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Nourishing number of some associated graphs

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Abstract

Let $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$ and $\mathcal{P}(\mathbf{N}_0)$ be the power set. An injection $f : V(G) \to \mathcal{P}(\mathbf{N}_0)$ is an integer additive set-indexer (IASI) of a graph G if the induced map $f^+ : E(G) \to \mathcal{P}(\mathbf{N}_0)$ given by $f^+(uv) = f(u)+f(v)$ is also an injection, where $f(u)+f(v)$ is the sumset of $f(u)$ and $f(v)$. Moreover, if $|f^+(uv)| = |f(u)|\,|f(v)|$, for all uv in $E(G)$, then f is a strong IASI of G. The nourishing number of a graph G is the minimum order of the maximal complete subgraph of G such that G admits a strong IASI. In this paper we investigate the admissibility of strong IASI for some associated graphs and calculate their nourishing number. In addition, we obtain the nourishing number of powers of the associated graphs.

Keywords: *Integer additive set-indexers; strong integer additive set-indexers; nourishing number of a graph; sumset.*

MSC (2020): *05C78, 11B13.*

1. Introduction

For terms and notations of graphs, we refer to [1]. For terms in graph labeling and sumset, we follow [2] and [3] respectively. For detailed work in strong integer additive set-valued graph, we suggest [4]. Let G be a simple, finite, connected and undirected graph of order n. The vertex set and the edge set of G are denoted by $V(G)$ and $E(G)$ respectively.

If $A, B \subset \mathbf{N}_0 = \mathbf{N} \cup \{0\}$, the sumset of A and B is $A + B = \{a + b : a \in A, b \in B\}$. For $A \subset \mathbf{N}_0$, A is finite and $|A|$ is its cardinality.

Definition 1.1. *[5] An injection $f : V(G) \to \mathcal{P}(\mathbf{N}_0)$ is an integer additive set-indexer (IASI) of a graph G if the induced map $f^+ : E(G) \to \mathcal{P}(\mathbf{N}_0)$ given by $f^+(uv) = f(u) + f(v)$ is also an injection. If G has such a map f then G is called an integer additive set-indexed graph (IASI graph).*

Definition 1.2. *[6] If f is a set-indexer of G and satisfies $|f^+(uv)| = |f(u)||f(v)|$ for all vertices u and v of G, then f is called a strong IASI of G. Such a G is called a strong integer additive set-indexed graph (strong IASI graph).*

In [6], the following notation is used. If $A, B \subset \mathbf{N}_0$ and $A, B \neq \phi$, then $A < B$ is used in the sense that $A \cap B = \emptyset$ and the sequence $A_1 < A_2 < A_3 < \ldots < A_n$ conveys that the sets are pairwise disjoint. $D_A = \{|a - b| : a, b \in A, a \neq b\}$ is the difference set of A.

Lemma 1.3. *[6] If $A, B \subset \mathbf{N}_0$ and $A, B \neq \phi$ then $|A + B| = |A||B| \iff$ the relation $D_A < D_B$ holds.*

Theorem 1.4. *[6] If each vertex v_i of K_n is labeled by the set $A_i \in \mathcal{P}(\mathbf{N}_0)$, then K_n admits a strong IASI $\iff$ for the difference set D_i of the set-label A_i of v_i there exists a finite sequence $D_1 < D_2 < D_3 < \ldots < D_n$.*

Theorem 1.5. *[6] A connected graph G (on n vertices) admits strong IASI if and only if each vertex v_i of G is labeled by a set A_i in $\mathcal{P}(\mathbf{N}_0)$ and there exists a finite sequence $D_1 < D_2 < D_3 < \ldots < D_m$, where $m \leq n$ is a positive integer and D_i is the difference set of A_i.*

Definition 1.6. *[7] The nourishing number of a graph G is the minimum order of the maximal complete subgraph of G such that G admits a strong IASI. It is denoted by $\kappa(G)$.*

Theorem 1.7. *[7]*

(a) $\kappa(G) = n$, if G is complete;

(b) $\kappa(G) = 2$, if G is bipartite or triangle-free.

