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Hermite-Hadamard type inequalities via weighted integral operators

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Abstract

In this paper, we consider general convex functions of various type. We establish some new integral inequalities of Hermite-Hadamard type for (h, s, m)-convex and (h, m)-convex functions, using generalized integrals. We also investigate differentiable functions with general convex derivative. The proven results generalize many results previously known from the literature.

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1. Introduction

A function $f : I \to \mathbf{R}$, $I := [a, b]$ is said to be convex on I, if $f\Big(\lambda x + (1 - \lambda)y\Big) \leq \lambda f(x) + (1 - \lambda)f(y)$ holds for all $x, y \in I$ and $\lambda \in [0, 1]$. If the reversed inequality holds, then f is said to be concave on I. Convex functions have been the object of attention in recent decades and the original notion has been extended and generalized in various directions, such functions are important in many parts of analysis and geometry and their properties have been studied in detail. Readers interested in the aforementioned development, can consult [33], where a panorama, practically complete, of these branches is presented.

One of the most important inequalities for convex functions, is the famous Hermite-Hadamard inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

holds for any function f convex on the interval $[a, b]$. This inequality was published by Hermite ([19]) in 1883 and, independently, by Hadamard in 1893 ([18]). It gives an estimation of the mean value of a convex function, and it is important to note that it also provides a refinement to the Jensen inequality. Several results can be consulted in [2, 3, 6, 7, 8, 12, 13, 14, 17, 20, 25, 28, 30, 34, 44] and references therein for more information and other extensions of the Hermite-Hadamard inequality (1.1).

Toader in [50] defined m-convexity in the following way:

Definition 1.1. *Function $f : [0, b] \to \mathbf{R}$, $b > 0$, is said to be m-convex, where $m \in [0, 1]$, if*

$$f\,(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y)$$

holds for all $x, y \in [0, b]$ and $t \in [0, 1]$. If the above inequality holds in reverse, then we say that f is m-concave.

The following definitions are successive extensions of the concept of m-convex functions.

Definition 1.2. *[52] Let $s \in [-1, 1]$ and $m \in [0, 1]$ be real numbers. Function $f : [0, b] \to [0, \infty)$ with $b > 0$ is said to be extended (s, m)-convex if*

$$f(tx + m(1-t)y) \leq t^s f(x) + m(1-t)^s f(y)$$

holds for all $x, y \in [0, b]$ and $t \in (0, 1)$.

In [29], the class of (α, m)-convex functions is presented as follows.

Definition 1.3. *The function* $f : [0, b] \to [0, \infty)$ *with* $b > 0$ *is said to be* (α, m)*-convex, where* $\alpha, m \in [0, 1]$*, if, for every* $x, y \in [0, b]$ *and* $t \in [0, 1]$*,*

$$f(tx + m(1-t)y) \leq t^{\alpha} f(x) + m(1-t^{\alpha}) f(y).$$

In [32] the authors presented the classes of (α, s, m)-convex functions as follows ("redefined" in [51]).

Definition 1.4. *Function* $f : [0, \infty) \to [0, \infty)$ *is said to be* (α, s, m)*-convex in the first sense, if for all* $x, y \in [0, \infty)$ *and* $t \in [0, 1]$*, we have the following inequality:*

$$f(tx + m(1-t)y) \leq t^{\alpha s} f(x) + m(1-t^{\alpha s}) f(y),$$

where $\alpha, m \in [0, 1]$ *and* $s \in (0, 1]$*.*

Note that the above Definition 1.4 is equivalent to Definition 1.3 that involves the (α, m)-convex functions, therefore we omit to work with the concept (α, s, m)-convex in the first sense.

