Let the newly pebbled edge be denoted by $e^*$. The remaining pebbles on $S_1$ is sufficient to place a pebble on the corresponding edges of $e^{''}$ and $e^*$ on the opposite face by a sequence of pebbling moves and hence the result follows.

**Case 2.2:** $e^{'}$ and $e^{''}$ are at a distance two
Since the edges $e^{'}$ and $e^{''}$ are at a distance two, we have two following possibilities: Edges $e^{'}$ and $e^{''}$ can be on the same face and edges $e^{'}$ and $e^{''}$ can be on the adjacent faces.

**Case 2.2.1:** $e^{'}$ and $e^{''}$ are on the same face
Let us assume that both the pebbled edges $e^{'}$ and $e^{''}$ are on the same face $f^{'}$.
Consider the case where $S_1$ has a minimum of nine pebbles on it. By placing a pebble on two edges that are of distance two from $e^{'}$ but are not adjacent to $e^{''}$ by a pebbling move, we are done.
Consider the case where $S_1$ has exactly eight pebbles on it. Place a pebble on the adjacent edges of $e^{'}$ which is not on the face $f^{'}$ by a pebbling move. Let the pebbled edges $e^{'''}$ and $e^{''''}$ be on the face $f^*$. It is possible to place a pebble on the corresponding edges of $e^{'''}$ and $e^{''''}$ on the opposite $f^*$ from $S_1$ and $S_2$ by a sequence of pebbling moves.
Consider the case where $S_1$ has six or seven pebbles on it. Place a pebble on the adjacent edges of $e^{'}$ not on the face $f_1$ by a pebbling move from $S_1$. Let the pebbled edges be $e^{'''}$ and $e^{''''}$ be on the face $f^*$. By placing a pebble on the corresponding edges $e^{'''}$ and $e^{''''}$ on the opposite face $f^*$ from $S_2$ by a pebbling move, the result follows.

**Case 2.2.2:** $e^{'}$ and $e^{''}$ are on the adjacent faces
Let the edge $e^{'}$ belongs to the face $f^{'}$ and the edge $e^{''}$ belongs to the face $f^{''}$.
Consider the case where $S_1$ has a minimum of eight pebbles on it. Place a pebble on the edge which is of distance two from both the edges $e^{'}$ and $e^{''}$ that belongs to the face $f^{''}$ from $S_1$ by a pebbling move. Let the pebbled edge be denoted by $e^*$ and let the pebbled edges $e^{''}$ and $e^*$ be on the face $f^*$. Placing a pebble on the corresponding edges of $e^{''}$ and $e^*$ on the opposite face of $f^*$ by using the remaining pebbles from $S_1$, we are done.
Consider the case where the edge $e^{'}$ has seven pebbles on it. Place a pebble on the edge which is distance two from both the edges $e^{'}$ and $e^{''}$ that belongs to the face $f^{'}$ from $S_1$ by a pebbling move. Let the pebbled edge be denoted by $e^*$. Now, by placing the pebbles on the corresponding edges of $e^{'}$ and $e^*$ on the opposite face from $S_2$ by a sequence of pebbling moves, we are

done.
Consider the case where the edge $e^{'}$ has six pebbles on it. Place a pebble on the edge which is distance two from $e^{'}$ where the target edge belongs to the same face $f^{'}$ by a pebbling move by using the pebbles from $S_1$. Let the pebbled edge be denoted by $e^{'''}$. Now, by placing a pebble on the corresponding edges of $e^{'}$ and $e^{'''}$ on the opposite face of $f^{'}$ from $S_2$ by a pebbling move, the result follows.

**Case 3:** Eleven pebbles are distributed to any three edges
The proof is divided into the following subcases based on the adjacency of pebbled edges on each face. Let the pebbled edge be denoted by $e^{'}$, $e^{''}$ and $e^{'''}$ and the number of pebbles on each edge $e^{'}$, $e^{''}$ and $e^{'''}$ be denoted by $S_1$, $S_2$ and $S_3$.

**Case 3.1:** Three pebbled edges are adjacent
Without loss of generality, assume that the edge $e^{''}$ is adjacent to both the edges $e^{'}$ and $e^{''}$. The proof is divided into the following subcases based on the number of pebbled edges on each face.

**Case 3.1.1:** Three pebbled edges are on the same face
Let the pebbled edges be on the face $f_1$. If there exist at least four pebbles on $e^{'}$ and $e^{'''}$, then there is nothing to prove.
So, let us assume that $S_1 < 4$ and $S_3 < 4$. Then the different possibilities of $S_1$ and $S_2$ are as follows:{(1,1) , (2,2), (3,3), (1,2), (1,3), (2,3)}. Note that the possibility (1,2) and (2,1) are the same.
The case where both $S_1$ and $S_3$ have a minimum of two pebbles can be proved as follows. Place a pebble on the adjacent edges of $e^{'}$ and $e^{'''}$ in such a way that the newly pebbled edges should be on the same face and opposite to each other. Let the newly pebbled edges $e_l$ and $e_m$ be on the face $f^*$. Now with the remaining pebbles from $S_2$, it is always possible to place a pebble on the corresponding edges $e_l$ and $e_m$ on the opposite face of $f^*$.
Consider the case where at least one of $S_1$ or $S_3$ has a minimum of one pebble. Without loss of generality, assume that the edge $e^{'}$ has one pebble on it. Place a pebble on $e^{'}$ from $S_2$ by a pebbling move, and then the proof proceeds as discussed above.

**Case 3.1.2:** Two pebbled edges are in one face and the third pebbled edge is adjacent to both the pebbled edges but belongs to an adjacent face
Let the pebbled edges on the face $f_1$ be denoted by $e^{'}$ & $e^{''}$ and the third

pebbled edge be denoted by $e^{'''}$. Let the number of pebbles on the edges $e^{'}$,$e^{''}$ and $e^{'''}$ be denoted by $S_1$, $S_2$ and $S_3$ respectively.

