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Study of multiplicative derivation and its additivity

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Abstract

In this paper, we modify the result of M. N. Daif [1] on multiplicative derivations in rings. He showed that the multiplicative derivation is additive by imposing certain conditions on the ring $\Re$. Here, we have proved the above result with lesser conditions than M. N. Daif for getting multiplicative derivation to be additive.

Key words: *Associative ring, derivation, Peirce decomposition.*

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1. Notations and Introduction

Many results on derivations of rings have been obtained in recent years. The derivation of ring $\Re$, we means an additive map $d : \Re \to \Re$ such that $\forall\ x, y \in \Re$, $d(xy) = d(x)y + xd(y)$. If d is a multiplicative derivation of $\Re$, it is said to be non additive derivation of $\Re$. In 1969, Martindale [4] gave a remarkable result. He demonstrated that under the existence of a family of idempotent object in $\Re$ that satisfy certain conditions, every anti-automorphism and multiplicative isomorphism on $\Re$ is additive. Martindale's work influenced Daif and he expanded his findings upon multiplicative derivation and raised the question: when is multiplicative derivation is additive ? In 1991, Daif [1] answered the question raised by him by using same Martindale's condition, for details see [2, 5, 6]. Our main objective in this paper is to improve the result of M. N. Daif (when is multiplicative derivation is additive?). He imposed 3 conditions to show that multiplicative derivation is additive. In this manuscript, we used only 2 condition to proved the additivity of multiplicative derivation, the conditions are as follows:

(1) $x\Re e = 0 \Rightarrow x = 0$ (and hence $x\Re = 0 \Rightarrow x = 0$).

(2) $x\Re_{12} = 0 \Rightarrow x = 0$.

Let $0, 1 \neq e$ be an idempotent element of ring $\Re$ (not necessarily having identity element). We will formally set $e = e_1$ and $1-e = e_2$. The following is how $\Re$ can be decomposed:

$$\Re = e_1\Re e_1 \oplus e_1\Re e_2 \oplus e_2\Re e_1 \oplus e_2\Re e_2. \tag{1.1}$$

Above expression of $\Re$ is known as two-sided Peirce decomposition determined by idempotent e_1 and e_2 (for details see [3]). So letting $\Re_{mn} = e_m\Re e_n$; where $m, n \in \{1, 2\}$. Then the decomposition takes the form $\Re = \Re_{11} \oplus \Re_{12} \oplus \Re_{21} \oplus \Re_{22}$, where $\Re_{ij}$ are subring of $\Re$ for all $i, j \in \{1, 2\}$. Moreover, an element of the subring $\Re_{mn}$ will be denoted by x_{mn} for all $m, n \in \{1, 2\}$.

Daif [1] has defined some basic concept in his result before the main result. We will be going to use those concept which was stated by Daif [1] as follow, that $d(0) = d(00) = d(0)0 + 0d(0) = 0$. Moreover, we have $d(e_1) = d(e_1^2) = d(e)e + ed(e)$. So, we can write $d(e_1) = d_{11} + d_{12} + d_{21} + d_{22}$ and consequently, we have $d(e_1) = d_{12} + d_{21}$. We define g be the inner derivation of $\Re$ observed by the element $d_{12} - d_{21}$. Therefore, $g(e_1) = [e_1, d_{12} - d_{21}] = d_{12} + d_{21}$. Further, we substitute d by $d - g$ which is also a multiplicative derivation, which is denoted as D i.e., $D = d - g$, without losing generality. As a result we get $D(e) = 0$, this simplicity is quite important for upcoming results.

2. The Results

Before, proving our main theorem we will need the following lemmas.

Lemma 2.1. *Let $\Re$ be a ring containing an idempotent e and D be a derivation of $\Re$ which satisifies the given conditions. Then the following are true:*

(*i*) $D(\Re_{11}) \subset \Re_{11}$

(*ii*) $D(\Re_{12}) \subset \Re_{12}$

(*iii*) $D(\Re_{21}) \subset \Re_{21}$

(*iv*) $D(\Re_{22}) \subset \Re_{22}$.

