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Resistance distance of generalized wheel and dumbbell graph using symmetric $\{1\}$-inverse of Laplacian matrix

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Abstract

A new class of graphs called dumbbell graphs, denoted by $\mathbf{DB}(W_{m,n})$ *is the graph obtained from two copies of generalized wheel graph* $W_{m,n}, m \geq 2, n \geq 3$. *It is a graph on* $2(m+n)$ *vertices obtained by connecting* m*-vertices in one copy with the corresponding vertices in the other copy. The resistance distance between two vertices* v_i *and* v_j, *denoted by* r_{ij}, *is defined as the effective electrical resistance between them if each edge of* G *is replaced by 1 ohm resistor. The Kirchhoff index is the sum of the resistance distances between all pairs of vertices in the graph. In this paper, we formulate the resistance distance of* $W_{m,n}$ *and* $\mathbf{DB}(W_{m,n})$ *using Symmetric* $\{1\}$*-inverse of Laplacian matrices. We provide examples to illustrate the proposed method and also obtain the Kirchhoff indices for these examples.*

Keywords: *Dumbbell graph; Resistance distance; Laplacian matrix; Block matrices; Moore-Penrose inverse; Schur complement*

1. Introduction

We consider an undirected and connected graph $G = (V, E)$, where V is the vertex set and E is the edge set, on n vertices. A graph G is *regular* if every vertex has the same degree. The maximum distance between any two vertices of a graph G is called the *diameter* of a graph G. The *complement* of G is the graph whose vertex set is same as that of G and two vertices are adjacent in $\overline{G}$ if only if they are not adjacent in G. The *union* of two graphs G_1 and G_2, denoted by $G_1 \cup G_2$ is the graph whose vertex set is $V(G_1) \cup V(G_2)$ and the edge set is $E(G_1) \cup E(G_2)$. The *join* of G_1 and G_2, denoted by $G_1 \nabla G_2$ is the graph obtained from $G_1 \cup G_2$ by adding all possible edges from the vertices of G_1 to those in G_2.

The adjacency matrix $A(G)$ of the graph G is a square matrix of order n, whose(i, j)-entry is equal to 1 if the vertices v_i and v_j are adjacent and is equal to 0 otherwise. Let $deg_G(v_i)$ be the degree of vertex v_i in G. The degree matrix $D(G)$ of the graph G is a diagonal matrix of order n with diagonal entries as the degrees of the vertices. The *Laplacian matrix* of G is defined as $L(G) = D(G) - A(G)$.

The standard distance between two vertices v_i and v_j, denoted by $d(v_i, v_j)$, is the length of the shortest path between them. In 1993, Klein and Randic [7] introduced a new distance function named resistance distance based on electrical network analysis. The *resistance distance* between two vertices v_i and v_j, denoted by r_{ij}, is defined as the effective electrical resistance between them if each edge of G is replaced by *1 ohm* resistor. The *Kirchhoff index* is the sum of the resistance distances between all pairs of vertices in the graph. The Kirchhoff index has a wide range of applications in physics, chemistry, and network science.

The resistance distance, unlike the shortest path distance, has the property that two vertices v_i and v_j that are connected by more than one path are closer than if they are only connected by the shortest path. The resistance distance has certain mathematical implications, which can be described in terms of random walks on graphs [9, 26], the number of spanning trees and spanning bi-trees [21], and the generalized inverse of the Laplacian matrix [7]. Resistance distance has extensive uses in chemistry, in addition to being an intrinsic graph metric and an important component of electrical circuit theory. The resistance distance is better for dealing with network wave-like movements, such as chemical molecule communication [8].

The resistance distance has received a lot of attention in the mathematical, chemical, and physical literature. For a long time, this has been a classic problem in electrical network theory that has been studied by many researchers. Resistance distances have been computed for a variety of interesting classes of graphs so far, with a focus on electrical networks and chemical graphs. Resistance distances have been obtained for some particular classes of graphs, for example, regular graphs [22], circulant graphs [11], distance regular networks [12], wheels and fans [31], Cayley graphs [40], complete graph minus N edges [27], complete n-partite graphs [34], Cayley graphs on symmetric groups [24], some class of graphs [37], pseudo-distance regular [13], almost complete bipartite graphs [23], ring clique network [35], and so on.

It is interesting to note that a good deal of attention has been paid to resistance distances in plane networks, such as fullerene graphs [29], Möbius ladder graphs [28], ladder graphs [44], Apollonian network [41], Sierpinski Gasket Network [43], simple cubic network lattices [30], straight linear 2-trees [38], Flower networks [42], Path Network [45], class of plane hexagonal networks [36], linear octogonal networks [16], and linear polyacene graphs [5]. Many formulae, such as combinatorial formulae, algebraic formulae, probabilistic formulae and so forth have been putforth for calculating resistance distance.