It is always possible to produce a maximal matching cover solution if there exists at least 2 pebbles on any one $S_i, i = 1, 2, 3$ and six pebbles on any other remaining $S_i, i = 1, 2, 3$.
Since, $S_1 + S_2 + S_3 = 11$, the remaining possibilities are as follows: $(1, 1, 9), (5, 3, 3), (4, 5, 2), (5, 5, 1)$.

Consider the case where the distribution of pebbles to the edges are (1,1,9). Move one pebble from the edge which has nine pebbles to one of the remaining pebbled edges by a pebbling move. Let the newly pebbled edge be denoted by $e^*$. Now, it is possible to place a pebble on the opposite edge of the pebbled edge which has one pebble on the same face by using the pebbles from $e^*$. Consequently, we have placed pebbles on two independent edges. Now the remaining seven pebbles are sufficient to place a pebble on the opposite face by a sequence of pebbling moves.
All the other cases can be proved in a similar manner.

**Case 3.2:** Two pebbled edges are adjacent
Let the pebbled edge on the face $f_1$ be denoted by $e^{'}$ and $e^{''}$.
Let the proof be divided into the following cases based on the distance of the third pebbled edge from either $e^{'}$ and $e^{''}$

**Case 3.2.1:** Third pebbled edge $e^{'''}$ is placed at distance two from both $e^{'}$ and $e^{''}$
Let the pebbled edge which is opposite to the edge $e^{'''}$ on the face $f_1$ be $e^{''}$. Then, if there exist at least six pebbles on the edge $e^{'}$, it is always possible to place a pebble on the corresponding edges of $e^{''}$ and $e^{'''}$ on the opposite face by a pebbling move. So, consider the case where the number of pebbles on the edge $e^{'}$ is less than six.
Let the number of pebbles distributed to the edge $e^{'}$ be five. Then, if there exists at least three pebbles on the adjacent pebbled edge of $e^{'}$, we are done. So, consider the case where the edge $e^{''}$ has a maximum of two pebbles on it. Let the pebbled edges $e^{''}$ and $e^{'''}$ be on the face $f^{'}$.
Now, let the edge $e^{''}$ has two pebbles on it. Move the pebbles from the edge $e^{''}$ to $e^{'}$ and hence it is possible to place a pebble on the corresponding edges of $e^{''}$ and $e^{'''}$ on the opposite face of $f^{'}$ by a sequence of pebbling moves. Since the sum of all the pebbles distributed to the edges $e^{'}$, $e^{''}$ and $e^{'''}$ is eleven, we have four pebbles on the edge $e^{'''}$. By placing a pebble on the

adjacent edges of $e^{'''}$ on the face $f^{'}$ by a pebbling move, we are done. Now, let the edge $e^{''}$ has only one pebble on it. Since, the sum of all the pebbles distributed to the edges is eleven, we have five pebbles on the edge $e^{'''}$. Retain one pebble on $e^{'''}$ itself. Transfer the remaining pebbles to the edge $e^{'}$ by a sequence of pebbling moves. Eventually, edge $e^{'}$ has six pebbles on it and hence the result follows.

**Case 3.2.2:** Third pebbled edge $e^{'''}$ is placed at distance two from either $e^{'}$ or $e^{''}$
Without loss of generality, let the edge $e^{'''}$ be placed at a distance two from the edge $e^{'}$. Let the pebbled edges $e^{'}$ and $e^{''}$ be on the face $f^{'}$. Then, the result is obvious if there exists at least

- four pebbles on the edge $e^{'''}$ and two pebbles on either $e^{'}$ or $e^{''}$
- six pebbles on any of the edges $e^{'}$ or $e^{''}$
- two pebbles on the edge $e^{'}$ and four pebbles on the edge $e^{''}$
- two pebbles on the edge $e^{''}$ and four pebbles on the edge $e^{'}$

Now, we are left with only one possibility where both the edges $e^{'}$ and $e^{''}$ have only one pebble on it. Consequently, edge $e^{'''}$ has nine pebbles on it. Then, by placing a pebble on the adjacent edges of $e^{'''}$ which is on the opposite face of $f^{'}$ and by placing a pebble on the adjacent edge of either $e^{'}$ or $e^{''}$ on the face $f^{'}$, a maximal matching cover solution can be obtained.

**Case 3.3:** Three pebbled edges are not adjacent
In order to complete the proof, let the proof be divided into the following cases based on the number of pebbled edges on each face.

**Case 3.3.1:** Two pebbled edges on the same face and the third pebbled edge at distance two from the other pebbled edges
Let the pebbled edges $e^{'}$ and $e^{''}$ be on the face $f_1$ and the third pebbled edge which is distance two from $e^{'}$ and $e^{''}$ be denoted by $e^{'''}$. The result is obvious if

- any two pebbled edges have at least four pebbles on it
- the edge $e^{'''}$ has four pebbles on it
- any one of the edge $e^{'}$ or $e^{''}$ has five pebbles on it

- any one of the edge $e^{'}$ or $e^{''}$ has two pebbles and another has four pebbles on it

Since, the sum of the pebbles distributed to the edges is eleven, any one of the above conditions should satisfy and hence the result follows.