Proof. (*i*) Let x_{11} be an arbitrary element of $\Re_{11}$, we have

$$\begin{aligned} D(x_{11}) &= D(e_1 x_{11} e_1) = D(e_1 x_{11}) e_1 + e_1 x_{11} D(e_1) \\ &= D(e_1 x_{11}) e_1 = D(e_1) x_{11} e_1 + e_1 D(x_{11}) e_1 \\ &= e_1 D(x_{11}) e_1 \in \Re_{11}. \end{aligned}$$

Since, x_{11} be an arbitrary element of $\Re_{11}$, so we have $D(\Re_{11}) \subset \Re_{11}$.
(*ii*) Let x_{12} be an arbitrary element of $\Re_{12}$, we obtain

$$D(x_{12}) = D(e_1 x_{12}) = D(e_1) x_{12} + e_1 D(x_{12}) = e_1 D(x_{12}). \tag{2.1}$$

This implies that

$$D(x_{12}) = e_1 D(x_{12}). \tag{2.2}$$

By using Peirce decomposition of $\Re$, $D(x_{12})$ can be written as $D_{11} + D_{12} + D_{21} + D_{22}$. Putting these value in (2.2), we see that

$$D_{11} + D_{12} + D_{21} + D_{22} = e_1(D_{11} + D_{12} + D_{21} + D_{22}) = D_{11} + D_{12}. \tag{2.3}$$

From (2.3), we find that

$$D_{21} + D_{22} = 0. \tag{2.4}$$

Putting the above relation in the value of $D(x_{12})$, we have

$$D(x_{12}) = D_{11} + D_{12}. \tag{2.5}$$

Since we know that $0 = D(0) = D(x_{12}e_1) = D(x_{12})e_1 + x_{12}D(e_1) = D(x_{12})e_1$. Using the value of D from (2.5) in previous equation, we receive $0 = (D_{11} + D_{12})e_1$ which implies that $D_{11} = 0$. Using these relation in (2.5), we get $D(x_{12}) = D_{12} \in \Re_{12}$. Since x_{12} be an arbitrary element of $\Re_{12}$, we have $D(\Re_{12}) \subset \Re_{12}$.

(iii) Using the similar argument as we have done in *(ii)*, we get the result.

(iv) Let $D(x_{22}) = y_{11} + y_{12} + y_{21} + y_{22}$ for all $x_{22} \in \Re_{22}$, we have

$$0 = D(0) = D(e_1x_{22}) = D(e_1)x_{22} + e_1D(x_{22}). \tag{2.6}$$

Above relation yields that

$$0 = e_1D(x_{22}). \tag{2.7}$$

Using the value of $D(x_{22})$ in above relation, we get

$$0 = e_1(y_{11} + y_{12} + y_{21} + y_{22}) = y_{11} + y_{12}. \tag{2.8}$$

Substituting (2.8) in the value of $D(x_{22})$ then we obtain $D(x_{22}) = y_{21} + y_{22}$. Again, we have $0 = D(x_{22}e_1) = D(x_{22})e_1 + x_{22}D(e_1) = D(x_{22})e_1$. Here we use the obtained value of $D(x_{22})$, we get $0 = (y_{21} + y_{22})e_1 = y_{21}$, this implies $0 = y_{21}$. Substituting this in $D(x_{22})$, we have $D(x_{22}) = y_{22} \in \Re_{22}$. Since, x_{22} be an arbitrary element of $\Re_{22}$, this yields $D(\Re_{22}) \subset \Re_{22}$. □

Lemma 2.2. *For any $x_{11} \in \Re_{11},\ x_{12} \in \Re_{12},\ x_{21} \in \Re_{21}$ and $x_{22} \in \Re_{22}$, we have*

(i) $D(x_{11} + x_{12}) = D(x_{11}) + D(x_{12})$

(ii) $D(x_{22} + x_{12}) = D(x_{22}) + D(x_{12})$

(iii) $D(x_{11} + x_{21}) = D(x_{11}) + D(x_{21})$

(iv) $D(x_{22} + x_{21}) = D(x_{22}) + D(x_{21})$.