The resistance distance for some graph operations was studied in recent years, i.e., the subdivision-vertex join and subdivision-edge join graphs [2], $R-$ vertex join and $R-$ edge join of two graphs [39], the subdivision-vertex and subdivision-edge coronae graphs [14], the $H-$ join of graphs [20], the corona and neighborhood corona graphs [15], the double corona based on $R-$ graphs [18], and Tensor Product of P_2 and K_n graphs [25]. Motivated by these, we have obtained the resistance distance of the Generalized wheel graph and Dumbbell graph using Symmetric $\{1\}$-inverse of Laplacian matrices in this article. Also, we have provided examples for generalized wheel and dumbbell graph and we have obtained the Kirchhoff indices for these graphs.

2. Preliminaries

For an $m \times m$ matrix P, the $\{1\}$-*inverse* of P is an $m \times m$ matrix X such that $PXP = P$. If P is singular then it has infinite $\{1\}$-inverses. If X is the unique matrix satisfying $PXP = P$, $XPX = X$ and $PX = XP$ then $X = P^{\#}$ is the *group inverse* of P. It is known that $P^{\#}$ exists if only if $rank\,(P) = rank\,(P^2)$.

If P is real symmetric then $P^{\#}$ exists and $P^{\#}$ is a symmetric $\{1\}$-inverse of P. Exactly, $P^{\#}$ is equal to the Moore-Penrose inverse of P [2, 31].

The existence and the representation of the group inverse for block matrices with an invertible subblock were given by the authors Bu, Zhang and Zheng [4].

Lemma 2.1. *[4] Let* $P = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ *be an* $m \times m$ *matrix, where* A *is an invertible* $n \times n$ *matrix and* $S = D - CA^{-1}B$. *If* $S^{\#}$ *exists then*

1. $P^{\#}$ *exists if only if* R *is invertible, where* $R = A^2 + BS^{\pi}C$ *and* $S^{\pi} = I_{m-n} - SS^{\#}$;
2. *If* $P^{\#}$ *exists then*
   $P^{\#} = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$,
   where
   $X = AR^{-1}\left(A + BS^{\#}C\right)R^{-1}A$,
   $Y = AR^{-1}\left(A + BS^{\#}C\right)R^{-1}BS^{\pi} - AR^{-1}BS^{\#}$,
   $Z = S^{\pi}CR^{-1}\left(A + BS^{\#}C\right)R^{-1}A - S^{\#}CR^{-1}A$,
   $W = S^{\pi}CR^{-1}\left(A + BS^{\#}C\right)R^{-1}BS^{\pi} - S^{\#}CR^{-1}BS^{\pi} - S^{\pi}CR^{-1}BS^{\#} + S^{\#}$.

Lemma 2.2. *[1, 2, 3] Let* $L = \begin{pmatrix} L_1 & L_2 \\ L_2^T & L_3 \end{pmatrix}$ *be the Laplacian matrix of a connected graph, where* L_1 *is non-singular. Denote* $S = L_3 - L_2^T L_1^{-1} L_2$. *Then*

1. $\begin{pmatrix} L_1^{-1} + L_1^{-1} L_2 S^{\#} L_2^T L_1^{-1} & -L_1^{-1} L_2 S^{\#} \\ -S^{\#} L_2^T L_1^{-1} & S^{\#} \end{pmatrix}$ *is a symmetric* $\{1\}$*-inverse of* L;

2. *If each column vector of* L_2 *is* **-1** *or a zero vector,* $\begin{pmatrix} L_1^{-1} & 0 \\ 0 & S^{\#} \end{pmatrix}$ *is a symmetric* $\{1\}$*-inverse of* L.

The next lemma is useful for computing the inverse of non-singular matrices.

Lemma 2.3. *[2] Let* $P = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ *be a non-singular matrix. If* A *and* D *are non-singular then*

$$P^{-1} = \begin{pmatrix} A^{-1} + A^{-1} B S^{-1} C A^{-1} & -A^{-1} B S^{-1} \\ -S^{-1} C A^{-1} & S^{-1} \end{pmatrix},$$

where $S = D - CA^{-1}B$ *is the Schur complement of* A *in* P.

Lemma 2.4. *[33]. If* A *and* $A + B$ *are invertible and* B *has rank 1 then let* $g = trace\left(BA^{-1}\right)$. *If* $g \neq -1$ *then*

$$(A + B)^{-1} = A^{-1} - \tfrac{1}{1+g} A^{-1} B A^{-1}.$$

Lemma 2.5. *[2]. Let* G *be a connected graph and* (A_{ij}) *be the* (i, j)*-entry of a matrix* A. *Then, for all* $1 \leq i, j \leq n$,

$$r_{ij}(G) = \left(L(G)^{\#}\right)_{ii} - 2\left(L(G)^{\#}\right)_{ij} + \left(L(G)^{\#}\right)_{jj}$$

Lemma 2.6. *[2, 39]. Let* L *be the Laplacian matrix of a graph of order* n. *For any* $a, b > 0$ *satisfying* $b \neq n$, *we have*

1. $\left(L + aI_n - \frac{a}{n} J_{n\times n}\right)^{\#} = (L + aI)^{-1} - \frac{1}{an} J_{n\times n}$

2. $\left(L + aI_n - \frac{a}{b} J_{n\times n}\right)^{\#} = (L + aI)^{-1} - \frac{1}{a(b-n)} J_{n\times n}$

3. Resistance distance of generalized wheel graph

In 1988, Fred Buckley and Frank Harary [10] defined the generalized wheel graph $W_{m,n}$ as the join $\overline{K_m}\nabla C_n,\ m \geq 2,\ n \geq 3$, where, $\overline{K_m}$ is an empty graph on m vertices and C_n is the cycle graph on n vertices.