**Case 3.3.2:** Two pebbled edges on the same face and the third pebbled edge at a distance three from any one pebbled edge
Let the pebbled edges on the face $f_1$ be denoted by $e^{'}$ and $e^{''}$ and let the third pebbled edge be denoted by $e^{'''}$. Without loss of generality, let us assume that the edges $e^{'}$ and $e^{'''}$ be on the face $f^{''}$. The result is obvious if,

- any two of the pebbled edges have at least four pebbles on it
- any one of the edges $e^{''}$ or $e^{'''}$ has four pebbles on it
- the edge $e^{'}$ has nine pebbles on it

Now we are left with the possibilities where the edges $e^{''}$ and $e^{'''}$ has pebbles as follows: (3,3), (3,2), (3,1), (2,3) ,(2,2) ,(2,1) ,(1,3) ,(1,2).
Consider the first case where the number of pebbles distributed to the edges $e^{''}$ and $e^{'''}$ be three. Consequently, edge $e^{'}$ has five pebbles on it. Then, a maximal matching cover solution can be obtained as follows: Place a pebble on the edge which is of distance two from $e^{'}$ and on the opposite face of $f_1$ from $e^{'}$ by a pebbling move. Let the pebbled edge be denoted by $e^{*}$. Now by placing a pebble on the adjacent edges of $e^{''}$ which is opposite to $e^{*}$ from $e^{'''}$ by a pebbling move, we are done. The case where the distribution of pebbles to the edges $e^{''}$ and $e^{'''}$ by a pebbling move is similar. Also, the case where the distribution of pebbles to the edges $e^{''}$ and $e^{'''}$ are (3,2) , (3,1) ,(2,3) ,(2,2) ,(2,1) can be proved in a similar manner. Consider the case where the distribution of pebbles to the edges $e^{''}$ and $e^{'''}$ are one and three respectively. Eventually, edge $e^{'}$ has seven pebbles on it. Place a pebble on the edge which is of distance two from $e^{'}$ and on the opposite face of $f^{'}$ from $e^{'}$ by a pebbling move. Let the pebbled edge be denoted by $e^{*}$. Now place a pebble on the adjacent edge of $e^{''}$ which is opposite to the edge $e^{*}$ from $e^{''}$ by a pebbling move and hence we are done. The case where the distribution of pebbles to the edges $e^{''}$ and $e^{'''}$ are one and two respectively can be proved in a similar way.
Let $C_1 = \{e_1, e_3, e_5, e_7\}, C_2 = \{e_2, e_4, e_6, e_8\}$ and $C_3 = \{e_9, e_{10}, e_{11}, e_{12}\}$ denote the classes of the hypercube. Then, clearly $C_1, C_2, C_3$ forms a maximal matching for the hypercube $Q_3$.

**Case 4:** Eleven pebbles are distributed to any four edges of $Q_3$
Let the four pebbled edges of $Q_3$ be denoted by $e_a, e_b, e_c$ and $e_d$. If all the four pebbled edges belong to any one of the class $C_i$, i = 1,2,3 then there is nothing to prove. So, consider the case where none of the classes have the four pebbled edges. The proof is divided into the following cases, based on the number of pebbled edges on each class.

**Case 4.1:** Two classes having two pebbled edges each
The proof is again divided into the following cases based on the number of pebbled edges on each face.

**Case 4.1.1:** All the four pebbled edges belongs to the same face
Let the pebbled edges $e_a, e_b, e_c, e_d$ (clockwise direction) belong to the face $f^*$. Then the result is obvious if there exists at least four pebbles on the edges $e_a$ and $e_b$ (or) $e_c$ and $e_d$ (or) $e_d$ and $e_a$ (or) $e_a$ and $e_c$ (or) $e_b$ and $e_d$ (or) $e_b$ and $e_c$ (or) if there exists at least two pebbles on each edge $e_a$, $e_b$, $e_c$ and $e_d$.
Eventually, after a sequence of pebbling moves any one of the above conditions should satisfy since we have distributed eleven pebbles on four edges where the edges form a cycle $C_4$.

**Case 4.1.2:** Three pebbled edges on one face and the fourth pebbled edge is at distance two from all the remaining three pebbled edges
Let the edges $e_a, e_b, e_c$ (clockwise direction) belong to the face $f^*$ and let the fourth pebbled edge $e_d$ be of distance two from the edges $e_a, e_b$ and $e_c$. Then the result is obvious if there exists at least,

- four pebbles on the edge $e_d$ (or) $e_a$ (or) $e_c$
- four pebbles on the edges $e_a$ and $e_b$ (or) $e_b$ and $e_c$
- two pebbles on the edges $e_a$ and $e_c$
- four pebbles on the edge $e_b$ and two pebbles on each $e_a$ and $e_c$

If not, consider the case where the edge $e_d$ has a maximum of three pebbles on it. Consequently, the sum of pebbles on the edges $e_a$, $e_b$ and $e_c$ is eight. Hence, after a sequence of pebbling moves any one of the above conditions should satisfy since we have placed eight pebbles on the path $P_4$ formed by the edges $e_a$, $e_b$ and $e_c$.

**Case 4.1.3:** Three pebbled edges on one face and the fourth pebbled edge at distance two from the two pebbled edges and at distance three from the remaining pebbled edge
The proof follows in a similar manner as in Case 4.2.