Definition 1.5. *Function* $f : [0, \infty) \to [0, \infty)$ *is said to be* (α, s, m)*-convex in the second sense, or shortly,* (α, s, m)*-convex, if for all* $x, y \in [0, +\infty)$ *and* $t \in [0, 1]$*, we have the following inequality:*

$$f(tx + m(1-t)y) \leq (t^{\alpha})^s f(x) + m(1-t^{\alpha})^s f(y),$$

where $\alpha, m \in [0, 1]$ *and* $s \in (0, 1]$*.*

In [28] the following definition is introduced.

Definition 1.6. *Let* $h : [0, 1] \to [0, \infty)$ *with* $h \not\equiv 0$*. Function* $f : [0, b] \to [0, \infty)$ *with* $b > 0$ *is said to be* (h, m)*-convex on* $[0, b]$*, if inequality*

$$f\left(tx + m(1-t)y\right) \leq h(t) f(x) + m h(1-t) f(y)$$

is fulfilled for $m \in [0, 1]$*, for all* $x, y \in [0, b]$ *and* $t \in [0, 1]$*.*

Note that if the given inequality is reversed in the previous definitions, then f is said to be extended (s, m)-concave etc., respectively.

On the basis of these definitions, we will present the class of functions that will be the basis of our work (see also [27]).

Definition 1.7. *Let $h : [0,1] \to [0,\infty)$, $h \not\equiv 0$. Function $f : I \subseteq [0,\infty) \to [0,\infty)$ is said to be (h,s,m)-convex on I if inequality*

$$f\left(tx + m(1-t)y\right) \leq h^s(t)f(x) + m(1-h(t))^s f(y)$$

is fulfilled for $m \in [0,1]$, $s \in [-1,1]$, for all $x, y \in I$ and $t \in [0,1]$.

Remark 1.8. *From Definition 1.7 we can define $N^s_{h,m}[a,b]$ as the set of functions (h,s,m)-convex on $[a,b]$, characterized by the triple $(h(t),s,m)$. Note that if $(h(t),s,m)$ is equal to*

1. *$(h(t),0,0)$, then we have the increasing functions ([9]).*
2. *$(t,s,0)$, then we have the s-starshaped functions ([9]).*
3. *$(t,1,0)$, then we have the starshaped functions ([9]).*
4. *$(t^\alpha,1,s)$, then f is an (α,s,m)-convex function.*
5. *(t,s,m), then f is an (s,m)-convex function.*
6. *$(t,1,m)$, then f is an m-convex function.*
7. *$(t,1,1)$, then f is a convex function.*

Note that concept of (h,s,m)-convex functions is not a generalization of the (h,m)-convex functions, but an extension, therefore we investigate both type of generalized convex functions.

All through the work we utilize the functions $\Gamma(z)$ (see [40, 41, 53, 54]) and $\Gamma_k(z)$ (see [10]):

$$\begin{aligned} \Gamma(z) &= \textstyle\int_0^\infty t^{z-1}e^{-t}\,dt, \quad \Re(z) > 0, \\ \Gamma_k(z) &= \textstyle\int_0^\infty t^{z-1}e^{-t^k/k}\,dt, \quad \Re(z) > 0, k > 0. \end{aligned}$$

Unmistakably, if $k \to 1$, then we have $\Gamma_k(z) \to \Gamma(z)$, furthermore, $\Gamma_k(z) = k^{\frac{z}{k}-1}\Gamma\left(\frac{z}{k}\right)$ and $\Gamma_k(z+k) = z\Gamma_k(z)$.

To encourage comprehension of the subject, we present the definition of the Riemann-Liouville fractional integrals ($[a,b] \subseteq [0,\infty)$), see e.g. [38]).

Definition 1.9. *Let $f \in L^1[a,b]$. Then the Riemann-Liouville fractional integrals (left and right, respectively) of order $\alpha \in \mathbf{C}$, $\Re(\alpha) > 0$ are defined by*

$$\begin{aligned} I^\alpha_{a+}f(x) &= \tfrac{1}{(\alpha)}\textstyle\int_a^x (x-t)^{\alpha-1}f(t)\,dt, \quad x > a, \\ I^\alpha_{b-}f(x) &= \tfrac{1}{(\alpha)}\textstyle\int_x^b (t-x)^{\alpha-1}f(t)\,dt, \quad x < b. \end{aligned}$$

Next we present the weighted integral operators, which will be the basis of our work.

