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On weakly $(m, n)-$closed $\delta-$primary ideals of commutative rings

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Abstract

Let R be a commutative ring with $1 \neq 0$. In this article, we introduce the concept of weakly $(m, n)-$closed $\delta-$primary ideals of R and explore its basic properties. We show that a proper ideal I of R is a weakly $(m, n)-$closed $\gamma \circ \delta-$primary ideal of R if and only if I is an $(m, n)-$closed $\gamma \circ \delta-$primary ideal of R, where δ and γ are expansions ideals of R with $\delta(0)$ is an $(m, n)-$closed $\gamma-$primary ideal of R. Furthermore, we provide examples to demonstrate the validity and applicability of our results.

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1. Introduction

Throughout this article all rings are commutative with nonzero identity, all modules are unitary and all ring homomorphisms preserve the identity. Recently, ring theorists have been interested in the class of prime ideals and modules and its generalizations. The notion of weakly prime ideals was introduced by Anderson and Smith in [2] and further studied by several authors, (see for example [1, 10]). Badawi in [6] generalized the notion of prime ideals in a different way, called $2-$absorbing ideal. Many authors studied on this issue, (see for example [4, 7, 9, 13, 15, 17]). The notion of $\delta-$primary ideals in commutative rings was introduced by Zhao in [18]. This concept is considered to unify prime and primary ideals. Many results of prime and primary ideals are extended to these structures, (see for example [11], [16]). Our aim of this article is to introduce a new class of ideals that is closely to the class of weakly $n-$absorbing ideals, called weakly $(m,n)-$closed $\delta-$primary ideals. The motivation of this article is to complete what has been studied in [12] and to develop related results. The remains of this article is organized as follows. In section 2, we give some basic concepts and results that are indispensable in the sequel of this article. Section 3, is devoted to the main results. Section 4, concerns the conclusion.

2. Preliminaries

In this section, we state some basic concepts and results related to weakly $(m,n)-$closed $\delta-$primary ideals. We hope that this will improve the readability and understanding of this article. Let R be a commutative ring and I be an ideal of R. An ideal I is called proper if $I \neq R$. Let I be a proper ideal of R. Then, the radical of I is defined by $\{x \in R \mid \exists\, n \in \mathbf{N},\ x^n \in I\}$, denoted by $\sqrt{I}$ (note that $\sqrt{R} = R$ and $\sqrt{0}$ is the ideal of all nilpotent elements of R). For the ring R, we shall use $Nil(R)$, $J(R)$ and $char(R)$ to denote the set of all nilpotent elements of R, the set of all maximal ideals of R and the characteristic of R, respectively.

Definition 2.1. *([6], [8]) A proper ideal I of R is called a $2-$absorbing ideal (respectively, $2-$absorbing primary ideal) of R if whenever $abc \in I$ (respectively, $0 \neq abc \in I$) for some $a, b, c \in R$, implies $ab \in I$ or $ac \in I$ or $bc \in I$ (respectively, $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$).*

Definition 2.2. *[4] Let n be a positive integer. A proper ideal I of R is called an $n-$absorbing ideal (respectively, strongly $n-$absorbing ideal)* *of R if whenever $x_1...x_{n+1} \in I$ for some $x_1,...,x_{n+1} \in R$ (respectively, $I_1...I_{n+1} \subseteq I$ for some ideals $I_1,...,I_{n+1}$ of R), then there are n of the x_i 's (respectively, n of the I_i 's) whose product is in I.*

Thus, a $1-$absorbing ideal is just a prime ideal.

Definition 2.3. *[15] Let n be a positive integer. A proper ideal I of R is called a weakly $n-$absorbing ideal (respectively, strongly weakly $n-$absorbing ideal) of R if whenever $0 \neq x_1...x_{n+1} \in I$ for some $x_1,...,x_{n+1} \in R$ (respectively, $0 \neq I_1...I_{n+1} \subseteq I$ for some ideals $I_1,...,I_{n+1}$ of R), then there are n of the x_i 's (respectively, n of the I_i 's) whose product is in I.*

Thus, a weakly $1-$absorbing ideal is just a weakly prime ideal.

Definition 2.4. *[3] Let m and n be positive integers. A proper ideal I of R is called a semi$-n-$absorbing ideal of R if whenever $x^{n+1} \in I$ for some $x \in R$ implies $x^n \in I$. More generally, a proper ideal I of R is called an $(m,n)-$closed ideal of R if whenever $x^m \in I$ for some $x \in I$ implies $x^n \in I$.*

Definition 2.5. *[5] Let m and n be positive integers. A proper ideal I of R is called a weakly semi$-n-$absorbing ideal of R if whenever $0 \neq x^{n+1} \in I$ for some $x \in R$ implies $x^n \in I$. More generally, a proper ideal I of R is called a weakly $(m,n)-$closed ideal of R if whenever $0 \neq x^m \in I$ for some $x \in I$ implies $x^n \in I$.*

Definition 2.6. *[18] Let $Id(R)$ be the set of all ideals of R. A function $\delta : Id(R) \longrightarrow Id(R)$ is called an expansion function of $Id(R)$ if it has the following two properties: $I \subseteq \delta(I)$ and if $I \subseteq J$ for some ideals, I, J of R, then $\delta(I) \subseteq \delta(J)$. A proper ideal I of R is called a $\delta-$primary ideal of R if whenever $xy \in I$ for some $x,y \in R$ implies $x \in I$ or $y \in \delta(I)$.*