**Case 4.1.4:** Two pebbled edges on one face and two pebbled edges on the opposite face where each pebbled edge on the first face is at a distance of two to any of the pebbled edges on the opposite face
Let $e_a$ and $e_b$ belong to face $f^*$ and $e_c$ and $e_d$ belong to the opposite face $f^{**}$ where edge $e_a$ is at distance two from the edge $e_c$ and the edge $e_b$ is at distance two from the edge $e_d$. Then, the result is obvious if there exist at least

- two pebbles on the edges $e_a$ and $e_c$
- four pebbles on the edges $e_a$ and $e_d$ (or) $e_b$ and $e_c$ (or) $e_b$ and $e_d$
- six pebbles on the edge $e_a$ (or) $e_b$ (or) $e_c$ (or) $e_d$

If not, then the left- out possibility is as follows:
Placing one pebble on the edges $e_a$ and $e_b$ and the remaining pebbles on the edges $e_c$ and $e_d$ or placing one pebble on the edges $e_c$ and $e_d$ and the remaining pebbles on the edges $e_a$ and $e_b$.
So, consider the case of placing a single pebble on the edges $e_a$ and $e_b$ and placing the remaining pebbles on the edges $e_c$ and $e_d$. Consequently, edges $e_c$ and $e_d$ receives nine pebbles. The case where the possible distribution of pebbles to the edges $e_c$ and $e_d$ as (1,8), (2,7), (3,6) are obvious as discussed above. Note that the distribution of pebbles as (1,8) and (8,1) can be proved similarly. So, consider the distribution of pebbles to the edges $e_c$ and $e_d$ as four and five. By placing a pebble on the adjacent non-pebbled edge of the face $f^{**}$ and by transferring the remaining pebbles to the edge $e_d$, the result follows.
The case where placing a single pebble to the edges $e_c$ and $e_d$ and placing the remaining pebbles on the edges $e_a$ and $e_b$ can be proved similarly as discussed above.

**Case 4.1.5:** Two pebbled edges on one face and two pebbled edges on the opposite face where both the pebbled edges on the opposite face are at distance two to one of the pebbled edges on the first face.
This case can be proved in a similar manner as discussed in Case 4.1.4.

**Case 4.2:** One class with three pebbled edges and another class with a single pebbled edge
The proof is divided into the following cases based on the number of pebbled edges on each face. In this case, we are left with only two possibilities.

**Case 4.2.1:** Three pebbled edges on one face and the fourth edge is on the opposite face where two edges on the first face are at distance two and the third edge is at distance three from the fourth pebbled edge
Let the pebbled edges $e_a$, $e_b$ and $e_c$ (clockwise direction) be on the face $f^*$ and the pebbled edge $e_d$ be on the opposite face $f^{**}$ where the edges $e_a$ and $e_b$ are at distance two from the edge $e_d$. Then, the result is obvious if there exists at least

- four pebbles on the edge $e_d$ (or) $e_b$ (or) $e_c$
- two pebbles on the edges $e_b$ and $e_c$ (or) $e_a$ and $e_c$
- six pebbles on the edge $e_a$

Any one of the above conditions should satisfy, otherwise, results in a contradiction to the assumption that eleven pebbles are placed on the four edges.

**Case 4.2.2:** Two pebbled edges on one face where the edges are non-adjacent and two pebbled edges on the opposite face where the edges are adjacent
Let the pebbled edges $e_a$ and $e_b$ belong to the face $f^*$ and $e_c$ and $e_d$ belong to face $f^{**}$ where the corresponding edge of $e_a$ on the opposite face $f^{**}$ be $e_c$. Then, the result is obvious if there exist at least two pebbles on the edge $e_d$ (or) $e_c$ (or) if there exist at least four pebbles on the edge $e_a$ (or) $e_b$. Then, any one of the above conditions should satisfy, otherwise, it results in a contradiction to the assumption that eleven pebbles are placed on the four edges.

**Case 4.3:** Two classes with one pebbled edge and the third class with two pebbled edge
Let the proof be divided into the following cases based on the number of pebbled edges on each face.

**Case 4.3.1:** Three pebbled edges on one face and the fourth pebbled edge on the adjacent face

Let the pebbled edges $e_a$, $e_b$, $e_c$ (clockwise direction) be on the face $f^*$ and the edge $e_d$ be on the adjacent face $f^{**}$. The proof is divided into the following subcases based on the adjacency of pebbled edges.

**Case 4.3.1.1:** $e_a$ and $e_d$ are adjacent
Then, the result is obvious if there exists at least

- two pebbles on the edges $e_a$, $e_b$ and $e_c$
- six pebbles on the edge $e_d$
- two pebbles on the edge $e_a$ and four pebbles on the edge $e_b$ (or) two pebbles on the edge $e_a$ and four pebbles on the edge $e_c$ (or) two pebbles on the edge $e_c$ and four pebbles on the edge $e_b$ (or) two pebbles on the edge $e_d$ and four pebbles on any of the edges $e_a$, $e_b$, or $e_c$

Then, any one of the above conditions should satisfy, otherwise it results in a contradiction to the assumption that eleven pebbles are placed on the four edges. The case where the edges $e_a$, $e_b$ and $e_d$ are adjacent can be proved similarly.

**Case 4.3.2:** Two pebbled edges on one face and two pebbled edges on adjacent faces
Let the pebbled edge $e_a$ and $e_b$ be on the face $f^*$. The case can be divided into the following cases based on the adjacency of pebbled edges on the first face.

**Case 4.3.2.1:** $e_a$ and $e_b$ are non-adjacent
Let the remaining pebbled edges $e_c$ and $e_d$ be on the face $f^{**}$ and $f^{***}$ respectively. Here, we need to consider two cases based on the adjacency of the edges $e_c$ and $e_d$.

**Case 4.3.2.1.1:** $e_c$ and $e_d$ are non-adjacent
Let the edges $e_a$ and $e_c$ be adjacent. Then, the result is obvious if there exist at least

- four pebbles on the edge $e_d$ (or) $e_b$
- two pebbles on the edge $e_c$
- three pebbles on the edge $e_a$

If any of the above conditions are not satisfied, then it results in a contradiction to the assumption that eleven pebbles are placed on the four edges.

**Case 4.3.2.1.2:** $e_c$ and $e_d$ are adjacent
The result holds in a similar manner as in Case 4.3.2.1.1.

**Case 4.3.2.2:** $e_a$ and $e_b$ are adjacent
Based on the adjacency of edges $e_a$ and $e_d$, the proof is divided into the following cases.