Definition 1.8. *[8] If $r \in \mathbf{N}$ then r^{th} power of G, represented by G^r, is the graph with vertices as of G and two vertices in G^r are adjacent if they are at a distance atmost r in G.*

Theorem 1.9. *[9] If d is the diameter of G, then G^d is complete.*

Definition 1.10. *[8] For any vertex v of G, the neighbourhood set of v in G is the set of all vertices adjacent to v in G. It is denoted by $N_G(v)$ or $N(v)$.*

Definition 1.11. *[2] If G is connected and $G_1, G_2, \dots, G_m$ are m copies of G then the graph formed by joining each vertex u in G_i to the neighbours of the corresponding vertex v in $G_j, 1 \leq i, j \leq m$ is called the m-shadow graph $D_m(G)$.*

Definition 1.12. *[2] A graph constructed by adding to each vertex v of G new m vertices, say $v_1, v_2, v_3, \dots, v_m$ such that $v_i, 1 \leq i \leq m$, is adjacent to each vertex that is adjacent to v in G is called the m-splitting graph $Spl_m(G)$.*

Definition 1.13. *[2] The middle graph $M(G)$ of a graph G is the graph whose vertex set is $V(G) \cup E(G)$ and two vertices x, y in the vertex set of $M(G)$ are adjacent in $M(G)$ in case one of the following holds:*

1. $x, y \in E(G)$ and x, y are adjacent in G;

2. $x \in V(G)$, $y \in E(G)$ and x, y are incident in G.

Definition 1.14. *[2] A vertex switching graph G_v is the graph created from G by eliminating all edges incident to v and introducing edges connecting v to every non-adjacent vertex of$v$ in G.*

Integer additive set-indexers and strong integer additive set-indexers have been intensively studied. For literature on this topics, we refer [5], [6], [7], [10], [11], and [12]. For work on the nourishing number of set-labeled graphs, we refer [7], [13], and [14]. In this paper, we investigate the admissibility of strong IASI for some associated graphs and calculate their nourishing number. In addition, we obtain the nourishing number of powers of the associated graphs.

2. Main Results

Theorem 2.1. *Let G be a strong IASI graph and $G_i, 1 \leq i \leq m$ be a copy of G. Then $D_m(G)$ admits a strong IASI $\iff$ the difference set of the set-label of each vertex in G_i is pairwise disjoint with the difference set of the set-label of the vertices adjacent to the corresponding vertex in G_j, for $1 \leq j \leq m$ and $j \neq i$.*

Proof. Assume that $D_m(G)$ admits a strong IASI. Fix a vertex v_i in G_i. Let $v_i^{(j)}$ be the corresponding vertex in the j^{th} copy G_j of G with $j \neq i$. Since $D_m(G)$ is a strong IASI graph, the difference set of the set-labels of any two adjacent vertices are pairwise disjoint. In particular, v_i and the vertices in $N(v_i^{(j)})$ are adjacent in $D_m(G)$. Thus, the difference set of the set-labels of v_i and the vertices in $N(v_i^{(j)})$ must be disjoint.
Conversely, suppose that the difference set of the set-label of each vertex in G_i is pairwise disjoint with the difference sets of the set-labels of the vertices adjacent to the corresponding vertex in G_j for $1 \leq j \leq m, i \neq j$. Since G_i is a strong IASI graph, the difference sets of the set-labels of any two adjacent vertices in G_i are pairwise disjoint. Hence for adjacent vertices u and v in $D_m(G)$, $D_u < D_v$. Therefore, $D_m(G)$ is a strong IASI graph. □

Theorem 2.2. *Let G be a strong IASI graph. Then $\kappa(D_m(G)) = \kappa(G)$.*

Proof. Let H be the maximal complete subgraph of G of order $\kappa(G)$. Then H is also a complete subgraph of $D_m(G)$. So, $\kappa(D_m(G)) \geq \kappa(G)$. Suppose $D_m(G)$ has a complete subgraph, say H_m, of order k where $k > \kappa(G)$. Observe that all the vertices of H_m cannot be in a single copy of G. Let $v_i, 1 \leq i \leq k$ be the vertices in H_m. Fix a copy of G, say G_j, containing the most number of vertices of H_m. If v is in $V(H_m)$ but not in $V(G_j)$, then its corresponding vertex in G_j will be adjacent to every vertex in H_m that is in G_j. Since v is any arbitrary vertex, H_m is isomorphic to a complete subgraph of G_j and it is of order k. A contradiction since H is maximal in G. Thus, $\kappa(D_m(G)) = \kappa(G)$. □

Theorem 2.3. *If $D_m(G)^r$ represents the r^{th} power of $D_m(G)$, then*

$$\kappa(D_m(G)^r) = \begin{cases} m\kappa(G^r), & \text{if } 1 < r < d; \\ mn, & \text{if } r \geq d \end{cases} \tag{2.1}$$

where n is the order of graph G.