Definition 2.7. *[11] A proper ideal I of R is called a $2-$absorbing $\delta-$primary ideal of R if whenever $xyz \in I$ for some $x,y,z \in R$ implies $xy \in I$ or $yz \in \delta(I)$ or $xz \in \delta(I)$. A proper ideal I of R is called a strongly $2-$absorbing $\delta-$primary ideal of R if whenever I_1, I_2, I_3 are ideals of R, $I_1I_2I_3 \subseteq I$, $I_1I_3 \nsubseteq I$ and $I_2I_3 \nsubseteq \delta(I)$, then $I_1I_2 \subseteq \delta(I)$.*

The notions of $n-$absorbing $\delta-$primary ideals and weakly $n-$absorbing $\delta-$primary ideals are generalizations of the notions of $n-$absorbing primary ideals and weakly $n-$absorbing primary ideals respectively. Recall the following definition.

Definition 2.8. *[16] A proper ideal I of R is called an $n-$absorbing $\delta-$primary ideal (respectively, weakly $n-$absorbing $\delta-$primary ideal) of R if whenever $x_1...x_{n+1} \in I$ (respectively, $0 \neq x_1...x_{n+1} \in I$) for some $x_1, ..., x_{n+1} \in R$ implies $x_1...x_n \in I$ or there exists $1 \leq k < n$ such that $x_1...\hat{x}_k...x_{n+1} \in \delta(I)$, where $x_1...\hat{x}_k...x_{n+1}$ denotes the product of $x_1...x_{k-1}x_{k+1}...x_{n+1}$.*

The notion of $(m, n)-$closed $\delta-$primary ideals is a generalization of the notion of $n-$absorbing $\delta-$primary ideals. Recall the following definition.

Definition 2.9. *[12] Let R be a commutative ring, δ be an expansion function of $Id(R)$, and m and n be positive integers. A proper ideal I of R is called a semi$-n-$absorbing $\delta-$primary ideal of R if whenever $a^{n+1} \in I$ for some $a \in R$, then $a^n \in \delta(I)$. More generally, a proper ideal I of R is called an $(m, n)-$closed $\delta-$primary ideal of R if whenever $a^m \in I$ for some $a \in R$, then $a^n \in \delta(I)$.*

3. Main Results

We start by the following definition.

Definition 3.1. *Let R be a commutative ring, I be a proper ideal of R, δ be an expansion function of $Id(R)$, and m and n be positive integers.*
(1) I is called a weakly semi$-n-$absorbing $\delta-$primary ideal of R if whenever $0 \neq a^{n+1} \in I$ for some $a \in R$, then $a^n \in \delta(I)$.
(2) I is called a weakly $(m, n)-$closed $\delta-$primary ideal of R if whenever $0 \neq a^m \in I$ for some $a \in R$, then $a^n \in \delta(I)$.

Clearly, a proper ideal is weakly $(m, n)-$closed $\delta-$primary for $1 \leq m \leq n$; so we usually assume that $1 \leq n < m$.

Theorem 3.2. *Let R be a commutative ring, I be a proper ideal of R, δ be an expansion function of $Id(R)$, and m and n be positive integers. Then,*

(1) I is a weakly semi$-n-$absorbing $\delta-$primary ideal of R if and only if I is a weakly $(n + 1, n)-$closed $\delta-$primary ideal of R.

(2) If I is a weakly $n-$absorbing $\delta-$primary ideal of R, then I is a weakly semi$-n-$absorbing $\delta-$primary ideal of R.

(3) If I is a weakly $(m, n)-$closed $\delta-$primary ideal of R, then I is a weakly $(m, k)-$closed $\delta-$primary ideal of R for every positive integer $k \geq n$.

(4) A weakly $n-$absorbing ideal of R is a weakly $(m, n)-$closed $\delta-$primary ideal of R for every positive integer m.

Proof. Follows directly from the definitions. □

In the following example, we give some expansion functions of ideals of a ring R.

Example 3.3. *(1) The identity function δ_I, where $\delta_I(I) = I$ for every $I \in Id(R)$, is an expansion function of ideals of R.*

(2) For each ideal I, define $\delta_{\sqrt{I}}(I) = \sqrt{I}$. Then $\delta_{\sqrt{I}}$ is an expansion function of ideals of R.

Remark 3.4. *(1) Let R be a commutative ring, δ_I be an expansion function of $Id(R)$ and m and n be positive integers, then a proper ideal I of R is a weakly $(m, n)-$closed δ_I-primary ideal of R if and only if I is a weakly $(m, n)-$closed ideal of R.*

(2) It is clear that any $(m, n)-$closed $\delta-$primary ideal of R is weakly $(m, n)-$closed $\delta-$primary ideal. The converse need not hold in general.

(3) A weakly $(m, n)-$closed $\delta-$primary ideal does not be weakly $(\acute{m}, n)-$closed $\delta-$primary for $\acute{m} < n$.