Now, we give the $\{1\}$-inverse representation of the Laplacian matrix of generalized wheel graph $\overline{K_m}\nabla C_n,\ m \geq 2,\ n \geq 3$.

Theorem 3.1. *Let $\overline{K_m}$ be an empty graph on $m \geq 2$ vertices and C_n, $n \geq 3$ be the cycle graph on n vertices. Then the symmetric $\{1\}$-inverse of $L(W_{m,n})$ is*

$$L^{\{1\}}(W_{m,n}) = \left[\begin{array}{c|c} \frac{1}{n}I_{m\times m} & 0_{m\times n} \\ \hline 0_{n\times m} & S^{\#}_{n\times n} \end{array}\right]$$

where $S^{\#}_{n\times n} = [L(C_n) + mI_n]^{-1} - \frac{1}{mn}J_{n\times n}$.

Proof. Let $\overline{K_m}$ be an empty graph of order $m, m \geq 2$ and $C_n, n \geq 3$ be the cycle graph (2-regular) of order n and the generalized wheel graph $W_{m,n} = \overline{K_m}\nabla C_n,\ m \geq 2,\ n \geq 3$. Clearly, the diameter of $W_{m,n}$ is two.

Let $V\left(\overline{K_m}\right) = \{u_1, u_2, \ldots, u_m\}$ and $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ be the vertex sets of graphs $\overline{K_m}$ and C_n, respectively.

Consider the labelled vertices of $W_{m,n}$, such that the first m vertices are from $\overline{K_m}, m \geq 2$ and n vertices are from $C_n, n \geq 3$

For all $u_i \in V\left(\overline{K_m}\right), i = 1, 2, \ldots, m$ in $W_{m,n}$, we have $deg_G(u_i) = n$ and

For all $v_i \in V(C_n), i = 1, 2, \ldots, n$ in $W_{m,n}$, we have $deg_G(v_i) = m + 2$

The Laplacian matrix of $W_{m,n}$ is

$$L(W_{m,n}) = \left[\begin{array}{c|c} L_1 & L_2 \\ \hline L_2^T & L_3 \end{array}\right] = \left[\begin{array}{c|c} nI_{m\times m} & -J_{m\times n} \\ \hline -J_{n\times m} & L\left(C_n\right) + mI_{n\times n} \end{array}\right]$$

where $J_{m\times n}$ is an all ones matrix, I_n is the identity matrix of order n, $L\left(C_n\right)$ is the Laplacian matrix of C_n.

We observe that to obtain the symmetric $\{1\}$-inverse of $L\left(W_{m,n}\right)$, we can use part 2 of Lemma 2.2 and we have

$$L_1^{-1} = \tfrac{1}{n} I_{m\times m}$$

Then

$$\begin{aligned} S &= L_3 - L_2^T L_1^{-1} L_2 \\ &= L\left(C_n\right) + mI_{n\times n} - \left(-J_{n\times m}\right) \tfrac{1}{n} I_{m\times m} \left(-J_{m\times n}\right) \\ &= L\left(C_n\right) + mI_{n\times n} - \tfrac{1}{n} J_{n\times m} I_{m\times m} J_{m\times n} \\ &= L\left(C_n\right) + mI_{n\times n} - \tfrac{1}{n}\left(mJ_{n\times n}\right) \\ &= L\left(C_n\right) + mI_{n\times n} - \tfrac{m}{n} J_{n\times n} \end{aligned}$$

Using part 1 of Lemma 2.6, for a graph G of order n, where $a = m$, we have

$$S^{\#} = \left[L\left(C_n\right) + mI_{n\times n}\right]^{-1} - \tfrac{1}{mn} J_{n\times n}$$

Therefore, the symmetric $\{1\}$-inverse of $L\left(W_{m,n}\right)$ is

$$L^{\{1\}}\left(W_{m,n}\right) = \left[\begin{array}{c|c} \tfrac{1}{n} I_{m\times m} & 0_{m\times n} \\ \hline 0_{n\times m} & S^{\#}_{n\times n} \end{array}\right]$$

where $S^{\#}_{n\times n} = \left[L\left(C_n\right) + mI_{n\times n}\right]^{-1} - \tfrac{1}{mn} J_{n\times n}$.

Hence the result. $\square$

Corollary 3.2. *The symmetric $\{1\}$-inverse of wheel graph $L\left(W_{1,n}\right)$ is*

$$L^{\{1\}}\left(W_{1,n}\right) = \left[\begin{array}{c|c} \tfrac{1}{n} I_{1\times 1} & 0_{1\times n} \\ \hline 0_{n\times 1} & S^{\#}_{n\times n} \end{array}\right]$$

where $S^{\#}_{n\times n} = \left[L\left(C_n\right) + I_{n\times n}\right]^{-1} - \tfrac{1}{n} J_{n\times n}$.

