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An application of the Stone-Weierstrass theorem

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Abstract

Let (X, τ) be a topological space, we will denote by $|X|$,$|X|_K$, $|X|_\tau$ and $|X|_\delta$, the cardinalities of X; the family of compacts in X; the family of closed in X, and the family of G_δ-closed in X, respectively. The purpose of this work is to establish relationships between these four numbers and conditions under which two of them coincide or one of them is $\leq c$, where c denotes, as usual, the cardinality of the set of real numbers **R**. *We will use the Stone-Weierstrass theorem to prove that: Let (X, τ) be a compact Hausdorff topological space, then $|X|_\delta \leq |X|^{\aleph_0}$.*

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1. Introduction

Weierstrass approximation theorem states that every continuous real function f over a closed interval $[a, b]$ can be uniformly approximated by polynomials (see [15]).

The most relevant generalization of the Weierstrass theorem is that of Stone, known as the Stone-Weierstrass theorem [13], which characterizes in simple terms the algebras of continuous functions that are uniformly dense on $C(X)$, for compact X.

An important aspect in analysis is the study of the closed, compact subsets G_δ and G_δ-closed of a topological space (X, τ). As an example, we can mention the problem of finding the number of compact subsets of a topological space X, this problem was worked by D. K. Burke and R. E. Hodel in [4], their work was based on finding the cardinality $|X|_K$ in terms of other invariant cardinals, they proved that if X is a separable Hausdorff space, then $|X|_K \leq 2^{\aleph_0}$.

Another very important result in this theory is the one given by P. Roy, on the cardinality of the first countable spaces see [11], in this work, Roy proves that if X is a Hausdorff space that is Lindelöf over every closed subset L, when $|L| \leq c$ and if, furthermore, the first axiom of countability holds in X, then $|X| \leq c$.

The purpose of the following work is to present a new alternative solution to the problem of establishing relations regarding the cardinality of a topological space (X, τ) and some of its subsets, from the *Stone-Weierstrass theorem.* We will also prove a metrization theorem (Arhangel'skiis original, see Theorem 3, [2]) on spaces whose family of compacts has cardinality $< c$.

2. Preliminaries

The terminology of [8] and [16], is used throughout.

Throughout this paper we shall assume that (X, τ) is a topological space.

Definition 1. *An topological space (X, τ) is a Lindelöf space, or has the Lindelöf property if and only if every open cover of X has a countable subcover. X is a hereditarily Lindelöf if and only if all open subspaces of X have the Lindelöf property.*

Apparently Theorem 2 below is well-known. We include it here for easy reference.

Theorem 2. *Let (X, τ) be a topological space. The following conditions are equivalent:*

1. *The space X is hereditarily Lindelöf.*
2. *Every open subspace of X is Lindelöf.*
3. *For every uncountable subspace Y of X, there exists a point $y \in Y$ such that every open subset of X containing y contains an uncountable number of points of Y .*

Proof. The implications of (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) can be found in detail in ([16], P 110).

(3) $\Rightarrow$ (1) Let's proceed by contrapositive. Suppose T is not a Lindelöf subspace of (X, τ). Let $\mathcal{U}$ be an open cover of T such that no countable subcollection of $\mathcal{U}$ can cover T. By a transfinite inductive process, let us choose a set of points $\{t_\alpha \in T \colon \alpha < \omega_1\}$ and a collection of open sets $\{U_\alpha \in \mathcal{U} \colon \alpha < \omega_1\}$ such that for each $\alpha < \omega_1$, $t_\alpha \in U_\alpha$ and $t_\alpha \notin \bigcup\{U_\beta \colon \beta < \alpha\}$. The inductive process is possible since no countable subcollection of $\mathcal{U}$ can cover T .

On the other hand, for each α, let V_α be an open subset of (X, τ) such that $U_\alpha = V_\alpha \cap Y$. We can now conclude that for any point t_α of Y, there exists an open set V_α containing t_α such that V_α contains only a countable number of points of Y. □

Remark 3. *The condition* 3 *indicates that every uncountable set has some special kind of limit points. Let $p \in X$. We say that p is a limit point of the set $Y \subset X$ if every open set containing p contains a point of Y other than p.*

In some situations, it will be enough to apply the following corollary obtained from the condition 3.

Corollary 4. *If the space (X, τ) is hereditarily Lindelöf, then every uncountable subspace Y of X contains one of its limit points.*

Definition 5. *A family $\mathcal{V}$ of open neighborhoods of a point $x \in X$ is a* local basis in x *if every neighborhood of x contains an element of $\mathcal{V}$. (X, τ) is* first-countable *if every $x \in X$ has a locally countable basis.*

Theorem 6. *If (X, τ) is T_2, Lindelöf and first-countable, then $|X| \leq c$ (see [2], Corollary 2.1).*

From the previous theorem, the following corollary follows.