Example 3.5. *Let $R = \mathbf{Z}_8$. Then $I = (0)$, the zero ideal is clearly a weakly $(3, 1)-$closed $\delta_{\sqrt{I}}-$primary ideal of $\mathbf{Z}_8$. However, I is not $(3, 1)-$closed $\delta_{\sqrt{I}}-$primary ideal of $\mathbf{Z}_8$ since $2^3 = 0 \in I$ and $2 \notin I$. More generally, $I = (0)$ is a weakly $(n+1, n)-$closed $\delta_{\sqrt{I}}-$primary ideal of $\mathbf{Z}_{2^{n+1}}$, by definition, but it is easy to see that I is not an $(n + 1, n)-$closed $\delta_{\sqrt{I}}-$primary ideal of $\mathbf{Z}_{2^{n+1}}$.*

Example 3.6. *Let $R = \mathbf{Z}_8$ and δ an expansion function of $Id(R)$ and $I = \{0, 4\}$. Then I is weakly $(3,1)-$closed $\delta-$primary ideal of $\mathbf{Z}_8$, since $r^3 = 0$ for every nonunit $r \in R$. However, I is not weakly $(2,1)-$closed $\delta-$primary ideal since $0 \neq 2^2 = 4 \in I$ and $2 \notin I$.*

In the following example, we have a weakly $(m,n)-$closed $\delta-$primary ideal of R, that is neither a weakly $(m,n)-$closed ideal of R nor an $(m,n)-$ closed $\delta-$primary ideal of R.

Example 3.7. *Let $A = \mathbf{Z}_{2^m}[\{X_m\}_{m\in\mathbf{N}}]$ and $I = (X_{m-1}^m A)$ be an ideal of A. Let $R = A/I$ and define $\delta : Id(R) \longrightarrow Id(R)$ such that $\delta(K) = K + (xA + I)/I$. It is clear that δ is an expansion function of $Id(R)$. Let $J = (X_m^m A)/I$. We show that J is not a weakly $(m,n)-$closed ideal of R. Now, in the ring R, we have $0 \neq X_m^m + I \in J$, but $X_m^n + I \notin J$ for every positive integers $1 \leq n < m$. Thus, J is not a weakly $(m,n)-$closed ideal of R. We show that J is not an $(m,n)-$closed $\delta-$primary ideal of R. Let $x = 2 + I \in R$. Then $x^m = 0 + I \in J$, but $x^n \notin \delta(J)$ for every positive integers $1 \leq n < m$. Thus, J is not an $(m,n)-$closed $\delta-$primary ideal of R. By the construction of δ, one can easily see that J is a weakly $(m,n)-$closed $\delta-$primary ideal of R.*

Theorem 3.8. *Let R be a commutative ring, δ and γ be expansion functions of $Id(R)$ with $\delta(I) \subseteq \gamma(I)$, m and n be integers with $1 \leq n < m$, and I be a proper ideal of R.*

(1) If I is a weakly $(m,n)-$closed $\delta-$primary ideal of R, then I is a weakly $(m,n)-$closed $\gamma-$primary ideal of R.

(2) If $\delta(I)$ is a weakly $(m,n)-$closed ideal of R, then I is a weakly $(m,n)-$closed $\delta-$primary ideal of R.

Proof.

(1) Since $\delta(I) \subseteq \gamma(I)$ and I is a weakly $(m,n)-$closed $\delta-$primary ideal of R, then the claim follows.

(2) Let $0 \neq a^m \in I \subseteq \delta(I)$ for some $a \in R$. Since $\delta(I)$ is a weakly $(m,n)-$closed ideal of R, then $a^n \in \delta(I)$. Therefore, I is a weakly $(m,n)-$closed $\delta-$primary ideal of R.

□

Let R be a commutative ring, δ_1 and δ_2 are expansion functions of $Id(R)$. Let $\delta : Id(R) \longrightarrow Id(R)$ such that $\delta(I) = (\delta_1 \circ \delta_2)(I) = \delta_1(\delta_2(I))$. Then, δ is an expansion function of ideals of R, such a function is denoted by $\delta_\circ$.

Theorem 3.9. *Let R be a commutative ring, δ and γ be expansion functions of $Id(R)$, m and n be positive integers, I be a proper ideal of R, and $\delta(0)$ be an $(m,n)-$closed $\gamma-$primary ideal of R. Then, I is a weakly $(m,n)-$closed $\gamma \circ \delta-$primary ideal of R if and only if I is an $(m,n)-$closed $\gamma \circ \delta-$primary ideal of R*

Proof. Assume that I is a weakly $(m,n)-$closed $\gamma \circ \delta-$primary ideal of R. Assume that $a^m \in I$ for some $a \in R$. If $0 \neq a^m \in I$, then $a^n \in \gamma(\delta(I))$. Hence, assume that $a^m = 0$. Since $\delta(0)$ is an $(m,n)-$closed $\gamma-$primary ideal of R, we conclude that $a^n \in \gamma(\delta(0))$. Since $\gamma(\delta(0)) \subseteq \gamma(\delta(I))$, we conclude that $a^n \in \gamma(\delta(0)) \subseteq \gamma(\delta(I))$. Therefore, I is an $(m,n)-$closed $\gamma \circ \delta-$primary ideal of R. The converse is clear. □

Theorem 3.10. *Let R be a commutative ring, δ and γ be expansion functions of $Id(R)$, m and n be positive integers, I be a proper ideal of R, and $\gamma(I)$ be a weakly $(m,n)-$closed $\delta-$primary ideal of R. Then, I is a weakly $(m,n)-$closed $\delta \circ \gamma-$primary ideal of R.*

Proof. Assume that $0 \neq a^m \in I \subseteq \gamma(I)$ for some $a \in R$. Since $\gamma(I)$ is a weakly $(m,n)-$closed $\delta-$primary ideal of R. Then, $a^n \in \delta(\gamma(I))$. Therefore, I is a weakly $(m,n)-$closed $\delta \circ \gamma-$primary ideal of R. □