Proof. In Theorem 3.1, substituting $m = 1$, we obtain the symmetric $\{1\}$-inverse of wheel graph $L(W_{1,n}) = L\left(\overline{K_1} \nabla C_n\right), n \geq 3$. Hence the result. $\square$

Using the elements of the symmetric $\{1\}$-inverse of $L(W_{m,n})$ and $L(W_{1,n})$ in Lemma 2.5, we can obtain the resistance distance between any two vertices of the generalized wheel graph $W_{m,n}$ and wheel graph $W_{1,n}$ respectively.

Example 3.3. *Consider the generalized wheel graph $W_{2,3}$. Refer Figure 1.*

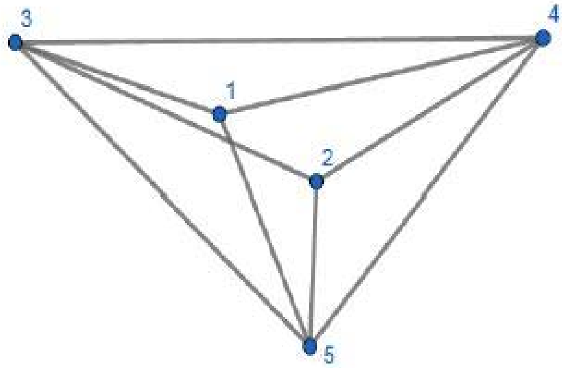

Figure 1. $W_{2,3}$

The Laplacian matrix of $W_{2,3}$ is

$$L(W_{2,3}) = \left[\begin{array}{cc|ccc} 3 & 0 & -1 & -1 & -1 \\ 0 & 3 & -1 & -1 & -1 \\ \hline -1 & -1 & 4 & -1 & -1 \\ -1 & -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & -1 & 4 \end{array}\right]$$

Using Theorem 3.1, we obtain the symmetric $\{1\}$-inverse of $L(W_{2,3})$ as

$$L^{\#}(W_{2,3}) = \left[\begin{array}{cc|ccc} \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 \\ \hline 0 & 0 & \frac{2}{15} & \frac{-1}{15} & \frac{-1}{15} \\ 0 & 0 & \frac{-1}{15} & \frac{2}{15} & \frac{-1}{15} \\ 0 & 0 & \frac{-1}{15} & \frac{-1}{15} & \frac{2}{15} \end{array}\right]$$

Using the symmetric $\{1\}$-inverse of $L(W_{2,3})$, we can find the resistance distance between any pair of vertices of $W_{2,3}$.
For example, the resistance distance between the vertices 1 and 4 is

$$r_{14} = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 0 \\ \\ 0 & \frac{2}{15} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{7}{15}.$$

Similarly, the resistance distances between all pairs of vertices of $W_{2,3}$ can be obtained.

The Kirchhoff index for $W_{2,3}$ is as follows

$$K_f(W_{2,3}) = \Sigma_{i<j} \quad r_{ij} = 4.66$$

4. Resistance distance of dumbbell graph

We now formulate the resistance distance for the dumbbell graph.

Definition 4.1. *Given $m \geq 2$ and $n \geq 3$, the dumbbell graph, $\mathbf{DB}(W_{m,n})$, is obtained from two copies of generalized wheel graph $W_{m,n}$ by connecting m vertices in one copy with the corresponding vertices in the other copy.*

The diameter of the $\mathbf{DB}(W_{m,n})$ graph is three.

In 2017, the name "dumbbell graph" was used by Bojana Borovicanin et al.[18] for another class of graphs. They have defined the dumbbell graph as a graph obtained by connecting two cycles by paths. This is different from the dumbbell graph introduced by us.

Theorem 4.2. *Let $\mathbf{DB}(W_{m,n})$ on $2(m+n)$ vertices be the dumbbell graph. Then the symmetric $\{1\}$-inverse of $L(\mathbf{DB}(W_{m,n}))$ is*

$$L^{\#}[\mathbf{DB}(W_{m,n})] = \begin{pmatrix} L_1^{-1} + L_1^{-1}L_2S^{\#}L_2^TL_1^{-1} & -L_1^{-1}L_2S^{\#} \\ -S^{\#}L_2^TL_1^{-1} & S^{\#} \end{pmatrix}$$

where

$$L_1^{-1} + L_1^{-1}L_2S^{\#}L_2^TL_1^{-1} =$$

$$\left[\begin{array}{c|c} \frac{n+1}{n(n+2)} I_{m\times m} + \frac{\left(2mn+m^2\right)\left(n^2+n-1\right)+n^2(n+1)^2}{mn(n+2)(n+m)^2} J_{m\times m} & \frac{(n+m)^2+n}{m(n+m)^2} J_{m\times n} \\ \hline \frac{(n+m)^2+n}{m(n+m)^2} J_{n\times m} & \left[L\left(C_n\right) + mI_{n\times n}\right]^{-1} + \frac{(n+m)^2+n}{m(n+m)^2} J_{n\times n} \end{array}\right]$$