**Case 4.3.2.2.1:** $e_c$ and $e_d$ are non-adjacent
Let the edges $e_a$ and $e_b$ be adjacent to $e_c$ and is not adjacent to the edge $e_d$. Then, the result is obvious if there exist at least

- two pebbles on the edges $e_a$ and $e_c$ (or) $e_b$ and $e_c$ (or) $e_a$ and $e_b$
- four pebbles on the edge $e_c$ (or) $e_a$ (or) $e_b$

If any of the above conditions are not satisfied, then it results in a contradiction to the assumption that eleven pebbles are placed on the four edges.

**Case 4.3.2.2.2:** $e_c$ and $e_d$ are adjacent
In this case, the following are the two possibilities:

- edges $e_c$ and $e_d$ are not adjacent to $e_a$ or $e_b$
- edges $e_c$ and $e_d$ are adjacent to any one of $e_a$ or $e_b$

Both the cases can be proved in a similar manner as in Case 4.3.2.2.1.

**Case 5:** Eleven pebbles are distributed to any five edges of $Q_3$
Let the pebbled edges be denoted by $e_a, e_b, e_c, e_d$ and $e_e$. If any one of the classes have four pebbled edges or if any of the four pebbled edges forms an independent edge set, then there is nothing to prove. So, the proof is divided into the following cases based on the number of pebbled edges on each class.

**Case 5.1:** Two classes have a single pebbled edge and the third class has one pebbled edge

The proof is divided into the following cases based on the number of pebbled edges on each face.

**Case 5.1.1:** Three pebbled edges on one face and two pebbled edges on the adjacent face
Let the edges $e_a, e_b, e_c$ (clockwise direction) be on the face $f^*$ and edges $e_d, e_e$ be on the adjacent faces. This case can be divided into the following subcases based on the adjacency of edges $e_d$ and $e_e$.

**Case 5.1.1.1:** $e_d$ and $e_e$ are adjacent
Let the edge $e_d$ be adjacent to $e_a$. Then, the result is obvious if there exist at least

- four pebbles on the edge $e_d$ (or) $e_e$
- two pebbles on the edges $e_a, e_b$ and $e_c$ (or) $e_b, e_c$ and $e_d$ (or) $e_a, e_c$ and $e_d$
- four pebbles on the edge $e_b$ and two pebbles on the edge $e_a$ (or) four pebbles on the edge $e_c$ and two pebbles on the edge $e_a$

Then, any one of the above conditions should be satisfied, otherwise it results in a contradiction to the assumption that eleven pebbles are placed on the five edges.
The case where the edges $e_a, e_b$ and $e_d$ are adjacent can be proved in a similar manner.

**Case 5.1.1.2:** $e_d$ and $e_e$ are non-adjacent
Let the edges $e_c$ and $e_e$ be adjacent and let the corresponding edge of $e_a$ on the opposite face of $f^*$ be $e_d$. Then, the result is obvious if there exists at least,

- two pebbles on the edge $e_e$
- four pebbles on the edge $e_d$
- four pebbles on the edges $e_a$ and $e_c$
- two pebbles on the edges $e_a, e_b$ and $e_c$
- three pebbles on the edge $e_c$
- four pebbles on the edge $e_a$ and two pebbles on the edge $e_c$ (or) four pebbles on the edge $e_b$ and two pebbles on the edge $e_c$

Then, any one of the above conditions should be satisfied, otherwise it results in a contradiction to the assumption that eleven pebbles are placed on the five edges.

**Case 5.1.2:** Two pebbled edges on adjacent faces
Let the edges $e_a$ and $e_b$ be on face $f^*$ and let the edges $e_c$ and $e_d$ be on the face $f^{**}$ where face $f^{**}$ is the opposite face of $f^*$ of $Q_3$. Let the edge $e_e$ be on the adjacent face of $f^*$ and $f^{**}$. The case where the edges $e_b, e_d, e_e$ are adjacent and the case where the edge $e_b$ and $e_e$ are adjacent can be proved in a similar manner. Without loss of generality, let us assume that the edges $e_b, e_d$ and $e_e$ are adjacent. Then the result is obvious if there exist at least two pebbles on any of the edges $e_d$ (or) $e_e$ (or) $e_c$ (or) $e_b$ (or) four pebbles on the edge $e_a$. Then, any one of the above conditions should be satisfied, otherwise it results in a contradiction to the assumption that we have placed eleven pebbles on five edges.

**Case 5.2:** One class with three pebbled edges and another class with two pebbled edges
The proof is divided into the following cases based on the number of pebbled edges on each face.

**Case 5.2.1:** Four pebbled edges on one face and a fifth pebbled edge on the adjacent face
Let the edges $e_a, e_b, e_c$ and $e_d$ (clockwise direction) be on the face $f^*$ and the edge $e_e$ be on the opposite face of $f^*$ where the corresponding edge of $e_a$ on the opposite face of $f^*$ is $e_e$. Then, the result is obvious if there exist at least

- four pebbles on the edge $e_e$ (or) $e_c$ (or) $e_a$ (or) $e_b$ (or) $e_d$
- two pebbles on the edges $e_b$ and $e_c$ (or) $e_c$ and $e_d$ (or) $e_a$ and $e_b$ (or) $e_a$ and $e_c$ (or) $e_a$ and $e_d$

Then, any one of the above conditions should be satisfied, otherwise it results in a contradiction to the assumption that we have placed eleven pebbles on five edges.