Definition 3.11. *[16] Let $f : R \longrightarrow A$ be a ring homomorphism and δ, γ expansion functions of $Id(R)$ and $Id(A)$ respectively. Then, f is called a $\delta\gamma-$homomorphism if $\delta(f^{-1}(I)) = f^{-1}(\gamma(I))$ for all ideals I of A.*

Remark 3.12. (1) *If γ_r is a radical operation on a commutative ring A and δ_r is a radical operation on a commutative ring R, then any homomorphism from R to A is an example of $\delta_r\gamma_r-$homomorphism. Also, if f is a $\delta\gamma-$epimorphism and I is an ideal of R containing $Ker(f)$, then $\gamma(f(I)) = f(\delta(I))$. In particular, if f is a $\delta\gamma-$ring$-$isomorphism, then $f(\delta(I)) = \gamma(f(I))$ for every ideal I of R.*
(2) *Let R be a commutative ring, δ be an expansion function of $Id(R)$ and I be a proper ideal of R. The function $\delta_q : R/I \longrightarrow R/I$ defined by $\delta_q(J/I) = \delta(J)/I$ for ideals $I \subseteq J$, becomes an expansion function of R/I.*

Theorem 3.13. *Let R and A be commutative rings, m and n be positive integers, and $f : R \longrightarrow A$ be a $\delta\gamma-$homomorphism, where δ is an expansion function of $Id(R)$ and γ is an expansion function of $Id(A)$.*

(1) If f is injective and J is a weakly $(m, n)-$closed $\gamma-$primary ideal of A, then $f^{-1}(J)$ is a weakly $(m, n)-$closed $\delta-$primary ideal of R.

(2) If f is surjective and I is a weakly $(m, n)-$closed $\delta-$primary ideal of R containing $Ker(f)$, then $f(I)$ is a weakly $(m, n)-$closed $\gamma-$primary ideal of A.

Proof.

(1) Assume that J is a weakly $(m, n)-$closed $\gamma-$primary ideal of A and $0 \neq a^m \in f^{-1}(J)$ for some $a \in R$. Since, $Ker(f) \neq 0$, we get $0 \neq f(a^m) = [f(a)]^m \in J$. By our assumption, we conclude that $[f(a)]^n \in \gamma(J)$. Thus, $a^n \in f^{-1}(\gamma(J))$. Since $\delta(f^{-1}(J)) = f^{-1}(\gamma(J))$, we get $a^n \in \delta(f^{-1}(J))$. Therefore, $f^{-1}(J)$ is a weakly $(m, n)-$closed $\delta-$primary ideal of R.

(2) Assume that I is a weakly $(m, n)-$closed $\delta-$primary ideal of R and $0 \neq b^m \in f(I)$ for some $b \in A$. Since f is epimorphism, we have $f(a^m) = b^m$ for some $a \in R$ and $0 \neq f(a^m) = b^m \in f(I)$. Since $Ker(f) \subseteq I$, we have $0 \neq a^m \in I$. As I is an $(m, n)-$closed $\delta-$primary ideal of R, we have $a^n \in \delta(I)$. Then, we have $b^n \in f(\delta(I)) \subseteq \gamma(f(I))$. Therefore, $f(I)$ is a weakly $(m, n)-$closed $\gamma-$primary ideal of A.

$\square$

Theorem 3.14. *Let R be a commutative ring, δ be an expansion function of $Id(R)$, m and n be positive integers, and $I \subseteq J$ be proper ideals of R.*

(1) If J is a weakly $(m, n)-$closed $\delta-$primary ideal of R, then J/I is a weakly $(m, n)-$closed δ_q-primary ideal of R/I.

(2) If I is an $(m, n)-$closed $\delta-$primary ideal of R and J/I is a weakly $(m, n)-$closed δ_q-primary ideal of R/I, then J is an $(m, n)-$closed $\delta-$primary ideal of R.

(3) If I is a weakly $(m,n)-$closed $\delta-$primary ideal of R and J/I is a weakly $(m,n)-$closed δ_q-primary ideal of R/I, then J is a weakly $(m,n)-$closed $\delta-$primary ideal of R.

Proof.

(1) Follows directly from Theorem 3.13 (2).

(2) Let $a^m \in J$ for $a \in R$. If $a^m \in I$, then $a^n \in \delta(I) \subseteq \delta(J)$. Assume that $a^m \notin I$. Then, we have $I \neq (a+I)^m \in J/I$. Since J/I is a weakly $(m,n)-$closed δ_q-primary ideal of R/I. Then, $(a+I)^n = a^n + I \in \delta_q(J/I)$. Thus, $a^n \in \delta(J)$. Therefore, J is an $(m,n)-$closed $\delta-$primary ideal of R.

(3) Let $0 \neq a^m \in J$ for $a \in R$. Then by a similar argument as in part (2), one can show that J is a weakly $(m,n)-$closed $\delta-$primary ideal of R.