Corollary 7. *If (X,τ) is T_2, hereditarily Lindelöf and first-countable, then $|X|_\tau \leq c$.*

Definition 8. *Let (X,τ) be a topological space.*

1. *A subset A of a space X is a set G_δ if it is a countable (finite or infinite) intersection of open spaces of X;*
2. *A subset A of a space X is a set closed-G_δ in X, if and only if, for every point $p \notin A$ there exists a set G_δ, D_p, such that $p \in D_p$ and $D_p \cap A = \emptyset$.*

Definition 9. *A topological space (X,τ) is normal iff whenever A and B are disjoint closed sets in X, there are disjoint open sets U and V with $A \subset U$ and $B \subset V$. A space (X,τ) is perfectly normal iff it is normal and each closed set in X is a G_δ-set. A set $A \subseteq X$ is called zero set if there exists a continuous function $f\colon X \to [0,1]$ such that $A = f^{-1}(0)$.*

The result that follows is well known in set topology.

Theorem 10. *Let (X,τ) be a normal space and $A \subseteq X$ closed. Then A is a zero set if and only if A is G_δ-set.*

A space (X,τ) is *Čech-complete* if it is homeomorphic to a subspace G_δ of a Hausdorff compact.

We will have the opportunity to use the following result:

Theorem 11. *(see [2], Lemma 2.) Every Čech-complete and hereditarily Lindelöf X space is either first-countable or has cardinality c.*

Proof. Suppose (X,τ) is uncountable, with at most a countable infinity of exceptions, all points of X are condensation points of X. Therefore, without loss of generality, we can assume that X is also self-dense. Suppose $X = \bigcap\limits_{n=1}^{\infty} W_n$, where $W_1, W_2, \ldots$ are open in a compact and Hausdorff space Z. Let's pick two distinct points $a_0, a_1 \in X$. Let V_0, V_1 be foreign open in Z such that $a_0 \in V_0 \subseteq \overline{V}_0 \subseteq W_1$, $a_1 \in V_1 \subseteq \overline{V}_1 \subseteq W_1$ (locks are taken at Z). Since X is dense in itself, there are distinct points $a_{00}, a_{01} \in V_0 \cap X$ and distinct points $a_{10}, a_{11} \in V_1 \cap X$. Let V_{00}, V_{01}, V_{11} be mutually unrelated open in Z such that $a_{00} \in V_{00} \subseteq \overline{V}_{00} \subseteq V_0 \cap W_2$; $a_{01} \in V_{01} \subseteq \overline{V}_{01} \subseteq V_0 \cap W_2$;

$a_{10} \in V_{10} \subseteq \overline{V}_{10} \subseteq V_1 \cap W_2$ and $a_{11} \in V_{11} \subseteq \overline{V}_{11} \subset V_1 \cap W_2$. Let's now choose distinct points $a_{ij0}, a_{ij1} \in V_{ij} \cap X$ $(i, j = 0, 1)$ and let V_{ij0}, V_{ij1} be open outside in Z such that:

$$a_{ij0} \in V_{ij0} \subseteq \overline{V}_{ij0} \subseteq V_{ij} \cap W_3$$
$$a_{ij1} \in V_{ij1} \subseteq \overline{V}_{ij1} \subseteq V_{ij} \cap W_3, \qquad i, j = 0, 1.$$

This process can be continued inductively. For each $(i_1, i_2, \ldots) \in 2^W$, let us choose a point $x(i_1, i_2, \ldots)$ in $V_{i_1} \cap V_{i_1 i_2} \cap V_{i_1 i_2 i_3} \cap \cdots$. As we can see $(i_1, i_2, \ldots) \to x(i_1, i_2, \ldots)$ defines an injective function from 2^W into X. Therefore, $|X| \geq c$. But by the theorem 6, $|X| \leq c$ (note that every $\check{C}$ech-complete and hereditarily Lindelöf space X is first-countable). Thus, $|X| = c$. $\square$

Definition 12. *For each compact and Hausdorff space X, let us denote by $C(X, \mathbf{R})$ the set of all continuous functions $f: X \to \mathbf{R}$ with the topology induced by the metric:*

$$d(f, g) = \sup\{|f(x) - g(x)|: x \in X\}, f, g \in C(X, \mathbf{R}).$$

Definition 13.