$$-L_1^{-1} L_2 S^{\#} = \left[\begin{array}{c|c} \frac{1}{n(n+2)} I_{m\times m} - \frac{(2n+m)m-n^2(n+1)}{mn(n+2)(n+m)^2} J_{m\times m} & -\frac{1}{(n+m)^2} J_{m\times n} \\ \hline \frac{n}{m(n+m)^2} J_{n\times m} & -\frac{1}{(n+m)^2} J_{n\times n} \end{array}\right]$$

$$-S^{\#} L_2^T L_1^{-1} = \left[\begin{array}{c|c} \frac{1}{n(n+2)} I_{m\times m} - \frac{(2n+m)m-n^2(n+1)}{mn(n+2)(n+m)^2} J_{m\times m} & \frac{n}{m(n+m)^2} J_{m\times n} \\ \hline -\frac{1}{(n+m)^2} J_{n\times m} & -\frac{1}{(n+m)^2} J_{n\times n} \end{array}\right]$$

$$S^{\#} =$$

$$\left[\begin{array}{c|c} \frac{n+1}{n(n+2)} I_{m\times m} - \frac{\left[\left(2mn+m^2\right)(n+1)\right]-n^2}{mn(n+2)(n+m)^2} J_{m\times m} & -\frac{1}{(n+m)^2} J_{m\times n} \\ \hline -\frac{1}{(n+m)^2} J_{n\times m} & \left[L\left(C_n\right) + mI_{n\times n}\right]^{-1} - \frac{2m+n}{m(n+m)^2} J_{n\times n} \end{array}\right]$$

Proof. The Laplacian matrix of dumbbell graph $\mathbf{DB}(W_{m,n})$, $m \geq 2, n \geq 3$, on $2(m+n)$ vertices is

$$L\left(\mathbf{DB}\left(W_{m,n}\right)\right) = \left[\begin{array}{c|c} L_1 & L_2 \\ \hline L_2^T & L_3 \end{array}\right] =$$

$$\left[\begin{array}{cc|cc} (n+1) I_{m\times m} & -J_{m\times n} & -I_{m\times m} & 0_{m\times n} \\ -J_{n\times m} & L\left(C_n\right) + mI_{n\times n} & 0_{n\times m} & 0_{n\times n} \\ \hline -I_{m\times m} & 0_{m\times n} & (n+1) I_{m\times m} & -J_{m\times n} \\ 0_{n\times m} & 0_{n\times n} & -J_{n\times m} & L\left(C_n\right) + mI_{n\times n} \end{array}\right]$$

Step 1. We start with the calculation of L_1^{-1} of $L\left(\mathbf{DB}\left(W_{m,n}\right)\right)$.

Let

$$L_1 = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|c} (n+1)\, I_{m\times m} & -J_{m\times n} \\ \hline -J_{n\times m} & L\,(C_n) + mI_{n\times n} \end{array}\right]$$

Since L_1 is non-singular, L_1^{-1} exists. By applying Lemma 2.3,

$$L_1^{-1} = \left[\begin{array}{c|c} A^{-1} + A^{-1}BS_1^{-1}CA^{-1} & -A^{-1}BS_1^{-1} \\ \hline -S_1^{-1}CA^{-1} & S_1^{-1} \end{array}\right]$$

Then

$$\begin{aligned} S_1 &= D - CA^{-1}B \\ &= L\,(C_n) + mI_{n\times n} - (-J_{n\times m})\,\tfrac{1}{(n+1)} I_{m\times m}\,(-J_{m\times n}) \\ &= L\,(C_n) + mI_{n\times n} - \tfrac{1}{(n+1)}\,(J_{n\times m})\, I_{m\times m}\,(J_{m\times n}) \\ &= L\,(C_n) + mI_{n\times n} - \tfrac{1}{(n+1)}\,(mJ_{n\times n}) \\ &= L\,(C_n) + mI_{n\times n} - \tfrac{m}{(n+1)} J_{n\times n} \end{aligned}$$

Using part 2 of Lemma 2.6, for a graph G of order n, where $a = m$, $b = n+1$, we have

$$S_1^{-1} = [L\,(C_n) + mI_{n\times n}]^{-1} + \frac{1}{m} J_{n\times n} \tag{4.1}$$

Next

$$\begin{aligned} -A^{-1}BS_1^{-1} &= -\left[\tfrac{1}{(n+1)} I_{m\times m}\right](-J_{m\times n})\left[[L\,(C_n) + mI_{n\times n}]^{-1} + \tfrac{1}{m} J_{n\times n}\right] \\ &= \tfrac{1}{(n+1)}\left[(J_{m\times n})\,[L\,(C_n) + mI_{n\times n}]^{-1} + \tfrac{1}{m} J_{m\times n}J_{n\times n}\right] \\ &= \tfrac{1}{(n+1)}\left[\tfrac{1}{m} J_{m\times n} + \tfrac{1}{m}\,(nJ_{m\times n})\right] \\ &= \tfrac{1}{(n+1)m}\,(n+1)\,J_{m\times n} \\ &= \tfrac{1}{m} J_{m\times n} \end{aligned}$$