□

Example 3.15. *Let D be an integral domain, I be a proper ideal of D, m, n be positive integers with $1 \leq n < m$, and δ be an expansion function of $Id(D)$. Then $< I, X >$ is a weakly $(m,n)-$closed $\delta-$primary ideal of $R[x]$ if and only if I is a weakly $(m,n)-$closed $\delta-$primary ideal of R. This is hold by Theorem 3.14* (1), (2), *since $< I, X > / < x > \approx I$ in $R[x]/ < x > \approx R$.*

Theorem 3.16. *Let R be a commutative ring, I be a proper ideal of R, m and n be positive integers, $S \subseteq R\backslash\{0\}$ be a multiplicative set with $I \cap S = \phi$, and δ_S be an expansion function of $Id(R_S)$ such that $\delta_S(I_S) = (\delta(I))_S$ where δ is an expansion function of $Id(R)$. If I is a weakly $(m,n)-$closed $\delta-$primary ideal of R, then I_S is a weakly $(m,n)-$closed δ_S-primary ideal of R_S.*

Proof. Let $0 \neq a^m \in I_S$ for $a \in R_S$. Then, $a = \frac{r}{s}$ for some $r \in R$ and $s \in S$, and thus $0 \neq a^m = \frac{r^m}{s^m} = \frac{i}{g}$ for some $i \in I$ and $g \in S$. Hence, $0 \neq r^m gz = s^m iz \in I$ for some $z \in S$, and thus $0 \neq (rgz)^m \in I$. Hence, $(rgz)^n \in \delta(I)$ since I is weakly $(m,n)-$closed $\delta-$primary ideal of R, and thus $a^n = \frac{r^n}{s^n} = \frac{r^n g^n z^n}{s^n g^n z^n} \in I_S \subseteq (\delta(I))_S = \delta_S(I_S)$. Therefore, I_S is a weakly $(m,n)-$closed δ_S-primary ideal of R_S. □

Corollary 3.17. *Let R be a commutative ring, I be a proper ideal of R, m and n be positive integers, $P \subseteq R \backslash \{0\}$ be a multiplicative set with $I \cap P = \phi$, and δ_P be an expansion function of $Id(R_P)$ such that $\delta_P(I_P) = (\delta(I))_P$ where δ is an expansion function of $Id(R)$. Then the following statements are equivalent.*

(1) I is a weakly $(m, n)-$closed $\delta-$primary ideal of R.

(2) I_P is a weakly $(m, n)-$closed δ_P-primary ideal of R_P for every prime (or maximal) ideal P of R.

Proof. (1) $\Longrightarrow$ (2) Follows directly from Theorem 3.16.
(2) $\Longrightarrow$ (1) Let $0 \neq a^m \in I$ for $a \in R$, consider the ideal $J = \{r \in R : ra^n \in \delta(I)\}$ of R and P be a prime ideal of R with $I \subseteq P$. Then, $0 \neq (\frac{a}{1})^m \in I_P$. Thus, $(\frac{a}{1})^n \in \delta_P(I_P)$ since I_P is a weakly $(m, n)-$closed δ_P-primary ideal of R_P. Thus, $xa^n \in \delta(I)$ for some $x \in R \backslash P$. Thus, $J \nsubseteq P$. In fact $J \nsubseteq L$ for every prime ideal L of R with $I \nsubseteq L$. Hence, $J = R$ and thus $a^n \in \delta(I)$. Therefore, I is a weakly $(m, n)-$closed $\delta-$primary ideal of R. □

Recall from [16] that an expansion function δ of $Id(R)$ satisfies the finite intersection property if $\delta(I_1 \cap ... \cap I_n) = \delta(I_1) \cap ... \cap \delta(I_n)$ for some ideals $I_1, ..., I_n$ of the commutative ring R.
Note that the radical operation on ideals of a commutative ring is an example of an expansion function satisfying the finite intersection property.

Theorem 3.18. *Let R be a commutative ring, δ be an expansion function of $Id(R)$ satisfying the finite intersection property, m and n be positive integers, and $I_1, ..., I_k$ be proper ideals of R. If $I_1, ..., I_k$ are weakly $(m, n)-$closed $\delta-$primary ideal of R, and $P = \delta(I_j)$ for all $j \in \{1, ..., k\}$, then $I_1 \cap ... \cap I_k$ is a weakly $(m, n)-$closed $\delta-$primary ideal of R.*

Proof. It is clear. □

Definition 3.19. *Let R be a commutative ring, δ be an expansion function of $Id(R)$, m and n be integers with $1 \leq n < m$, and I be a weakly $(m, n)-$ closed $\delta-$primary ideal of R. Then $a \in R$ is called a $\delta - (m, n)-$unbreakable-zero element of I if $a^m = 0$ and $a^n \notin \delta(I)$.*

(Thus, I has a $\delta-(m,n)-$unbreakable-zero element if and only if I is not $(m,n)-$closed $\delta-$primary ideal of R).

Theorem 3.20. *Let R be a commutative ring, δ be an expansion function of $Id(R)$, m and n be integers with $1 \leq n < m$, and I be a weakly $(m,n)-$closed $\delta-$primary ideal of R. If x is a $\delta-(m,n)-$unbreakable-zero element of I, then $(x+i)^m = 0$ for every $i \in I$.*

Proof. Assume that $(x+i)^m \neq 0$ for some $i \in I$. Then by Binomial Theorem, we have $(x+i)^m = x^m + mx^{m-1}i + ... + i^m = 0 + mx^{m-1}i + ... + i^m \in I$. Since I is a weakly $(m,n)-$ closed $\delta-$primary ideal of R, then $(x+i)^n \in \delta(I)$, a contradiction. Therefore, $(x+i)^m = 0$ for every $i \in I$. □