Definition 1.10. *Let $f \in L^1[a,b]$ and let $w : [0,1] \to [0,\infty)$ be a continuous function with a derivative piecewise continuous on $[0,1]$. Then the weighted fractional integrals (left and right, respectively) are defined by*

$$I^w_{a+}f(x) = \int_a^x w'\left(\tfrac{x-t}{x-a}\right) f(t)\,dt, \quad x > a,$$
$$I^w_{b-}f(x) = \int_x^b w'\left(\tfrac{t-x}{b-x}\right) f(t)\,dt, \quad x < b,$$

where the integrals are considered in Lebesgue's sense.

Remark 1.11. *To have a clearer idea of the amplitude of the previous definition, let us consider some particular cases:*

1. *Putting $w(t) = t$, we obtain the classical Lebesgue integral.*
2. *If $w(t) = \frac{(x-a)^{\alpha-1}}{\Gamma(\alpha+1)} t^\alpha$ or $w(t) = \frac{(b-x)^{\alpha-1}}{\Gamma(\alpha+1)} t^\alpha$, then we obtain the Riemann-Liouville fractional integral, left or right.*
3. *If $w(t) = \frac{(x-a)^{\frac{\alpha}{k}-1}}{\Gamma_k(\alpha+k)} t^{\frac{\alpha}{k}}$ or $w(t) = \frac{(b-x)^{\frac{\alpha}{k}-1}}{\Gamma_k(\alpha+k)} t^{\frac{\alpha}{k}}$, then we obtain the k-Riemann-Liouville fractional integral, left or right (see [31]).*

In the following, we obtain different extensions of the Hermite-Hadamard inequality (1.1) and its variants such as the following inequality given by Sarikaya and Yildirim in [46].

Theorem 1.12. *Let $f : [a,b] \subset [0,\infty) \to [0,\infty)$ be a convex function on $[a,b]$. If $f \in L^1[a,b]$ and $\alpha > 0$, then we have*

$$f\left(\frac{a+b}{2}\right) \le \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[I^\alpha_{\frac{a+b}{2}+} f(b) + I^\alpha_{\frac{a+b}{2}-} f(a)\right] \le \frac{f(a)+f(b)}{2}.$$

Throughout the paper, we use the framework of (h,s,m)-convex functions or (h,m)-convex functions, and generalized operators of Definition 1.10.

2. Inequalities for (h,s,m)-convex and (h,m)-convex functions

Our first extension of the Hermite-Hadamard inequality can be represented as follows.

Theorem 2.1. *Let $f:[0,\infty)\to[0,\infty)$ be an (h,s,m)-convex function on $\left[a,\frac{b}{m}\right]$ $(0\le a<b<\infty)$ and $n\in\mathbf{N}_0$. If $f\in L^1\left[a,\frac{b}{m}\right]$ and $h\in L^1[0,1]$, then we have the following inequality:*

$$\begin{aligned}
&f\left(\tfrac{a+b}{2}\right)(w(1)-w(0))\\
\le\;& \tfrac{n+1}{b-a}\left[h^s\left(\tfrac12\right)I^w_{\frac{a+nb}{n+1}+}f(b)+m\left(1-h\left(\tfrac12\right)\right)^s I^w_{\frac{na+b}{(n+1)m}-}f\left(\tfrac{a}{m}\right)\right]\\
\le\;& f(a)h^s\left(\tfrac12\right)\int_0^1 w'(t)h^s\left(\tfrac{t}{n+1}\right)dt\\
&+mf\left(\tfrac{a}{m}\right)\left(1-h\left(\tfrac12\right)\right)^s\int_0^1 w'(t)\left(1-h\left(\tfrac{t}{n+1}\right)\right)^s dt\\
&+f(b)h^s\left(\tfrac12\right)\int_0^1 w'(t)\left(1-h\left(\tfrac{t}{n+1}\right)\right)^s dt\\
&+mf\left(\tfrac{b}{m}\right)\left(1-h\left(\tfrac12\right)\right)^s\int_0^1 w'(t)h^s\left(\tfrac{t}{n+1}\right)dt.
\end{aligned}\tag{2.1}$$