Therefore

$$-A^{-1}BS_1^{-1} = \frac{1}{m} J_{m\times n} \tag{4.2}$$

Similarly,

$$-S_1^{-1}CA^{-1} = \frac{1}{m} J_{n\times m} \tag{4.3}$$

$$A^{-1} + A^{-1}BS_1^{-1}CA^{-1} = \frac{1}{n+1} I_{m\times m} + \frac{n}{m\,(n+1)} J_{m\times m} \tag{4.4}$$

From equations (4.1), (4.2), (4.3), (4.4), we have

$$L_1^{-1} = \begin{bmatrix} \frac{1}{n+1} I_{m\times m} + \frac{n}{m(n+1)} J_{m\times m} & \frac{1}{m} J_{m\times n} \\ \frac{1}{m} J_{n\times m} & [L(C_n) + mI_{n\times n}]^{-1} + \frac{1}{m} J_{n\times n} \end{bmatrix}$$

Step 2. Next, we obtain S using L_1^{-1}, L_2 and L_3 as

$$S = L_3 - L_2^T L_1^{-1} L_2$$

$$S = \begin{bmatrix} & \frac{n(n+2)}{n+1} I_{m\times m} - \frac{n}{m(n+1)} J_{m\times m} \\ -J_{m\times n} & \\ & -J_{n\times m} \\ L(C_n) + mI_{n\times n} & \end{bmatrix}$$

Step 3. We calculate $S^{\#}$ using Lemma 2.1.

Let

$$S = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|c} \frac{n(n+2)}{n+1} I_{m\times m} - \frac{n}{m(n+1)} J_{m\times m} & -J_{m\times n} \\ \hline -J_{n\times m} & L(C_n) + mI_{n\times n} \end{array}\right]$$

We get A^{-1} using Lemma 2.4.

$$A^{-1} = \frac{n+1}{n(n+2)} I_{m\times m} + \frac{1}{nm(n+2)} J_{m\times m}$$

Then we follow the procedure from Lemma 2.1 to obtain

$$\begin{aligned} S_0 &= D - CA^{-1}B \\ &= [L(C_n) + mI_{n\times n}] - (-J_{n\times m}) \left[\frac{n+1}{n(n+2)} I_{m\times m} + \frac{1}{nm(n+2)} J_{m\times m}\right] (-J_{m\times n}) \\ &= [L(C_n) + mI_{n\times n}] - \frac{m}{n} J_{n\times n} \end{aligned}$$

Using part 1 of Lemma 2.6, for a graph G of order n, where $a = m$ to obtain

$$S_0^{\#} = [L(C_n) + mI_{n\times n}]^{-1} - \frac{1}{mn} J_{n\times n}$$

By continuing the procedure using Lemma 2.1, we have S_0^{π}.

We know, $S_0^{\pi} = I_n - S_0 S_0^{\#}$. Then we get,

$$S_0^{\pi} = \frac{1}{n} J_{n\times n}$$

Next, we calculate $R = A^2 + BS_0^{\pi}C$

$$R = \frac{n^2(n+2)^2}{(n+1)^2} I_{m\times m} + \frac{mn(n+1)^2 - n^2(2n+3)}{m(n+1)^2} J_{m\times m}$$

Using Lemma 2.4 we get R^{-1} as

$$R^{-1} = \frac{(n+1)^2}{n^2(n+2)^2} I_{m\times m} - \frac{m(n+1)^2 - n(2n+3)}{mn^2(n+2)^2(n+m)} J_{m\times m}$$

We continue the process to obtain

$$X = AR^{-1}\left(A + BS_0^{\#}C\right)R^{-1}A$$

$$= \frac{n+1}{n(n+2)} I_{m\times m} - \frac{(2mn+m^2)(n+1) - n^2}{mn(n+2)(n+m)^2} J_{m\times m} \tag{4.5}$$

Also

$$Y = AR^{-1}\left(A + BS_0^{\#}C\right)R^{-1}BS_0^{\pi} - AR^{-1}BS_0^{\#}$$

$$= -\frac{1}{(n+m)^2} J_{m\times n} \tag{4.6}$$

We have

$$Z = S_0^{\pi}CR^{-1}\left(A + BS_0^{\#}C\right)R^{-1}A - S_0^{\#}CR^{-1}A$$

$$= -\frac{1}{(n+m)^2} J_{n\times m} \tag{4.7}$$

and

$$W = S_0^{\pi}CR^{-1}\left(A + BS_0^{\#}C\right)R^{-1}BS_0^{\pi} - S_0^{\#}CR^{-1}BS_0^{\pi} - S_0^{\pi}CR^{-1}BS_0^{\#} + S_0^{\#}$$

$$= \left[L(C_n) + mI_{n\times n}\right]^{-1} - \frac{2m+n}{m(n+m)^2} J_{n\times n} \tag{4.8}$$

From equations (4.5), (4.6), (4.7), (4.8), we have

$$S^{\#} = \begin{bmatrix} \frac{n+1}{n(n+2)} I_{m\times m} - \frac{(2mn+m^2)(n+1)-n^2}{mn(n+2)(n+m)^2} J_{m\times m} & -\frac{1}{(n+m)^2} J_{m\times n} \\ -\frac{1}{(n+m)^2} J_{n\times m} & \left[L(C_n) + mI_n\right]^{-1} - \frac{2m+n}{m(n+m)^2} J_{n\times n} \end{bmatrix}$$

Step 4. Using part 1 of Lemma 2.2, we obtain the symmetric $\{1\}$-inverse of L.