Theorem 3.21. *Let R be a commutative ring, δ be an expansion function of $Id(R)$, m and n be positive integers, and I be a weakly $(m,n)-$closed $\delta-$primary ideal of R that is not $(m,n)-$ closed $\delta-$primary. Then $I \subseteq Nil(R)$. Furthermore, if $char(R) = m$ is prime, then $x^m = 0$ for every $x \in I$.*

Proof. Since I is a weakly $(m,n)-$ closed $\delta-$primary ideal of R that is not $(m,n)-$ closed $\delta-$primary, then I has a $\delta-(m,n)-$unbreakable-zero element say a. Let $x \in I$. Then, $a^m = 0$ and $(a+x)^m = 0$ by Theorem 3.20. Thus, a, $a+x \in Nil(R)$. Thus, $x = (a+x)-a \in Nil(R)$. Therefore, $I \subseteq Nil(R)$. Now, assume that $char(R) = m$ is prime. Then, it is clear that $0 = (a+x)^m = a^m + x^m = x^m$. Therefore, $x^m = 0$ for every $x \in I$. □

Lemma 3.22. *Let R be a commutative ring, δ be an expansion function of $Id(R)$, n be a positive integer and $a \in J(R)$. Then the ideal $I = a^{n+1}R$ is a weakly semi$-n-$absorbing $\delta-$primary ideal of R if and only if $a^{n+1} = 0$.*

Proof. Assume that $a^{n+1} = 0$. Then, $I = (0)$ and thus I is a weakly semi$-n-$absorbing $\delta-$primary ideal of R by definition.
Conversely; assume that I is a weakly semi$-n-$absorbing $\delta-$primary ideal of R and $a^{n+1} \neq 0$. Since $a^{n+1} \in I\backslash\{0\}$, then $a^n \in \delta(I)$. Set $x = a^n \in I$. Then, $x = xar$ for some $r \in R$. Thus $x(1-ar) = 0$. Since $ar \in J(R)$, then $1-ar$ is a unit of R. Thus, $x = 0$ and then $a^{n+1} = 0$, a contradiction. Therefore, $a^{n+1} = 0$. □

Theorem 3.23. *Let D be an integral domain, δ be an expansion function of $Id(D)$, $I = p^k D$ be a principal ideal of D, where p is a prime element of D and k a positive integer. Let m be a positive integer such that $m < k$, and write $k = ma + b$ for some integers a, b, where $a \geq 1$ and $0 \leq b < m$. If $I/p^c D$ is a weakly $(m, n)-$closed $\delta-$primary ideal of $D/p^c D$ that is not $(m, n)-$closed $\delta-$primary ideal for positive integers $n < m$ and $c \geq k + 1$, then $b \neq 0$, $k + 1 \leq c \leq m(a + 1)$ and $n(a + 1) < k$.*

Proof. Assume that $I/p^c D$ is a weakly $(m, n)-$closed $\delta-$primary ideal of $D/p^c D$ that is not $(m, n)-$closed $\delta-$primary for positive integers $n < m$ and $c \geq k+1$. It is clear that $b \neq 0$, for if $b = 0$, then $0 \neq (p^a)^m + p^c D \in I/p^c D$, but $(p^a)^n + p^c D \notin I/p^c D$. Since $a + 1$ is the smallest positive integer such that $(p^{(a+1)})^m + p^c D \in I/p^c D$ and $I/p^c D$ is not $(m, n)-$closed $\delta-$primary ideal of $D/p^c D$, we have $0 = (p^{(a+1)})^m + p^c D \in I/p^c D$ and $(p^{(a+1)})^n + p^c D \notin \delta(I/p^c D)$. Thus, $(p^{(a+1)})^n + p^c D \notin I/p^c D$ and therefore, $n(a + 1) < k$ and $k + 1 \leq c \leq m(a + 1)$. □

Recall that an ideal of $R_1 \times R_2$ has the form $I_1 \times I_2$ for some ideals I_1 of R_1 and I_2 of R_2, where R_1 and R_2 are commutative rings. Let $R = R_1 \times ... \times R_k$, where R_i is a commutative ring with nonzero identity and δ_i be an expansion function of $Id(R_i)$ for each $i \in \{1, 2, ..., k\}$. Let $\delta_\times$ be a function of $Id(R)$, which is defined by $\delta_\times(I_1 \times I_2 \times ... \times I_k) = \delta_1(I_1) \times \delta_2(I_2) \times ... \times \delta_k(I_k)$ for each ideal I_i of R_i were $i \in \{1, 2, ..., k\}$. Then $\delta_\times$ is an expansion function of $Id(R)$. Note that every ideal of R is of the form $I_1 \times I_2 \times ... \times I_k$, where each ideal I_i is an ideal of R_i, $i \in \{1, 2, ..., k\}$. In the next results, we characterize weakly $(m, n)-$closed $\delta-$primary ideals of $R_1 \times R_2$.

Theorem 3.24. *Let $R = R_1 \times R_2$, where R_1 and R_2 are commutative rings, δ_i be an expansion function of $Id(R_i)$ for each $i \in \{1, 2\}$, I_1 be a proper ideal of R_1, I_2 be a proper ideal of R_2, and m and n be positive integers with $1 \leq n < m$. Then the following statements are equivalent.*

(1) $I_1 \times R_2$ is a weakly $(m, n)-$closed $\delta_\times-$primary ideal of R.

(2) I_1 is an $(m, n)-$closed δ_1-primary ideal of R_1.