Proof. (i) Consider the sum $D(x_{11}) + D(x_{12}) \in \Re$. Let $y_{1n} \in \Re_{1n}$ for $n = 1, 2$, we have

$$\begin{aligned}
[D(x_{11}) + D(x_{12})]y_{1n} &= D(x_{11})y_{1n} + D(x_{12})y_{1n} \\
&= D(x_{11})y_{1n} \\
&= D(x_{11}y_{1n}) - x_{11}D(y_{1n}) \\
&= D[(x_{11} + x_{12})y_{1n}] - x_{11}D(y_{1n}) \\
&= D(x_{11} + x_{12})y_{1n} + (x_{11} + x_{12})D(y_{1n}) - x_{11}D(y_{1n}) \\
&= D(x_{11} + x_{12})y_{1n}.
\end{aligned}$$

Thus, we have

$$[D(x_{11}) + D(x_{12}) - D(x_{11} + x_{12})]y_{1n} = 0. \tag{2.9}$$

In the similar fashion for any $y_{2n} \in \Re_{2n}$, we obtain

$$[D(x_{11}) + D(x_{12}) - D(x_{11} + x_{12})]y_{2n} = 0. \tag{2.10}$$

Since, y_{1n} and y_{2n} are an arbitrary element of $\Re_{1n}$ and $\Re_{2n}$ respectively. Therefore, from (2.9) and (2.10), we conclude that

$$[D(x_{11}) + D(x_{12}) - D(x_{11} + x_{12})]\Re = (0). \tag{2.11}$$

By using condition (1), we get

$$D(x_{11}) + D(x_{12}) - D(x_{11} + x_{12}) = 0. \tag{2.12}$$

Above relation implies that

$$D(x_{11} + x_{12}) = D(x_{11}) + D(x_{12}). \tag{2.13}$$

Proof of (i) is done.

(ii) Let x_{22} and x_{12} be an any element of $\Re_{22}$ and $\Re_{12}$, we assume the sum $D(x_{22}) + D(x_{12}) \in \Re$. Now, for $y_{12} \in \Re_{12}$ we have

$$[D(x_{22}) + D(x_{12})]y_{12} = D(x_{22})y_{12} + D(x_{12})y_{12}. \tag{2.14}$$

Using the definition of derivation in (2.14), it's yields that

$$\begin{aligned}
[D(x_{22}) + D(x_{12})]y_{12} &= D(x_{22}y_{12}) - x_{22}D(y_{12}) + D(x_{12}y_{12}) - x_{12}D(y_{12}) \\
&= 0 - x_{22}D(y_{12}) + 0 - x_{12}D(y_{12}) \\
&= D(0) - (x_{22} + x_{12})D(y_{12}) \\
&= D[(x_{22} + x_{12})y_{12}] - (x_{22} + x_{12})D(y_{12}) \\
&= D(x_{22} + x_{12})y_{12} + (x_{22} + x_{12})D(y_{12}) \\
&\quad -(x_{22} + x_{12})D(y_{12}) = D(x_{22} + x_{12})y_{12}.
\end{aligned}$$

Which implies that

$$[D(x_{22}) + D(x_{12})]y_{12} = D(x_{22} + x_{12})y_{12}. \tag{2.15}$$

This implies that

$$[D(x_{22}) + D(x_{12}) - D(x_{22} + x_{12})]y_{12} = 0. \tag{2.16}$$

Since, y_{12} be an arbitrary element of $\Re_{12}$, above relation yields that

$$[D(x_{22}) + D(x_{12}) - D(x_{22} + x_{12})]\Re_{12} = (0). \tag{2.17}$$

By using condition (2), we obatin

$$D(x_{22} + x_{12}) = D(x_{22}) + D(x_{12}). \tag{2.18}$$

(iii) Let x_{11} and x_{21} be an arbitrary element of $\Re_{11}$ and $\Re_{21}$, we consider the sum $D(x_{11}) + D(x_{21}) \in \Re$. Now, for $y_{12} \in \Re_{12}$ we have

$$[D(x_{11}) + D(x_{21})]y_{12} = D(x_{11})y_{12} + D(x_{21})y_{12}. \tag{2.19}$$