**Case 5.2.2:** Three pebbled edges on one face and two pebbled edges on the opposite face
Let the pebbled edges $e_a, e_b$ and $e_c$ be on the face $f^*$ and the edges $e_d$ and $e_e$ be on the opposite face $f^*$ where the corresponding edges of $e_a$ and $e_b$

on the opposite face of $f^*$ are $e_d$ and $e_e$ respectively. Then, the result is obvious if there exists at least,

- two pebbles on the edge $e_d$ (or) $e_e$
- two pebbles on the edges $e_b$ and $e_c$
- if there exist at least four pebbles on the edge $e_a$ (or) $e_b$ (or) $e_c$

Then, any one of the above conditions should be satisfied, otherwise it results in a contradiction to the assumption that we have placed eleven pebbles on five edges.

**Case 5.3:** Two classes with two pebbled edges and one class with one pebbled edge
The proof is divided into the following cases based on the number of pebbled edges on each face:

**Case 5.3.1:** Four pebbled edges on one face and one pebbled edge on any of the adjacent face

Let $e_a, e_b, e_c, e_d$ be the four pebbled edges on the face $f^*$ and let $e_e$ be the fifth pebbled edge on the adjacent face. Without loss of generality, assume that $e_a, e_b$ and $e_e$ are adjacent. Then, the result is obvious if there exists at least,

- two pebbles on the edges $e_a, e_b$ and $e_c$ (or) $e_b$, $e_c$ and $e_d$ (or) $e_c, e_d$ and $e_a$ (or) $e_a$, $e_d$ and $e_e$ (or) $e_c$, $e_d$ and $e_e$
- two pebbles on the edge $e_a$ and four pebbles on any of the edge $e_b$, $e_c$ or $e_d$ (or) two pebbles on the edge $e_b$ and four pebbles on any of the edge $e_a$, $e_c$ or $e_d$ (or) two pebbles on the edge $e_c$ and four pebbles on any of the edge $e_a$, $e_b$ or $e_d$ (or) two pebbles on the edge $e_d$ and four pebbles on any of the edge $e_a$, $e_b$ or $e_c$ (or) two pebbles on the edge $e_e$ and four pebbles on any of the the edge $e_a$, $e_b$, $e_c$ or $e_d$
- four pebbles on the edge $e_e$ and two pebbles on any of the the edge $e_a$, $e_b$, $e_c$ or $e_d$
- six pebbles on the edge $e_e$

One of the above conditions should be satisfied, otherwise it results in a contradiction to the assumption that we have placed eleven pebbles on five edges.
The case where three pebbled edges are on the same face is similar to the cases discussed in Case 5.1 and Case 5.2.

**Case 5.3.2:** Two pebbled edges on two different faces where the faces are opposite to each other and one pebbled edge on the adjacent face
Let the edges $e_a, e_b$ belong to face $f^*$ and the edges $e_c, e_d$ belong to the opposite face of $f^*$ and the edge $e_e$ belong to the adjacent face of $f^*$ where edge $e_a$ is adjacent to the edge $e_e$. Then, the result is obvious if there exists at least two pebbles on the edges $e_e$ and $e_a$ (or) $e_a$ and $e_c$ (or) $e_a$ and $e_d$ (or) $e_e$ and $e_d$ (or) $e_e$ and $e_c$ (or) $e_b$ and $e_e$ (or) $e_b$ and $e_c$ (or) $e_b$ and $e_d$.

Then, the left-out possibility is by placing a single pebble on all the edges except one edge and placing the remaining pebbles on the exceptional edge. In this case, it is always possible to produce a maximal matching cover solution by a sequence of pebbling moves.

All the other possibilities where two pebbled edges on two different faces in such a way that the faces are opposite to each other and one pebbled edge on the adjacent face can be proved in a similar manner. See Figure 3.

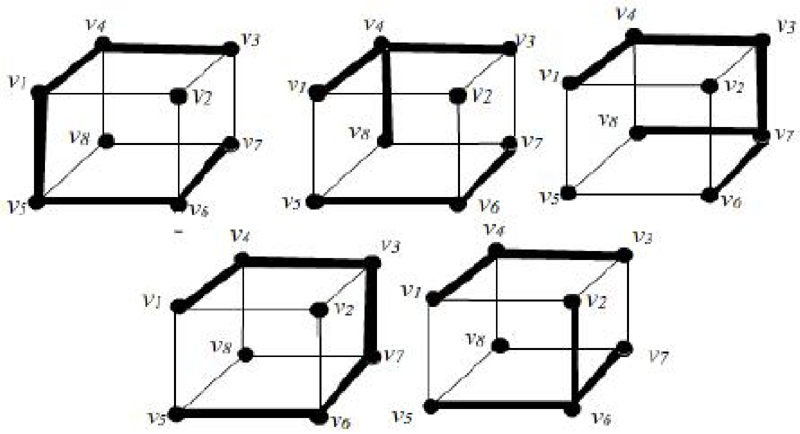

**Figure 3: Possible maximal matching edge sets for Case 5.3.2**

**Case 6:** Eleven pebbles are distributed to any six edges of $Q_3$
Let the pebbled edges be denoted by $e_a, e_b, e_c, e_d, e_e$ and $e_f$. If any one of the classes have four pebbled edges or if any of the four pebbled edges forms an independent edge set, then there is nothing to prove. So, the proof is divided into the following cases based on the number of pebbled edges on each class.

**Case 6.1:** Each class with two pebbled edges
In this case, the proof is divided into the following cases based on the number of pebbled edges on each face.

**Case 6.1.1:** Pebbled edges are placed on the adjacent faces where one face has four pebbled edges and the other face has three pebbled edges
Let the pebbled edges $e_a, e_b, e_c, e_d$ (clockwise direction) belong to face $f^*$ and edges $e_a, e_e, e_f$ belongs to face $f^{**}$. Then, the result is obvious if there exists at least,

- two pebbles on any two edges of the face $f^*$
- four pebbles on any edge of $Q_3$ on the face $f^*$
- two pebbles on the edges $e_f$ and $e_e$

Since, we have distributed eleven pebbles on six edges, where the pebbled edges are on the adjacent faces, then after a sequence of pebbling moves, any one of the above conditions should satisfy and hence the result follows.