Proof. If d is the diameter of G, then the diameter of $D_m(G)$ is also d. Consider the following two cases :

Case (i): If $r \geq d$, $\kappa(D_m(G)^r) = mn$, since $D_m(G)^r$ of order mn is complete.

Case (ii): Let $1 < r < d$. For j^{th} copy G_j of G, G_j^r is a subgraph of $D_m(G)^r$. Suppose that $\kappa(G_j^r) = k$. Therefore, the maximal complete subgraph, say H_j, of G_j^r contains k vertices. If v is a vertex in H_j then it is adjacent to all the vertices of $D_m(G)^r$ that corresponds to the vertices of H_j. Since v is arbitrary, it holds for every vertex in H_j. Thus, the set of all vertices of H_j and their corresponding vertices in $D_m(G)^r$ induces a complete subgraph H of order mk. Therefore, $\kappa(D_m(G)^r) \geq mk$.
In order to prove equality, it is enough to show that $D_m(G)^r$ has no complete subgraph of order greater than mk. Observe that if a vertex is in a complete subgraph of $D_m(G)^r$, then every corresponding vertex must also be in that subgraph. Therefore, the order of a larger subgraph of $D_m(G)^r$ must be a multiple of m. Now if H' is a subgraph of order mk' where $k' > k$, then for some j, G_j^r has a complete subgraph of order k', which is not possible. Hence, $\kappa(D_m(G)^r) = m\kappa(G_j^r)$. □

Theorem 2.4. *Let G be a strong IASI graph. Then $Spl_m(G)$ admits a strong IASI $\iff$ the difference set of the set-label of every vertex in the neighbourhood of u in $V(G)$ is pairwise disjoint with the difference set of the set-label of m vertices corresponding to u in $Spl_m(G)$.*

Proof. Suppose that $Spl_m(G)$ is a strong IASI graph. The difference sets of the set labels of any two adjacent vertices are pairwise disjoint. In particular, let u be any vertex in G and $u_1, u_2, \ldots, u_m$ be the vertices corresponding to u which are added in G to obtain $Spl_m(G)$. Since each vertex in $N_G(u)$ is also in $Spl_m(G)$, it is adjacent to m vertices $u_1, u_2, \ldots, u_m$ corresponding to u. Therefore, the difference set of the set-labels of every vertex in $N_G(u)$ and the m vertices corresponding to u are pairwise disjoint.
Conversely, assume that the difference set of the set-label of each vertex in $N_G(u)$ is pairwise disjoint with the difference set of the set-label of m vertices corresponding to u in $Spl_m(G)$. Since G is a strong IASI graph, the difference set of the set-label of any two adjacent vertices of G is pairwise disjoint. This condition holds when vertices are considered in $Spl_m(G)$. Thus, $Spl_m(G)$ admits a strong IASI. □

Theorem 2.5. *Let G be a strong IASI graph. Then $\kappa(Spl_m(G)) = \kappa(G)$.*

Proof. Let H be a maximal complete subgraph of G having order $\kappa(G)$. Since H is also a complete subgraph of $Spl_m(G)$, we have $\kappa(Spl_m(G)) \geq \kappa(G)$. Suppose $\kappa(Spl_m(G)) > \kappa(G)$. Let H_m be the the maximal complete subgraph of $Spl_m(G)$ having order $\kappa(Spl_m(G))$. Let $v_i^j, 1 \leq j \leq m$ be the vertices corresponding to $v_i \in V(G), 1 \leq i \leq |V(G)|$. Observe that any two such vertices are not adjacent in $Spl_m(G)$. Hence H_m can contain atmost one such vertex. Suppose H_m contains a vertex v_i^j for some i, j. Then v_i must be adjacent to every vertex in H_m except v_i^j. Therefore, a complete subgraph of G isomorphic to H_m is obtained having order greater than $\kappa(G)$. A contradiction since H is maximal in G. Hence, $\kappa(Spl_m(G)) = \kappa(G)$. □

Theorem 2.6. *Let $Spl_m(G)^r$ be the r^{th} power of $Spl_m(G)$. Then*

$$\kappa(Spl_m(G)^r) = \begin{cases} (m+1)\kappa(G^r), & \text{if } 1 < r < d; \\ mn, & \text{if } r \geq d \end{cases} \tag{2.2}$$

where n is the order of graph G.