Using the definition of derivation, above relation yields that

$$[D(x_{11}) + D(x_{21})]y_{12} = D(x_{11}y_{12}) - x_{11}D(y_{12}) + D(x_{21}y_{12}) - x_{21}D(y_{12}). \tag{2.20}$$

Since, $x_{11} \in \Re_{11}$ and $y_{12} \in \Re_{12}$ this implies $x_{11}y_{12} \in \Re_{12}$ and similarly we obtain $x_{21}y_{12} \in \Re_{22}$. Using previous part of this Lemma, above relation can be written as

$$[D(x_{11}) + D(x_{21})]y_{12} = D(x_{11}y_{12} + x_{21}y_{12}) - x_{11}D(y_{12}) - x_{21}D(y_{12}). \tag{2.21}$$

This implies that

$$[D(x_{11}) + D(x_{21})]y_{12} = D((x_{11} + x_{21})y_{12}) - (x_{11} + x_{21})D(y_{12}). \tag{2.22}$$

Using the definition of derivation in (2.22), we obtain

$$[D(x_{11}) + D(x_{21})]y_{12}$$

$$= D(x_{11}+x_{21})y_{12}+(x_{11}+x_{21})D(y_{12})-(x_{11}+x_{21})D(y_{12}).$$

Above equation can be rewritten as $[D(x_{11})+D(x_{21})]y_{12}=D(x_{11}+x_{21})y_{12}$.

This implies that

$$[D(x_{11})+D(x_{21})-D(x_{11}+x_{21})]y_{12}=0. \tag{2.23}$$

Since, y_{12} be an arbitrary element of $\Re_{12}$, above relation yields that

$$[D(x_{11})+D(x_{21})-D(x_{11}+x_{21})]\Re_{12}=(0). \tag{2.24}$$

By using condition (2), we obatin

$$D(x_{11}+x_{21})=D(x_{11})+D(x_{21}). \tag{2.25}$$

(iv) Consider the sum $D(x_{22})+D(x_{21})\in\Re$. Let $y_{1n}\in\Re_{1n}$ for $n=1,2$, we have

$$\begin{aligned}
[D(x_{22})+D(x_{21})]y_{1n} &= D(x_{22})y_{1n}+D(x_{21})y_{1n}\\
&= D(x_{21})y_{1n}\\
&= D(x_{21}y_{1n})-x_{21}D(y_{1n})\\
&= D[(x_{22}+x_{21})y_{1n}]-x_{21}D(y_{1n})\\
&= D(x_{22}+x_{21})y_{1n}+(x_{22}+x_{21})D(y_{1n})-x_{21}D(y_{1n})\\
&= D(x_{22}+x_{21})y_{1n}+x_{21}D(y_{1n})-x_{21}D(y_{1n})\\
&= D(x_{22}+x_{21})y_{1n}.
\end{aligned}$$

Thus, we have

$$[D(x_{22})+D(x_{21})-D(x_{22}+x_{21})]y_{1n}=0. \tag{2.26}$$

In a similar way for any $y_{2n}\in\Re_{2n}$, we obtain

$$[D(x_{22})+D(x_{21})-D(x_{22}+x_{21})]y_{2n}=0. \tag{2.27}$$

Since, y_{1n} and y_{2n} are an arbitrary element of $\Re_{1n}$ and $\Re_{2n}$, respectively. Therefore, from (2.26) and (2.27), we conclude that

$$[D(x_{22})+D(x_{21})-D(x_{22}+x_{21})]\Re=(0). \tag{2.28}$$

By using condition (1), we get

$$D(x_{22})+D(x_{21})-D(x_{22}+x_{21})=0. \tag{2.29}$$

Above relation implies that

$$D(x_{22}+x_{21}) = D(x_{22}) + D(x_{21}). \tag{2.30}$$

This complete the proof. □

Lemma 2.3. *D is additive on $\Re_{21}$*

Proof. Let $x_{21}, y_{21} \in \Re_{21}$. For any $z_{12} \in \Re_{12}$ and $z_{2n} \in \Re_{2n}$ for $n = 1, 2$ and using the definition of derivation, we have

$$D(x_{21}+y_{21})z_{12}z_{2n} = D((x_{21}+y_{21})z_{12}z_{2n}) - (x_{21}+y_{21})D(z_{12}z_{2n}). \tag{2.31}$$