The symmetric $\{1\}$-inverse of $L\left(\mathbf{DB}\left(W_{m,n}\right)\right)$ is

$$L^{\#}\left[\mathbf{DB}\left(W_{m,n}\right)\right] = \begin{pmatrix} L_1^{-1} + L_1^{-1}L_2S^{\#}L_2^TL_1^{-1} & -L_1^{-1}L_2S^{\#} \\ -S^{\#}L_2^TL_1^{-1} & S^{\#} \end{pmatrix}$$

where

$$L_1^{-1} + L_1^{-1}L_2S^{\#}L_2^TL_1^{-1} =$$

$$\begin{bmatrix} \frac{n+1}{n(n+2)}I_{m\times m} + \frac{\left(2mn+m^2\right)\left(n^2+n-1\right)+n^2(n+1)^2}{mn(n+2)(n+m)^2}J_{m\times m} & \frac{(n+m)^2+n}{m(n+m)^2}J_{m\times n} \\ \frac{(n+m)^2+n}{m(n+m)^2}J_{n\times m} & \left[L\left(C_n\right)+mI_n\right]^{-1} + \frac{(n+m)^2+n}{m(n+m)^2}J_{n\times n} \end{bmatrix}$$

$$-L_1^{-1}L_2S^{\#} = \begin{bmatrix} \frac{1}{n(n+2)}I_{m\times m} - \frac{(2n+m)m-n^2(n+1)}{mn(n+2)(n+m)^2}J_{m\times m} & -\frac{1}{(n+m)^2}J_{m\times n} \\ \frac{n}{m(n+m)^2}J_{n\times m} & -\frac{1}{(n+m)^2}J_{n\times n} \end{bmatrix}$$

$$-S^{\#}L_2^TL_1^{-1} = \begin{bmatrix} \frac{1}{n(n+2)}I_{m\times m} - \frac{(2n+m)m-n^2(n+1)}{mn(n+2)(n+m)^2}J_{m\times m} & \frac{n}{m(n+m)^2}J_{m\times n} \\ -\frac{1}{(n+m)^2}J_{n\times m} & -\frac{1}{(n+m)^2}J_{n\times n} \end{bmatrix}$$

$$S^{\#} = \begin{bmatrix} \frac{n+1}{n(n+2)}I_{m\times m} - \frac{\left[\left(2mn+m^2\right)(n+1)\right]-n^2}{mn(n+2)(n+m)^2}J_{m\times m} & -\frac{1}{(n+m)^2}J_{m\times n} \\ -\frac{1}{(n+m)^2}J_{n\times m} & \left[L\left(C_n\right)+mI_n\right]^{-1} - \frac{2m+n}{m(n+m)^2}J_{n\times n} \end{bmatrix}$$

Hence the result. □

Using the elements of the symmetric $\{1\}$-inverse of $L\left(\mathbf{DB}\left(W_{m,n}\right)\right)$, we can obtain the resistance distance between the vertices v_i and v_j in $\mathbf{DB}\left(W_{m,n}\right)$ as

$$r_{ij} = \left(L^{\#}\left(\mathbf{DB}\left(W_{m,n}\right)\right)\right)_{ii} - 2\left(L^{\#}\left(\mathbf{DB}\left(W_{m,n}\right)\right)\right)_{ij} + \left(L^{\#}\left(\mathbf{DB}\left(W_{m,n}\right)\right)\right)_{jj}.$$

Example 4.3. *Consider the dumbbell graph* $\mathbf{DB}\left(W_{2,3}\right)$*, given in Figure 2.*

The Laplacian matrix of $\mathbf{DB}\left(W_{2,3}\right)$ is

$$L = \left[\begin{array}{ccccc|ccccc}
4 & 0 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 \\
0 & 4 & -1 & -1 & -1 & 0 & -1 & 0 & 0 & 0 \\
-1 & -1 & 4 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & -1 & 4 & -1 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & -1 & -1 & 4 & 0 & 0 & 0 & 0 & 0 \\
\hline
-1 & 0 & 0 & 0 & 0 & 4 & 0 & -1 & -1 & -1 \\
0 & -1 & 0 & 0 & 0 & 0 & 4 & -1 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 & -1 & -1 & 4 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 4 & -1 \\
0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & 4
\end{array}\right].$$