(3) $I_1 \times R_2$ is an $(m, n)-$closed $\delta_\times-$primary ideal of R.

Proof. (1) $\Longrightarrow$ (2) I_1 is a weakly $(m,n)-$closed δ_1-primary ideal of R_1 by Theorem 3.13 (2). If I_1 is not an $(m,n)-$closed δ_1-primary ideal of R_1, then I_1 has a $\delta_1-(m,n)-$unbreakable-zero element x. Thus, $(0,0) \neq (x,1)^m \in I_1 \times R_2$, but $(x,1)^n \notin \delta_\times(I_1 \times R_2)$, a contradiction. Hence, I_1 is an $(m,n)-$closed δ_1-primary ideal of R_1.
(2) $\Longrightarrow$ (3) It is clear. (Analogue to the proof of [3], Theorem 2.12).
(3) $\Longrightarrow$ (1) It is clear by definition. □

Theorem 3.25. *Let $R = R_1 \times R_2$, where where R_1 and R_2 are commutative rings, δ_i be an expansion function of $Id(R_i)$ for each $i \in \{1,2\}$, I be a proper ideal of R, and m and n be positive integers with $1 \leq n < m$. Then the following statements are equivalent.*

(1) I is a weakly $(m,n)-$closed $\delta_\times-$primary ideal of R that is not (m,n) $-$closed $\delta_\times-$primary.

(2) $I = J_1 \times J_2$ for some proper ideals J_1 of R_1 and J_2 of R_2 such that either
(a) J_1 is a weakly $(m,n)-$closed δ_1-primary ideal of R_1 that is not $(m,n)-$closed δ_1-primary, $y^m = 0$ whenever $y^m \in J_2$ for some $y \in R_2$ (in particular, $j^m = 0 \ \forall\ j \in J_2$), and if $0 \neq x^m \in J_1$ for some $x \in R_1$, then J_2 is an $(m,n)-$closed δ_2-primary ideal of R_2, or
(b) J_2 is a weakly $(m,n)-$closed δ_2-primary ideal of R_2 that is not $(m,n)-$closed δ_2-primary, $y^m = 0$ whenever $y^m \in J_1$ for some $y \in R_1$ (in particular, $j^m = 0 \ \forall\ j \in J_1$), and if $0 \neq x^m \in J_2$ for some $x \in R_2$, then J_1 is an $(m,n)-$closed δ_1-primary ideal of R_1.

Proof. (1) $\Longrightarrow$ (2) Since I is not an $(m,n)-$closed $\delta_\times-$primary ideal of R. Then, by Theorem 3.24, we have $I = J_1 \times J_2$, where J_1 is a proper ideal of R_1 and J_2 is a proper ideal of R_2. Since I is not an $(m,n)-$closed $\delta_\times-$primary ideal of R, we have either J_1 is a weakly $(m,n)-$closed δ_1-primary ideal of R_1 that is not $(m,n)-$closed δ_1-primary or J_2 is a weakly $(m,n)-$closed δ_2-primary ideal of R_2 that is not $(m,n)-$closed δ_2-primary. Suppose that J_1 is a weakly $(m,n)-$closed δ_1-primary ideal of R_1 that is not $(m,n)-$closed δ_1-primary. Thus, J_1 has a $\delta_1-(m,n)-$unbreakable-zero element a. Suppose that $y^m \in J_2$ for some $y \in R_2$. Since a is a $\delta_1-(m,n)-$unbreakable-zero element of J_1 and $(a,y)^m \in I$, we have $(a,y)^m = (0,0)$. Thus, $y^m = 0$ (in particular, $j^m = 0 \ \forall\ j \in J_2$). Now,

suppose that $0 \neq x^m \in J_1$ for some $x \in R_1$. Let $y \in R_2$ be such that $y^m \in J_2$. Then, $(0,0) \neq (x,y)^m \in I$. Thus, $y^n \in \delta_2(J_2)$, and hence J_2 is an $(m,n)-$closed δ_2-primary ideal of R_2. Similarly, if J_2 is a weakly $(m,n)-$closed δ_2-primary ideal of R_2 that is not $(m,n)-$closed δ_2-primary, then $y^m = 0$ whenever $y^m \in J_1$ for some $y \in R_1$ (in particular, $j^m = 0 \; \forall \; j \in J_1$), and if $0 \neq x^m \in J_2$ for some $x \in R_2$, then J_1 is an $(m,n)-$closed δ_1-primary ideal of R_1.
(2) $\Longrightarrow$ (1) Assume that J_1 is a weakly $(m,n)-$closed δ_1-primary ideal of R_1 that is not $(m,n)-$closed δ_1-primary, $y^m = 0$ whenever $y^m \in J_2$ for some $y \in R_2$ (in particular, $j^m = 0 \; \forall \; j \in J_2$), and if $0 \neq x^m \in J_1$ for some $x \in R_1$, then J_2 is an $(m,n)-$closed δ_2-primary ideal of R_2. Let a be a $\delta_1 - (m,n)-$unbreakable-zero element of J_1. Then, $(a,0)$ is a $\delta_\times -(m,n)-$unbreakable-zero element of I. Thus, I is not an $(m,n)-$closed $\delta_\times-$primary ideal of R. Now, suppose that $(0,0) \neq (x,y)^m = (x^m, y^m) \in I$ for some $x \in R_1$ and some $y \in R_2$. Then, $(0,0) \neq (x,y)^m = (x^m, 0) \in I$ and $0 \neq x^m \in J_1$. Since J_1 is a weakly $(m,n)-$closed δ_1-primary ideal of R_1 and J_2 is an $(m,n)-$closed δ_2-primary ideal of R_2, we have $(x,y)^n \in \delta_\times(I)$. Similarly, suppose that J_2 is a weakly $(m,n)-$closed δ_2-primary ideal of R_2 that is not $(m,n)-$closed δ_2-primary, $y^m = 0$ whenever $y^m \in J_1$ for some $y \in R_1$ (in particular, $j^m = 0 \; \forall \; j \in J_1$), and if $0 \neq x^m \in J_2$ for some $x \in R_2$, then J_1 is an $(m,n)-$closed δ_1-primary ideal of R_1. Then again, I is a weakly $(m,n)-$closed $\delta_\times-$primary ideal of R that is not $(m,n)-$closed $\delta_\times-$primary. $\square$