Above equation can be written as

$$\begin{aligned} D(x_{21}+y_{21})z_{12}z_{2n} &= D((x_{21}+y_{21})z_{12}z_{2n}) \\ &\quad -(x_{21}+y_{21})D(z_{12}z_{2n}) \\ &= D((x_{21}z_{12}+y_{21})(z_{2n}+z_{12}z_{2n})) \\ &\quad -(x_{21}+y_{21})D(z_{12}z_{2n}) \\ &= D(x_{21}z_{12}+y_{21})(z_{2n}+z_{12}z_{2n}) \\ &\quad +(x_{21}z_{12}+y_{21})D(z_{2n}+z_{12}z_{2n}) \\ &\quad -(x_{21}+y_{21})D(z_{12}z_{2n}). \end{aligned}$$

In last relation, we use $(ii), (iii)$ and (iv) of Lemma 2.2, this yields

$$\begin{aligned} D(x_{21}+y_{21})z_{12}z_{2n} &= D(x_{21}z_{12})(z_{2n}+z_{12}z_{2n}) + D(y_{21})(z_{2n}+z_{12}z_{2n}) \\ &\quad +(x_{21}z_{12}+y_{21})D(z_{2n}) + (x_{21}z_{12}+y_{21})D(z_{12}z_{2n}) \\ &\quad -x_{21}D(z_{12}z_{2n}) - y_{21}D(z_{12}z_{2n}). \end{aligned}$$

Using Lemma 2.1 and after simplifying, we obtain

$$\begin{aligned} D(x_{21}+y_{21})z_{12}z_{2n} &= D(x_{21}z_{12})z_{2n} + D(y_{21})z_{12}z_{2n} + x_{21}z_{12}D(z_{2n}) \\ &\quad +y_{21}D(z_{12}z_{2n}) - x_{21}D(z_{12}z_{2n}) - y_{21}D(z_{12}z_{2n}). \end{aligned} \tag{2.32}$$

Using definition of derivation in (2.32), we see that

$$\begin{aligned} D(x_{21}+y_{21})z_{12}z_{2n} &= D(x_{21})z_{12}z_{2n} + x_{21}D(z_{12})z_{2n} + D(y_{21})z_{12}z_{2n} \\ &\quad +x_{21}z_{12}D(z_{2n}) \\ &\quad +y_{21}D(z_{12}z_{2n}) - x_{21}D(z_{12})z_{2n} - x_{21}z_{12}D(z_{2n}) \\ &\quad -y_{21}D(z_{12}z_{2n}). \end{aligned}$$

On simplification, this implies

$$[D(x_{21}+y_{21}) - D(x_{21}) - D(y_{21})]z_{12}z_{2n} = 0. \tag{2.33}$$

Since, z_{12} and z_{2n} are arbitrary element of $\Re_{12}$ and $\Re_{2n}$ respectively, (2.33) becomes

$$[D(x_{21}+y_{21})-D(x_{21})-D(y_{21})]\Re_{12}\Re_{2n}=(0). \tag{2.34}$$

Also, it is clear that

$$[D(x_{21}+y_{21})-D(x_{21})-D(y_{21})]\Re_{12}\Re_{1n}=(0). \tag{2.35}$$

Combining (2.34) and (2.35), we get

$$[D(x_{21}+y_{21})-D(x_{21})-D(y_{21})]\Re_{12}\Re=(0). \tag{2.36}$$

First, we use condition (1) and then (2), above relation yields that

$$D(x_{21}+y_{21})=D(x_{21})-D(y_{21}). \tag{2.37}$$

This completes the Lemma. □

Lemma 2.4. *D is additive on* $\Re_{12}$

Proof. Let $x_{12},\ y_{12}\in\Re_{12}$, $t_{1n}\in\Re_{1n}$ and using Lemma 2.1(ii), we have