**Case 6.1.2:** Pebbled edges are placed on three adjacent faces where each face has three edges

Let the edges $\{e_a, e_b, e_c\}$ belong to the face $f^*$, $\{e_a, e_d, e_e\}$ belong to the face $f^{**}$ and $\{e_b, e_e, e_f\}$ belong to the face $f^{***}$. Then, the result is obvious if there exists at least,

- four pebbles on the edge $e_c$ (or) $e_d$ (or) $e_b$
- two pebbles on the edge $e_f$ (or) $e_a$
- three pebbles on the edge $e_e$
- two pebbles on the edges $e_b$ and $e_c$ (or) $e_a$ and $e_c$

Any one of the above conditions should be satisfied, otherwise, it results in a contradiction to the assumption that we have placed eleven pebbles on six edges.

**Case 6.1.3:** Pebbled edges are placed on three adjacent faces where two faces have three edges each and the third face has two pebbled edges
Let the pebbled edges $\{e_a, e_b, e_c\}$ belong to the face $f^*$, $\{e_a, e_d, e_e\}$ belong to the face $f^{**}$ and $\{e_d, e_f, \}$ belongs to the face $f^{***}$. Then, the result is obvious if there exist at least two pebbles on the edge $e_f$ (or) $e_e$ (or) $e_a$ (or) $e_d$ (or) $e_b$ (or) at least four pebbles on the edge $e_c$. Then, any one of the above conditions should satisfy, otherwise it results in a contradiction to the assumption that we have placed eleven pebbles on six edges.

**Case 6.1.4:** Pebbled edges placed on three adjacent faces where two faces are having two edges each and the third face with three pebbled edges
Let the pebbled edges $\{e_a, e_b, e_c\}$ belong to the face $f^*$, $\{e_b, e_d\}$ belongs to the face $f^{**}$ and $\{e_e, e_f, \}$ belong to the face $f^{***}$. Then, the result is obvious if there exists at least,

- two pebbles on the edge $e_d$
- four pebbles on the edge $e_f$ (or) $e_e$ (or) $e_c$ (or) $e_b$ (or) $e_a$
- two pebbles on the edges $e_a$ and $e_b$ (or) $e_a$ and $e_c$ (or) $e_b$ and $e_f$ (or) $e_b$ and $e_c$

Thus, any one of the above conditions should be satisfied, otherwise, it results in a contradiction to the assumption that we have placed eleven pebbles on six edges.

**Case 6.1.5:** Pebbled edges on the adjacent faces where each face has three pebbled edges
Let $e_a, e_b, e_c$ be the pebbled edges on the face $f^*$ and $e_d, e_e, e_f$ be the pebbled edges on the face $f^{**}$ where the edges $e_a$, $e_d$ and $e_c$, $e_f$ are adjacent. Then, the result is obvious if there exists at least

- two pebbles on the edges $e_a$ and $e_c$ (or) $e_a$ and $e_b$ (or) $e_b$ and $e_c$ (or) $e_d$ and $e_f$ (or) $e_d$ and $e_e$ (or) $e_e$ and $e_f$
- four pebbles on the edge $e_a$ (or) $e_b$ (or) $e_c$ (or) $e_d$ (or) $e_e$ (or) $e_f$

Thus, any one of the above conditions should be satisfied, otherwise, it results in a contradiction to the assumption that we have placed eleven pebbles on six edges.

**Case 6.2:** Two classes having three pebbled edges each and the third class with zero pebbled edges
In this case, the proof is divided into the following case based on the number of pebbled edges on each face.

**Case 6.2.1:** Pebbled edges are on the opposite faces where one face has four pebbled edges and the other one has three pebbled edges
Let the pebbled edges on the face $f^*$ be $e_a, e_b, e_c, e_d$ (clockwise direction) and on the opposite face $f^{**}$ be $e_e, e_f$ (clockwise direction) where the corresponding edges of $e_a$ and $e_b$ on the opposite face $f^{**}$ be $e_e$ and $e_f$ respectively.
Then, the result is obvious if there exists at least

- two pebbles on the edge $e_f$ (or) $e_e$
- two pebbles on the edges $e_a, e_b, e_c$ and $e_d$
- four pebbles on the edge $e_a$ (or) $e_b$ (or) $e_c$ (or) $e_d$

Consider the case where any one of the above conditions does not hold. Consequently, a maximum of nine pebbles are distributed on the face $f^*$. Now, for any distribution of pebbles, it is always possible to produce a maximal matching cover solution and the result follows.

**Case 6.2.2:** Pebbled edges on the opposite faces where each face has three pebbled edges
Let the pebbled edges on the face $f^*$ be $e_a, e_b, e_c$ (clockwise direction) and on the face $f^{**}$ be $e_d, e_e, e_f$ (clockwise direction) where the corresponding edges of $e_a$ and $e_b$ on the opposite face $f^{**}$ be $e_e$ and $e_f$ respectively.

Since, we have distributed eleven pebbles on six edges, by the Pigeonhole principle, there should exist an edge with at least two pebbles and hence the result is obvious.

**Case 6.3:** One class with one pebbled edge and the other class with two pebbled edges and the third class with three pebbled edges
In this case, we have the possibility of either with four independent pebbled edges or the cases which we have discussed earlier.