1. *A family $\mathcal{A} \subset C(X)$, is an algebra, if for all $f, g \in \mathcal{A}$ and for all $c \in R$, we have $f + g \in \mathcal{A}, fg \in \mathcal{A}$ and $cf \in \mathcal{A}$.*
2. *Given a collection $\mathcal{D} \subset C(X)$, the subalgebra $\mathcal{A}(\mathcal{D})$ generated by $\mathcal{D}$, is the smallest of the subalgebras of $C(X)$ containing $\mathcal{D}$.*

The uniform closure of $\mathcal{A}$ is the set $\overline{\mathcal{A}}$ of the functions in $C(X)$ that can be uniformly approximated by elements of $\mathcal{A}$. The collection $\mathcal{D} \subset C(X)$ is said to separate points if given any two points $x \neq y$ in X, there exists a function $f \in D$ such that $f(x) \neq f(y)$.

The following version of the Stone-Weierstrass Theorem is not the most general. However, it will suffice for our purposes:

Theorem 14 (Stone-Weierstrass (see [10])). *Let $\mathcal{A} \subseteq C(X, \mathbf{R})$. Suppose $\mathcal{A}$ contains at least one non-zero constant function. If for each pair of distinct points $a, b \in X$, there exists $f \in \mathcal{A}$ such that $f(a) \neq f(b)$, then the subalgebra $\mathcal{A}^*$ of $\mathcal{A} \subseteq C(X, \mathbf{R})$ generated by $\mathcal{A}$ is dense in $C(X, \mathbf{R})$.*

3. Main results

Theorem 1. *Let (X, τ) be a compact Hausdorff topological space. Then $|X|_\delta \leq |X|^{\aleph_0}$.*

Proof. Let K_δ be the family of closed G_δ in X. By theorem 10, there exists an injective function $j\colon K_\delta \to C(X,\mathbf{R})$ such that for each $H \in K_\delta$, we can choose $f_H \in C(X,\mathbf{R})$ such that $f_H^{-1}(0) = H$. Let $j(H) = f_H$. It will then suffice to prove that $|C(X,\mathbf{R})| \leq |X|^{\aleph_0}$. For each pair of distinct points $a, b \in X$, we define $g_{a,b} \in C(X,\mathbf{R})$ such that $g_{a,b}(a) = 0$ and $g_{a,b}(b) = 1$. Let $\overline{\lambda} \in C(X,\mathbf{R})$ be the constant function $\overline{\lambda} \equiv 1$. We can note that $\mathcal{A}^*$, is a subalgebra generated by the family $\mathcal{A} = \{\overline{\lambda}\} \cup \{g_{a,b}\colon a, b \in X, a \neq b\}$, plus $\mathcal{A}^*$ satisfies the hypotheses of the Theorem 14. Therefore, the subalgebra $\mathcal{A}^*$ generated by $\mathcal{A}$ is dense in $C(X,\mathbf{R})$. The above allows us to state that $|\mathcal{A}^*| \leq \aleph_0 \cdot |\mathcal{A}| \cdot c \leq \max\{|X|, c\}$. Finally, since $C(X,\mathbf{R})$ is first countable, each $f \in C(X,\mathbf{R})$ is the limit of a sequence in $\mathcal{A}^*$ and thus $|C(X,\mathbf{R})| \leq |X|^{\aleph_0}$. □

Before stating some corollaries of the previous theorem, we need some definitions.

Definition 2. *A topological space (X,τ) is σ-compact if there are compacts $A_1, A_2, \ldots$ in X such that $X = \bigcup_{n=1}^{\infty} A_n$.*

Corollary 3. *Let (X,τ) be a σ-compact Hausdorff topological space, with points G_δ. Then $|X|_\delta \leq c$.*

Proof. Let $X_1, X_2, \ldots$ be compact in X such that $X = \bigcup_{n=1}^{\infty} X_n$. Let $K, K_1, K_2, \ldots$ be the families of closed G_δ in $X, X_1, X_2, \ldots$, respectively. Since each X_n is first countable (because it is compact and has points G_δ, see ([16], 16 **A**. 4)) , we have $|X_n| \leq c$ for each $n = 1, 2, \ldots$ (see Theorem 6). Therefore, each $|K_n| \leq c$ (see Theorem 1). Then the correspondence $K \to (K \cap X_1, K \cap X_2, \cdots)$ determines an injective function of K in $K_1 \times K_2 \times, \ldots$. Like $|K_1 \times K_2 \times \cdots| \leq c$, also in $|K| \leq c$, or $|X|_\delta \leq c$. □

A function $f\colon X \to Y$ is perfect if f is continuous, surjective, closed and every $f^{-1}(y)$ is compact.