$$[D(x_{12})+D(y_{12})]t_{1n}=D(x_{12})t_{1n}+D(y_{12})t_{1n}=0. \tag{2.38}$$

We have, $[D(x_{12})+D(y_{12})]t_{1n}=D(x_{12}+y_{12})t_{1n}$. Solving last relation, we arrives at

$$[D(x_{12})+D(y_{12})-D(x_{12}+y_{12})]t_{1n}=0. \tag{2.39}$$

Since, t_{1n} be an arbitrary element of $\Re_{1n}$. Equation (2.39) can be written as

$$[D(x_{12})+D(y_{12})-D(x_{12}+y_{12})]\Re_{1n}=(0). \tag{2.40}$$

Now, for an element $t_{2n}\in\Re_{2n}$ and by using $(i),(ii)$ and (iii) of Lemma 2.2, we have

$$\begin{aligned}D((x_{12}+y_{12})t_{2n}) &= D((e_1+x_{12})(e_2t_{2n}+y_{12}t_{2n}))\\ &= D(e_1+x_{12})(e_2t_{2n}+y_{12}t_{2n})+(e_1+x_{12})D(e_2t_{2n}+y_{12}t_{2n})\\ &= D(e_1)(e_2t_{2n})+D(e_1)(y_{12}t_{2n})+D(x_{12})(e_2t_{2n})+D(x_{12})(y_{12}t_{2n})\\ &\quad +e_1D(e_2t_{2n})+e_1D(y_{12}t_{2n})+x_{12}D(e_2t_{2n})+x_{12}D(y_{12}t_{2n}).\end{aligned}$$

By using $(i), (ii), (iii)$ and (iv) part of Lemma 2.1 and $D(e_1) = 0$ in above equation and after solving, we obtain

$$D((x_{12} + y_{12})t_{2n}) = D(x_{12})t_{2n} + D(y_{12})t_{2n} + y_{12}D(t_{2n}) + x_{12}D(t_{2n}). \tag{2.41}$$

Using the definition of derivation in left side of (2.40), we get

$$D(x_{12}+y_{12})t_{2n}+(x_{12}+y_{12})D(t_{2n}) = (D(x_{12})+D(y_{12}))t_{2n}+(y_{12}+x_{12})D(t_{2n}). \tag{2.42}$$

On solving, we find

$$D(x_{12} + y_{12})t_{2n} = (D(x_{12}) + D(y_{12}))t_{2n}. \tag{2.43}$$

Which implies that

$$[D(x_{12}) + D(y_{12}) - D(x_{12} + y_{12})]t_{2n} = 0. \tag{2.44}$$

Since, t_{2n} is an any element of $\Re_{2n}$, this gives

$$[D(x_{12}) + D(y_{12}) - D(x_{12} + y_{12})]\Re_{2n} = (0). \tag{2.45}$$

Combining (2.40) and (2.45), we get

$$[D(x_{12} + y_{12}) - D(x_{12}) + D(y_{12})]\Re = (0). \tag{2.46}$$

On using condition (1), we get the required result i.e., $D(x_{12} + y_{12}) = D(x_{12}) + D(y_{12})$. □

Lemma 2.5. *D is additive on $\Re_{11}$*

Proof. Let $x_{11},\ y_{11} \in \Re_{11}$ and $t_{12} \in \Re_{12}$, we get

$$[D(x_{11}) + D(y_{11})]t_{12} = D(x_{11})t_{12} + D(y_{11})t_{12}. \tag{2.47}$$

Using the definition of derivation in last relation, we obtain

$$\begin{aligned}[D(x_{11}) + D(y_{11})]t_{12} &= D(x_{11}t_{12}) \\ &\quad -x_{11}D(t_{12}) \\ &\quad +D(y_{11}t_{12}) - y_{11}D(t_{12}).\end{aligned}$$