**Case 7**: Eleven pebbles distributed to any seven edges of $Q_3$

Let the pebbled edges be denoted by $e_a, e_b, e_c, e_d, e_e, e_f$ and $e_g$. The proof is divided into the following cases based on the number of pebbled edges on each class. If any of the four pebbled edges belong to one class then there is nothing to prove. In this case, we are left with two possibilities as given below:

- Two of the classes have two pebbled edges and the other class with three pebbled edges
- Two of the classes have three pebbled edges and the other class with one pebbled edge

In this case, it is always possible to find three independent pebbled edges. If the fourth independent non-pebbled edge is adjacent to a pebbled edge with at least two pebbles, then we are done. If not, it is always possible to find a path from an edge that has more than one pebble to the fourth independent edge, since the number of pebbles on the path is greater than the length of the path and the result follows.

**Case 8**: Eleven pebbles distributed to any eight edges of $Q_3$
Let the pebbled edges be denoted by $e_a, e_b, e_c, e_d, e_e, e_f, e_g$ and $e_h$. The proof is divided into the following cases based on the number of pebbled edges on each class. If any of the four pebbled edges belong to any one class $C_i, i = 1, 2, 3$, then there is nothing to prove. In this case, we are left with two possibilities; i.e., two classes having three pebbled edges and the other class having one pebbled edge. See Figure 4. The proof follows similarly as in Case 7. □

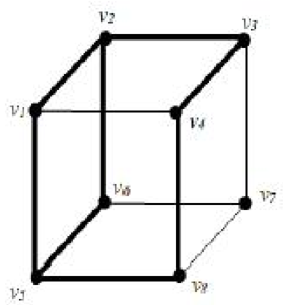


**Figure 4: Distribution of pebbles to eight edges**

**Theorem 3.** *The maximal matching cover pebbling number of a hypercube $Q_n$ is, $f_{mmcp}(Q_n) = 4(3^{n-2})$.*

**Proof.** It is observed that $Q_n$ has $2^n$ vertices and $n2^{n-1}$ edges. It is evident from Theorem 1 and Theorem 2 that the Stacking property is true for $Q_2$ and $Q_3$ and consequently holds true for $Q_n$[8]. Therefore, the worst-case scenario is by placing all the pebbles on a single edge. Since, the maximal matching number of a hypercube is $2^{n-1}$, the result follows. □

**Corollary 1.** *The maximal matching cover pebbling number for folded hypercube $FQ_n$ is given by, $f_{mmcp}(FQ_n) = 4(3^{n-2})$.*

**Corollary 2.** *The maximal matching cover pebbling number for crossed cube $CQ_n$ is given by, $f_{mmcp}(CQ_n) = 4(3^{n-2})$.*

**Corollary 3.** *The maximal matching cover pebbling number for folded crossed cube $FCQ_n$ is given by, $f_{mmcp}(FCQ_n) = 4(3^{n-2})$.*

## 4. Conclusion

The two main areas of research in graph theory are graph pebbling and matching. By combining these two graph invariants, one can find the solution to many real-world problems. In this paper, the maximal matching cover pebbling number for variants of hypercube are determined.

Given below are some interesting open problems.

- Finding the maximal matching cover pebbling number for other networks
- Finding the maximal matching cover pebbling number for directed graphs

## References

[1] J. A. Bondy and US Murty, *Graph theory with applications.* London: Macmillan, 1976.

[2] X. Cai and E. Vumar, “The super connectivity of folded crossed cubes”, *Information Processing Letters,* vol. 142, pp. 52-56, 2019. doi: 10.1016/j.ipl.2018.10.013

[3] F. R. K. Chung, “Pebbling in hypercubes”, *SIAM Journal on Discrete Mathematics,* vol. 2, no. 4, pp. 467-472, 1989. doi: 10.1137/0402041

[4] B. Crull, T. Cundiff, P. Feltman, GH. Hurlbert, L. Pudwell, Z. Szaniszlo and Z. Tuza, “The cover pebbling number of graphs”, *Discrete mathematics*, vol. 296, no. 1, pp. 15-23, 2005.

[5] J. Gardner, A.P. Godbole, A.M. Teguia, A.Z. Vuong, N. Watson and C.R. Yerger, “Domination cover pebbling: graph families”, *Journal of Combinatorial Mathematics and Combinatorial Computin,* vol. 64, pp. 255-271, 2008.

[6] G. H. Hurlbert, "A survey of graph pebbling", *Congressus Numerantium*, vol. 139, pp. 41-64, 1999.

[7] A. P. Paul, “On Edge Pebbling Number and Cover Edge Pebbling Number of Some Graphs”, *Journal of Information and Computational Science,* vol. 10, no. 6, pp. 337-344, 2020.

[8] T. L. Holt, J.E. Worley and A.P. Godbole, *Explorations in domination cover pebbling.* Preprint.

[9] S. Sarah and L. Mathew, “On Maximal Matching Cover Pebbling Number”, *Advances ad Applications in Mathematical Sciences,* vol. 21, no. 11, pp. 6281-6302, 2022.

[10] S. Sarah and L. Mathew, “On Total Secure Domination Cover Pebbling Number”, *Communications in Mathematics and Applications,* vol. 13, no. 1, pp. 117-127, 2022.

[11] S. Sarah and L. Mathew, “Secure Domination Cover Pebbling Number for Variants of Complete Graphs”, *Advances and Applications in Discrete Mathematics,* vol. 27, no. 1, pp. 105-122, 2021. doi: 10.17654/dm027010105

**S. Sarah Surya**
Department of Mathematics
Stella Maris College(Autonomous),
Chennai
India
e-mail: sarahsurya@stellamariscollege.edu.in

and

**Lian Mathew**
CHRIST (Deemed to be University)
Pune, Lavasa Campus
India
e-mail: lianmathew64@gmail.com
Corresponding author