Let R be a commutative ring, δ be an expansion function of $Id(R)$ and M be an $R-$module. As in [14], the trivial ring extension of R by M (or the idealization of M over R) is defined by $R(+)M = \{(a,b) : a \in R, \; b \in M\}$ is a commutative ring with identity $(1,0)$ under addition defined by $(a,b) + (c,d) = (a+c, b+d)$ and multiplication defined by $(a,b)(c,d) = (ac, ad+bc)$ for each $a, c \in R$ and $b, d \in M$. Note that $(\{0\}(+)M)^2 = \{0\}$, so $\{0\}(+)M \subseteq Nil(R(+)M)$.

We define a function $\delta_{(+)} : Id(R(+)M) \longrightarrow Id(R(+)M)$ such that $\delta_{(+)}(I(+)N) = \delta(I)(+)M$ for every ideal I of R and every submodule N of M. Then $\delta_{(+)}$ is an expansion function of ideals of $R(+)M$.

Theorem 3.26. *Let R be a commutative ring, δ be an expansion function of $Id(R)$, m and n be integers with $1 \leq n < m$, I be a proper ideal of R, and M be an $R-$module. Then the following statements are equivalent.*

(1) $I(+)M$ is a weakly $(m,n)-$closed $\delta_{(+)}-$primary ideal of $R(+)M$ that is not $(m,n)-$closed $\delta_{(+)}-$primary.

(2) I is a weakly $(m,n)-$closed $\delta-$primary ideal of R that is not $(m,n)-$closed $\delta-$primary and $m(x^{m-1}M)=0$ for every $(m,n)-$unbrekable-zero element x of I.

Proof. (1) $\Longrightarrow$ (2) Assume that $0 \neq r^m \in I$ for some $r \in R$. Then, $(0,0) \neq (r,0)^m = (r^m,0) \in I(+)M$. Thus, $(r,0)^n = (r^n,0) \in I(+)M$; so $r^n \in \delta(I)$. Thus, I is a weakly $(m,n)-$closed $\delta-$primary ideal of R. Since $I(+)M$ is not $(m,n)-$closed $\delta_{(+)}-$primary ideal of $R(+)M$, we have $I(+)M$ and hence I, has a $\delta-(m,n)-$unbrekable-zero element; so I is not an $(m,n)-$closed $\delta-$primary ideal of R. Let x be a $\delta-(m,n)-$unbrekable-zero element of I and $a \in M$. Then, $(x,a)^m = (x^m, m(x^{m-1}a)) \in I(+)M$. Since $a^n \notin \delta(I)$, we have $(x,a)^m = (x^m, m(x^{m-1}a)) = (0,0)$. Thus $m(x^{m-1}M)=0$.
(2) $\Longrightarrow$ (1) Since I is a weakly $(m,n)-$closed $\delta-$primary ideal of R that is not $(m,n)-$closed $\delta-$primary, we have I has a $\delta-(m,n)-$unbrekable-zero element x. Thus, $(x,0)$ is a $\delta_{(+)}-(m,n)-$unbrekable-zero element of $I(+)M$. Thus, $I(+)M$ is not an $(m,n)-$closed $\delta_{(+)}-$primary ideal of $R(+)M$. Assume that $(0,0) \neq (r,y)^m = (r^m, m(r^{m-1}y)) \in I(+)M$. Then, r is not a $\delta-(m,n)-$unbrekable-zero element of I by hypothesis. Hence, $(r^n, n(r^{n-1}y) = (r,y)^n \in \delta(I(+)M)$. Therefore, $I(+)M$ is a weakly $(m,n)-$closed $\delta_{(+)}-$primary ideal of $R(+)M$ that is not $(m,n)-$closed $\delta_{(+)}-$primary. $\square$

4. Conclusion

This article included the structure of weakly $(m,n)-$closed $\delta-$primary ideals of a commutative ring R. We have discussed and proved important results in this class of ideals. We proved that if R is a commutative ring, δ an expansion function of $Id(R)$, m and n positive integers, and I a weakly $(m,n)-$ closed $\delta-$primary ideal of R that is not $(m,n)-$ closed $\delta-$primary, then $I \subseteq Nil(R)$. Moreover, if $char(R)=m$ is prime, then $x^m=0$ for every $x \in I$. Also, we have studied the weakly $(m,n)-$ closed $\delta-$primary ideal I of the trivial ring extension of R by an $R-$module M. We shall transfer the notion studied in this article to the amalgamated algebras along an ideal in the next work.

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