Definition 4. *A Hausdorff topological space (X,τ) is perfect pre-image of a metric space Y (that is, X is said to be premetrizable), if there exists a perfect function $f\colon X \to Y$.*

Lemma 5. *Let (X,τ) be a premetrizable Lindelöf topological space, with points G_δ and let K be the family of closed sets G_δ and σ -compacts of X Then $|K| \leq c$.*

Proof. By hypothesis, since (X, τ) is a premetrizable topological space, there exists a metrizable space Y and a perfect function ϕ such that $\phi\colon X \to Y$. Since X is Lindelöf, Y is too, and therefore Y has a countable base. By the Corollary 7, $|Y|_\tau \leq c$ Let $\mathcal{H}^* = \{H \subseteq X\colon H = \phi^{-1}\phi(H),\ H$ closed and $\sigma-$ compact$\}$ Therefore, we can say that $|\mathcal{H}^*| \leq |Y|_\tau \leq c$ Furthermore, every $K_i \in K$ is a closed G_δ in some element of $\mathcal{H}^*$ (since $K_i \subseteq \phi^{-1}\phi(K_i)$ and $\phi^{-1}\phi(K_i) \in \mathcal{H}^*$), for $i = 1, 2, \ldots$ By the Corollary 3 , for each $H \in \mathcal{H}^*$ we have $|H|_\delta \leq c$, so $|K| \leq c \cdot c = c$. □

Corollary 6. *Let (X, τ) be a topological space, σ-compact, premetrizable, Lindelöf hereditary and with points G_δ. Then $|X|_\delta \leq c$.*

Proof. It is obtained directly from the corollary 5. □

The following metrization theorem is well known. The proof of it can be found in [3], Theorem 8.1.

Theorem 7. *A topological space (X, τ) is metrizable if and only if it is premetrizable and its diagonal $\Delta(X)$ is a set G_δ in $X \times X$.*

Corollary 8. *A premetrizable space X is metrizable if there is a continuous and injective function from X to a metrizable space Y.*

Proof. Just note that the assumptions imply that $\Delta(X)$ is a set G_δ in $X \times X$. □

The following theorem belongs to the field of set theory. However, we will have the opportunity to apply it in our context.

Theorem 9. *Let X be a set of cardinality c and $\mathcal{V}$ a subfamily of 2^X of cardinality $\leq c$ such that $V \in \mathcal{V}$ has cardinality c. Then there exist two alien sets A, B such that $X = A \cup B$ and such that for each $V \in \mathcal{V}$, $A \cap V \neq \emptyset \neq B \cap V$.*

Before obtaining some consequences of this theorem, we will give the following definitions.

A topological space (X, τ) is *totally imperfect* if every compact subspace of X is countable. A family $\mathcal{G}$ of subsets of X is a *grill* of X if everything open in X is a union of elements of $\mathcal{G}$.

Corollary 10. *Every premetrizable and hereditarily Lindelöf space X is a union of two totally imperfect subspaces.*

Proof. Suppose that X is not totally imperfect. Theorem 11 then implies that $|X| \geq c$. But according to the corollary 6 and theorem 1, also $|X| \leq c$. Therefore, $|X| = c$.

Let $\mathcal{V}$ be the family of compact subspaces of X of cardinality c. The corollary 6 implies that $|\mathcal{V}| \leq c$. Therefore, we can apply theorem 9 and obtain two alien sets A, B such that $X = A \cup B$ and with $A \cap V \neq \emptyset \neq B \cap V$ for each $V \in \mathcal{V}$. By theorem 11, A and B are totally imperfect. □

Before proving the metrization theorem alluded to in the introduction to this work, we need two simple theorems whose proof we omit:

Theorem 11. *Every regular space X and hereditarily Lindelöf is perfectly normal.*

Theorem 12. *If X has a countable grill, then $X \times X$ is hereditarily Lindelöf.*

Let us also remember the famous Urysohn metrization theorem:

Theorem 13. *Every regular space X, T_1, with countable base is metrizable.*

Let us now state and prove Arhangel'skii metrization theorem see Theorem 3, [2].

Theorem 14. *Every hereditarily premetrizable space X ($\equiv$ every subspace is premetrizable) and hereditarily Lindelöf X is metrizable with countable base.*

Proof. By the corollary 10, X has two totally imperfect subspaces A and B such that $X = A \cup B$. Let Y_A, Y_B be metrizable spaces and ϕ_A, ϕ_B perfect functions $\phi_A : A \to Y_A, \phi_B : B \to Y_B$. Since the fibers or inverse points of these functions are countable sets, A and B are countable unions of sets each of which admits a continuous and injective function to a metrizable space. Using the corollary 8, we deduce that X is a countable union of metrizable spaces with countable base. Therefore, X has a countable grill and by theorems 11 and 12, $\Delta(X)$ is a set G_δ in $X \times X$. Theorem 7 then implies that X is metrizable and, being Lindelöf, X has a countable base. □

We conclude this work with a question that, as far as we know, has not been answered.

Question 15. *Let (X, τ) be hereditarily premetrizable and perfectly normal. Is (X, τ) metrizable?*

4. Dedicatory

In memory of Adalberto García-Máynez

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