Proof. For $x,y\in\left[a,\frac{b}{m}\right]$ and $t=\frac12$, we have

$$f\left(\frac{x+y}{2}\right)\le h^s\left(\frac12\right)f(x)+m\left(1-h\left(\frac12\right)\right)^s f\left(\frac{y}{m}\right).$$

If we choose $x=\frac{t}{n+1}a+\frac{n+1-t}{n+1}b$ and $y=\frac{n+1-t}{n+1}a+\frac{t}{n+1}b$ with $t\in[0,1]$, we get

$$\begin{aligned}
f\left(\tfrac{a+b}{2}\right)\le\;& h^s\left(\tfrac12\right)f\left(\tfrac{t}{n+1}a+\tfrac{n+1-t}{n+1}b\right)\\
&+m\left(1-h\left(\tfrac12\right)\right)^s f\left(\tfrac{n+1-t}{n+1}\tfrac{a}{m}+\tfrac{t}{n+1}\tfrac{b}{m}\right).
\end{aligned}\tag{2.2}$$

Multiplying both members of the previous inequality by $w'(t)$, integrating with respect to t from 0 to 1, and changing variables we obtain the first inequality of (2.1), since

$$\begin{aligned}
\int_0^1 w'(t)f\left(\tfrac{t}{n+1}a+\tfrac{n+1-t}{n+1}b\right)dt\;&=\tfrac{n+1}{b-a}\int_{\frac{a+nb}{n+1}}^b w'\left(\tfrac{b-u}{\frac{b-a}{n+1}}\right)f(u)\,du\\
&=\tfrac{n+1}{b-a}I^w_{\frac{a+nb}{n+1}+}f(b)
\end{aligned}\tag{2.3}$$

and

$$\begin{aligned}\int_0^1 w'(t) f\left(\frac{n+1-t}{n+1}\frac{a}{m}+\frac{t}{n+1}\frac{b}{m}\right)dt &= \frac{n+1}{b-a}\int_{\frac{a}{m}}^{\frac{na+b}{(n+1)m}} w'\left(\frac{u-\frac{a}{m}}{\frac{b-a}{(n+1)m}}\right) f(u)\,du \\ &= \frac{n+1}{b-a} I^{w}_{\frac{na+b}{(n+1)m}-} f\left(\frac{a}{m}\right).\end{aligned} \tag{2.4}$$

For the right hand side of (2.2), we obtain

$$\begin{aligned}& h^s\left(\tfrac{1}{2}\right) f\left(\tfrac{t}{n+1}a+\tfrac{n+1-t}{n+1}b\right) \\ & +m\left(1-h\left(\tfrac{1}{2}\right)\right)^s f\left(\tfrac{n+1-t}{n+1}\tfrac{a}{m}+\tfrac{t}{n+1}\tfrac{b}{m}\right) \\ \leq\ & h^s\left(\tfrac{1}{2}\right)\left[h^s\left(\tfrac{t}{n+1}\right) f(a)+\left(1-h\left(\tfrac{t}{n+1}\right)\right)^s f(b)\right] \\ & +m\left(1-h\left(\tfrac{1}{2}\right)\right)^s\left[h^s\left(\tfrac{t}{n+1}\right) f\left(\tfrac{b}{m}\right)+\left(1-h\left(\tfrac{t}{n+1}\right)\right)^s f\left(\tfrac{a}{m}\right)\right].\end{aligned}$$