Proof. If d is the diameter of a graph G, then the diameter of $Spl_m(G)$ is also d. If $r \geq d$, $\kappa(Spl_m(G)^r) = mn$, since $Spl_m(G)^r$ of order mn is complete. If $r < d$, consider a subgraph G^r of $Spl_m(G)^r$. Let H be a maximal complete subgraph of G^r and has order $k = \kappa(G^r)$. Then H is also a complete subgraph of $Spl_m(G)^r$. Let $v_i, 1 \leq i \leq k$ be the vertices in H. Let $v_i^j, 1 \leq j \leq m$ denote the vertices in $Spl_m(G)$ corresponding to v_i in H. Fix a vertex v_p in H. Then the distance between the vertices $v_p^j, 1 \leq j \leq m$ and v_i is atmost r. So, they are adjacent in $Spl_m(G)^r$. Since p is arbitrary, a complete subgraph of $Spl_m(G)^r$ is obtained which contains the vertices $v_i^j, 1 \leq j \leq m$ along with the vertices of H. This subgraph, say H', is of order $(m+1)k$. It is enough to show that H' is maximal. Suppose there is a vertex u in $V(Spl_m(G)) \setminus V(H')$ which is adjacent to the vertices of H'. Observe that u cannot be in $V(G)$. So u corresponds to some vertex u' in $V(G)$ and is at a distance atmost r from all vertices of H'. But then u' is also at a distance atmost r from all vertices of H' and in particular of H. A contradiction since H is maximal. Therefore, H' is a maximal complete subgraph of $Spl_m(G)^r$ of order $(m+1)k$. Thus, $\kappa(Spl_m(G)^r) = (m+1)k$. □

Theorem 2.7. *Let G be a strong IASI graph. Then $M(G)$ does not admit a strong IASI.*

Proof. If $M(G)$ admits a strong IASI, then the difference set of the set-labels of any two adjacent vertices must be disjoint. Let v_i and v_j be the vertices in $M(G)$ corresponding to the adjacent edges e_i and e_j in G. Let u_k be the vertex in G incident with e_i and e_j. Let u_i and u_j be the other end vertex of e_i and e_j respectively. Let f be a strong IASI defined on G. If the sets $A = \{a_k + b_i \mid a_k \in f(u_k), b_i \in f(u_i)\}$ and $B = \{a_k + c_j \mid a_k \in f(u_k), c_j \in f(u_j)\}$ are assigned to e_i and e_j respectively, then in $M(G)$ they are assigned to v_i and v_j respectively. Fix $b_i \in f(u_i)$ and $c_j \in f(u_j)$. Then $a_k - a_{k'}$ is present in both D_A and D_B for $a_k, a_{k'} \in f(u_k)$. Hence, the difference set of the set-label of the vertices v_i and v_j have non-empty intersection. So, $M(G)$ is not a strong IASI graph. □

Theorem 2.8. *Let G be a strong IASI graph and $v \in V(G)$. Then G_v admits a strong IASI $\iff$ the difference set of the set-label of v is pairwise disjoint with the difference set of the set-label of the vertices in the neighbourhood of v in G_v.*

Proof. Suppose G_v is a strong IASI graph. Then the difference set of the set label of adjacent vertices in G_v are pairwise disjoint. In particular, the difference set of the set-label of v is pairwise disjoint with the difference set of the set-label of vertices adjacent to v.
Conversely it is enough to show that the difference set of the set labels of any two adjacent vertices in G_v are pairwise disjoint. Let v_i and v_j be adjacent vertices in G_v. If $v_i = v$ or $v_j = v$, then the difference sets of the set-labels of v and any adjacent vertices of v are pairwise disjoint. If $v_i, v_j \neq v$, then the difference set of any adjacent vertices v_i and v_j in G_v are pairwise disjoint since G admits strong IASI. Therefore, G_v admits a strong IASI. □