Since, $x_{11}t_{12}, y_{11}t_{12} \in \Re_{12}$ and by using Lemma 2.4 in (2.46), we find that

$$\begin{aligned}[D(x_{11}) + D(y_{11})]t_{12} &= D(x_{11}t_{12} + y_{11}t_{12}) - (x_{11} + y_{11})D(t_{12}) \\ &= D((x_{11} + y_{11})t_{12}) - (x_{11} + y_{11})D(t_{12}) \\ &= D(x_{11} + y_{11})t_{12} + (x_{11} + y_{11})D(t_{12}) \\ &\quad -(x_{11} + y_{11})D(t_{12}) \\ &= D(x_{11} + y_{11})t_{12}.\end{aligned}$$

Since, t_{12} be an arbitrary element of $\Re_{12}$, so we have

$[D(x_{11}) + D(y_{11}) - D(x_{11} + y_{11})]\Re_{12} = (0)$.

So, by applying condition (2), we get the result. □

Lemma 2.6. *D is additive on $\Re_{11} + \Re_{21} = \Re e$*

Proof. Let $x_{11}, y_{11} \in \Re_{11}$ and $x_{21}, y_{21} \in \Re_{21}$. Then we have

$$D((x_{11} + x_{21}) + (y_{11} + y_{21})) = D((x_{11} + y_{11}) + (x_{21} + y_{21})). \tag{2.48}$$

Using Lemma 2.2(iii) in (2.48), we arrives at

$$D((x_{11} + x_{21}) + (y_{11} + y_{21})) = D(x_{11} + y_{11}) + D(x_{21} + y_{21}). \tag{2.49}$$

By using Lemma 2.3 and Lemma 2.5, we find that

$$\begin{aligned}D((x_{11} + x_{21}) + (y_{11} + y_{21})) &= D(x_{11}) + D(y_{11}) + D(x_{21}) + D(y_{21}) \\ &= D(x_{11}) + D(x_{21}) + D(y_{11}) + D(y_{21}).\end{aligned}$$

Again we use Lemma 2.2(iii) in last relation, its yields that

$D((x_{11} + x_{21}) + (y_{11} + y_{21})) = D(x_{11} + x_{21}) + D(y_{11} + y_{21})$.

We are done. □

Theorem 2.7. *Let $\Re$ be a ring containing an idempotent e and $1-e$. If d be a multiplicative derivation of $\Re$ and suppose $\Re$ satisfies the following conditions:*

(1) $x\Re e = 0 \Rightarrow x = 0$ *(and hence $x\Re = 0 \Rightarrow x = 0$)*

(2) $x\Re_{12} = 0 \Rightarrow x = 0$.

Then d becomes additive.

Proof. As we mention earlier, we replace d by D. Let $u, v \in \Re$ and consider $D(u) + D(v)$ be an element of $\Re$. Take an element $t \in \Re e = \Re_{11} + \Re_{21}$. Thus, ut and vt are element of $\Re e = \Re_{11} + \Re_{21}$. Now, we find that

$$(D(u) + D(v))t = D(u)t + D(v)t. \tag{2.50}$$

Using definition of derivation in (2.50), we have

$$(D(u) + D(v))t = D(ut) - uD(t) + D(vt) - vD(t). \tag{2.51}$$

Since, $ut, vt \in \Re e = \Re_{11} + \Re_{21}$, then by using Lemma 2.6 in (2.51), we obtain

$$\begin{aligned}(D(u) + D(v))t &= D(ut + vt) - (u + v)D(t)\\ &= D((u + v)t) - (u + v)D(t)\\ &= D(u + v)t + (u + v)D(t) - (u + v)D(t)\\ &= D(u + v)t.\end{aligned}$$

Thus, we have

$$[D(u) + D(v) - D(u + v)]t = 0. \tag{2.52}$$

Since, t be an arbitrary element of $\Re e$. So, we see that

$$[D(u) + D(v) - D(u + v)]\Re e = (0). \tag{2.53}$$

By using condition (1), we obtain $D(u+v) = D(u)+D(v)$. This shows that D, and also d, is additive. □

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