Multiplying the members of the previous inequality by $w'(t)$ and integrating with respect to t, between 0 and 1, allows us to get the right member of (2.1). In this way the proof is complete. □

Remark 2.2. *If in the previous theorem, we consider convex functions, i.e. $s=m=1$ and $h(t)=t$, moreover we put $w(t)=t$ implying the classical Lebesgue integral and $n=0$, from (2.1), we obtain the Hermite-Hadamard inequality (1.1).*

Remark 2.3. *Analogously, working with convex functions ($s=m=1$ and $h(t)=t$), taking $w(t)=t^{\alpha}$ and $n=1$, we obtain Theorem 1.12.*

Remark 2.4. *If we consider $w'(t)=t^{-\alpha}$, the previous result gives us new results for non-conformable integral operators, see [34]. It is clear that we can consider other kernels $w(t)$ for which the results derived from the previous theorem are new.*

An alternative to Theorem 2.1 can be proved analogously.

Theorem 2.5. *Let $f:[0,\infty)\to[0,\infty)$ be an (h,m)-convex function on $\left[a,\frac{b}{m}\right]$ $(0\leq a<b<\infty)$ and $n\in\mathbf{N}_0$. If $f\in L^1\left[a,\frac{b}{m}\right]$, $h\in L^1[0,1]$ and $h\left(\frac{1}{2}\right)\neq 0$, then we have the following inequality:*

$$\begin{aligned}
&\frac{1}{h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right)(w(1)-w(0)) \\
\leq\ & \frac{n+1}{b-a}\left[I^w_{\frac{a+nb}{n+1}+} f(b) + m I^w_{\frac{na+b}{(n+1)m}-} f\left(\frac{a}{m}\right)\right] \\
\leq\ & f(a)\int_0^1 w'(t) h\left(\frac{t}{n+1}\right) dt + m f\left(\frac{a}{m}\right)\int_0^1 w'(t) h\left(\frac{n+1-t}{n+1}\right) dt \\
& + f(b)\int_0^1 w'(t) h\left(\frac{n+1-t}{n+1}\right) dt + m f\left(\frac{b}{m}\right)\int_0^1 w'(t) h\left(\frac{t}{n+1}\right) dt.
\end{aligned}$$

The following result will be basic from now on.

Lemma 2.6. *Let $f : [a,b] \subset \mathbf{R} \to \mathbf{R}$ f be differentiable on (a,b) and $n \in \mathbf{N}_0$. If $f' \in L^1[a,b]$, then we have the following equality:*

$$\begin{aligned}
&\frac{b-a}{n+1}\left[w(0)\left(f(a)+f(b)\right) - w(1)\left(f\left(\frac{a+nb}{n+1}\right) + f\left(\frac{na+b}{n+1}\right)\right)\right] \\
&+ \left[I^w_{\frac{a+nb}{n+1}+} f(b) + I^w_{\frac{na+b}{n+1}-} f(a)\right] \\
=\ & \frac{(b-a)^2}{(n+1)^2}\int_0^1 w(t)\left[f'\left(\frac{t}{n+1}a + \frac{n+1-t}{n+1}b\right) - f'\left(\frac{n+1-t}{n+1}a + \frac{t}{n+1}b\right)\right] dt.
\end{aligned} \tag{2.5}$$

Proof. First note that

$$\begin{aligned}
&\int_0^1 w(t)\left[f'\left(\frac{t}{n+1}a + \frac{n+1-t}{n+1}b\right) - f'\left(\frac{n+1-t}{n+1}a + \frac{t}{n+1}b\right)\right] dt \\
=\ & \int_0^1 w(t) f'\left(\frac{t}{n+1}a + \frac{n+1-t}{n+1}b\right) dt - \int_0^1 w(t) f'\left(\frac{n+1-t}{n+1}a + \frac{t}{n+1}b\right) dt \\
=\ & I_1 - I_2.
\end{aligned}$$