Theorem 2.9. *Let G be a graph that admits strong IASI and v be a vertex of G such that G_v is connected. If H is a unique maximal complete subgraph of G of order $\kappa(G)$, then*

$$\kappa(G_v) = \begin{cases} \kappa(G)+1, & \textit{if } v \notin V(H), v \notin N(u), \textit{for any } u \in V(H); \\ \kappa(G), & \textit{if } v \notin V(H), v \in N(u), \textit{for } u \in V(H) \textit{ or } v \in V(H), \\ & k = \kappa(G)-1, v \notin V(K), v \notin N(u) \textit{ for any } u \in V(K); \\ \kappa(G)-1, & \textit{if } v \in V(H) \textit{ and } k \leq \kappa(G)-2 \textit{ or } k = \kappa(G)-1, \\ & v \in V(K) \textit{ or } k = \kappa(G)-1, v \notin V(K), v \in N(u) \\ & \textit{for } u \in K \end{cases}$$

(2.3)

where K is a complete subgraph of highest order k such that $k < \kappa(G)$.

Proof. Let H be a unique maximal complete subgraph of G of order $\kappa(G)$. Let K be a complete subgraph of highest order k such that $k < \kappa(G)$. Let v be a vertex in G.

Case (i): $v \notin V(H)$

Sub-case (a) : If $v \notin N(u)$ for any $u \in V(H)$, then v will be adjacent to every vertex of H in G_v. So, a complete subgraph of G_v is obtained with order $\kappa(G)+1$. This will also be a maximal complete subgraph of G_v. Thus, $\kappa(G_v) = \kappa(G)+1$.

Sub-case (b) : If $v \in N(u)$ for some $u \in V(H)$ then v is adjacent to $\kappa(G)-k'$ vertices of H in G_v, where $1 \leq k' < \kappa(G)$. Therefore, a complete subgraph of order $\kappa(G)-k'+1$ is obtained in G_v. But H is also a complete subgraph of order $\kappa(G)$ in G_v. Since H is unique subgraph of G of order $\kappa(G)$, there is no subgraph of G_v with order greater than $\kappa(G)$. Hence, H is maximal in G_v. So, $\kappa(G_v) = \kappa(G)$.

Case (ii): $v \in V(H)$

If $v \in V(H)$, then v is not adjacent to any vertex of H in G_v and $H \setminus \{v\}$ is a complete subgraph of order $\kappa(G)-1$. Consider the subgraph K as defined previously. Let $k \leq \kappa(G)-2$. If $v \in V(K)$, then $K \setminus \{v\}$ in G_v is of order atmost $\kappa(G)-3$. Thus, $H \setminus \{v\}$ is maximal in G_v. If $v \notin V(K)$ then K, viewed as complete subgraph of G_v, is of order atmost $\kappa(G)-1$. Hence in any case, $\kappa(G_v) = \kappa(G)-1$. Let $k = \kappa(G)-1$. If $v \in V(K)$, $K \setminus \{v\}$ is complete in G_v but has order atmost $\kappa(G)-2$. But then $H \setminus \{v\}$ is maximal in G_v. So, $\kappa(G_v) = \kappa(G)-1$. If $v \notin V(K)$ and is adjacent to some vertices of K, then $H \setminus \{v\}$ is maximal in G_v. So, $\kappa(G_v) = \kappa(G)-1$. In the case $v \notin V(K)$ and is not adjacent to any vertices of K, then $K \cup \{v\}$ is complete in G_v and it is of order $\kappa(G)$. Since $H \setminus \{v\}$ is of order $\kappa(G)-1$, $\kappa(G_v) = \kappa(G)$. □

3. Conclusion

In this paper, we established the necessary and sufficient criteria for a m-shadow graph, m-splitting graph, and vertex switching graph to be strong IASI graphs. We calculated the nourishing number for each of these graphs. We investigated further and determined the nourishing number of powers of the m-shadow graph and the m-splitting graph. We proved that the middle graph is not a strong IASI graph.

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