Using Theorem 4.2, we obtain the symmetric $\{1\}$-inverse of $L\left(\mathbf{DB}\left(W_{2,3}\right)\right)$ as

$$L^{\#}\left(\mathbf{DB}\left(W_{2,3}\right)\right) = \left[\begin{array}{ccccc|ccccc}
\frac{52}{75} & \frac{32}{75} & \frac{14}{25} & \frac{14}{25} & \frac{14}{25} & \frac{7}{75} & \frac{2}{75} & \frac{-1}{25} & \frac{-1}{25} & \frac{-1}{25} \\
\frac{32}{75} & \frac{52}{75} & \frac{14}{25} & \frac{14}{25} & \frac{14}{25} & \frac{2}{75} & \frac{7}{75} & \frac{-1}{25} & \frac{-1}{25} & \frac{-1}{25} \\
\frac{14}{25} & \frac{14}{25} & \frac{43}{50} & \frac{33}{50} & \frac{33}{50} & \frac{3}{50} & \frac{3}{50} & \frac{-1}{25} & \frac{-1}{25} & \frac{-1}{25} \\
\frac{14}{25} & \frac{14}{25} & \frac{33}{50} & \frac{43}{50} & \frac{33}{50} & \frac{3}{50} & \frac{3}{50} & \frac{-1}{25} & \frac{-1}{25} & \frac{-1}{25} \\
\frac{14}{25} & \frac{14}{25} & \frac{33}{50} & \frac{33}{50} & \frac{43}{50} & \frac{3}{50} & \frac{3}{50} & \frac{-1}{25} & \frac{-1}{25} & \frac{-1}{25} \\
\hline
\frac{7}{75} & \frac{2}{75} & \frac{3}{50} & \frac{3}{50} & \frac{3}{50} & \frac{29}{150} & \frac{-11}{150} & \frac{-1}{25} & \frac{-1}{25} & \frac{-1}{25} \\
\frac{2}{75} & \frac{7}{75} & \frac{3}{50} & \frac{3}{50} & \frac{3}{50} & \frac{-11}{150} & \frac{29}{150} & \frac{-1}{25} & \frac{-1}{25} & \frac{-1}{25} \\
\frac{-1}{25} & \frac{-1}{25} & \frac{-1}{25} & \frac{-1}{25} & \frac{-1}{25} & \frac{-1}{25} & \frac{-1}{25} & \frac{-1}{25} & \frac{-1}{25} & \frac{-1}{25} \\
\frac{-1}{25} & \frac{-1}{25} & \frac{-1}{25} & \frac{-1}{25} & \frac{-1}{25} & \frac{-1}{25} & \frac{-1}{25} & \frac{-1}{25} & \frac{-1}{25} & \frac{-1}{25} \\
\frac{-1}{25} & \frac{-1}{25} & \frac{-1}{25} & \frac{-1}{25} & \frac{-1}{25} & \frac{-1}{25} & \frac{-1}{25} & \frac{-1}{25} & \frac{-1}{25} & \frac{-1}{25}
\end{array}\right]$$

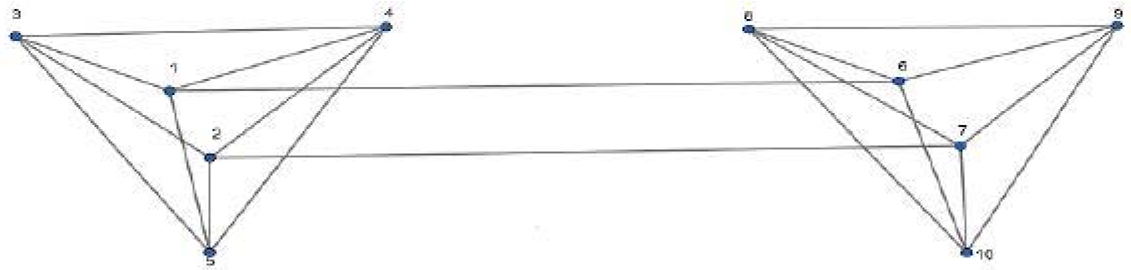

Figure 2. $\mathbf{DB}\,(W_{2,3})$

Using the symmetric $\{1\}$-inverse of $L\,(\mathbf{DB}\,(W_{2,3}))$, we can find the resistance distance between any pair of vertices of $\mathbf{DB}\,(W_{2,3})$.

For example, the resistance distance between the vertices 1 and 6 is

$$r_{16} = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} \frac{52}{75} & \frac{7}{75} \\ \\ \frac{7}{75} & \frac{29}{150} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{7}{10}.$$

Similarly, the resistance distances between all pairs of vertices of $\mathbf{DB}\,(W_{2,3})$ can be obtained.

The Kirchhoff index for $\mathbf{DB}\,(W_{2,3})$ is as follows

$$K_f\,(\mathbf{DB}\,(W_{2,3})) = \Sigma_{i<j} \quad r_{ij} = 32.8$$

5. Conclusion

The resistance distance which is the effective electrical resistance in a network has wide applications. Resistance distance is closely connected with practical applications in electrical circuit theory. For complex networks, resistance distance and Kirchhoff index are very important physical quantities. With these two quantities, the network topology can be optimized. For this reason, it has been widely explored by many authors. As we know, there exists a relationship between resistance distance and symmetric $\{1\}$-inverse of Laplacian matrix. In this paper, we have made use of the symmetric {1}-inverse of Laplacian matrix to obtain the resistance distance for two classes of graphs viz the generalized wheel and dumbbell graph. Making use of the resistance distance the Kirchhoff indices for these graphs have been computed.

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