Integrating by parts for I_1 yields

$$\begin{aligned}
I_1 =\ & -\frac{n+1}{b-a}\left[w(1) f\left(\frac{a+nb}{n+1}\right) - w(0) f(b)\right] \\
& + \frac{(n+1)}{(b-a)}\int_0^1 w'(t) f\left(\frac{t}{n+1}a + \frac{n+1-t}{n+1}b\right) dt \\
=\ & -\frac{n+1}{b-a}\left[w(1) f\left(\frac{a+nb}{n+1}\right) - w(0) f(b)\right] + \frac{(n+1)^2}{(b-a)^2} I^w_{\frac{a+nb}{n+1}+} f(b).
\end{aligned}$$

Analogously, for I_2, we have

$$I_2 = \frac{n+1}{b-a}\left[w(1) f\left(\frac{na+b}{n+1}\right) - w(0) f(a)\right] - \frac{(n+1)^2}{(b-a)^2} I^w_{\frac{na+b}{n+1}-} f(a).$$

From $I_1 - I_2$, and grouping appropriately, we obtain the required equality. □

To realize the scope and generality of our previous result, we will present several particular cases.

Remark 2.7. *Putting $w(t) = t^\alpha$ and considering convex functions and $n = 1$, we obtain the Lemma 3 of [46].*

Remark 2.8. *If $w(t) = t$, $n = 0$, we obtain a new result for the classical Lebesgue integral.*

Remark 2.9. *Let us consider $n = 0$. For various choices of the weight $w(t)$ and taking not the right member of (2.5), but only one of the integrals, we can obtain without difficulty a variant of Lemma 1 of [4], Lemma 2.1 of [11] (also see Lemma 2.1 of [22]), Lemma 2.3 of [16], Lemma 2.1 of [21], Lemma 1 of [23], Lemma 2.1 of [24], Lemma 1 of [35], Lemma 1 of [37], Lemma 3.1 of [48], and Lemma 2 of [45] (see also [36]) are obtained.*

Remark 2.10. *Also, the reader will be able to verify, without much difficulty, that under different variants of the weight $w(t)$ we can obtain Lemma 2 of [38], Lemma 1.1 of [49], Lemma 2.1 of [43], Lemma 2.1 of [56], Lemma 1.6 of [42], Lemma 2.1 of [1], Lemma 1 of [5], Lemma 2.1 of [47].*

Theorem 2.11. *Let $f : [0, \infty) \to [0, \infty)$ be differentiable on $\left[a, \frac{b}{m}\right]$ such that $|f'|$ is (h, s, m)-convex on $\left[a, \frac{b}{m}\right]$ $(0 \le a < b < \infty)$. If $f' \in L^1\left[a, \frac{b}{m}\right]$, $h \in L^1[0, 1]$ and $n \in \mathbf{N}_0$, then we have the following inequality:*

$$\begin{aligned} & \left|\mathcal{A} + \left[I^w_{\frac{a+nb}{n+1}+} f(b) + I^w_{\frac{na+b}{n+1}-} f(a)\right]\right| \\ (2.6) \quad \le\ & \tfrac{(b-a)^2}{(n+1)^2}\left(|f'(a)| + |f'(b)|\right)\int_0^1 w(t) h^s\left(\tfrac{t}{n+1}\right)dt \\ & + \tfrac{(b-a)^2}{(n+1)^2} m\left(\left|f'\left(\tfrac{a}{m}\right)\right| + \left|f'\left(\tfrac{b}{m}\right)\right|\right)\int_0^1 w(t)\left(1 - h\left(\tfrac{t}{n+1}\right)\right)^s dt \end{aligned}$$

with

$$\mathcal{A} = \frac{b-a}{n+1}\left[w(0)\left(f(a) + f(b)\right) - w(1)\left(f\left(\frac{a+nb}{n+1}\right) + f\left(\frac{na+b}{n+1}\right)\right)\right].$$

Proof. From Lemma 2.6, we obtain

$$\begin{aligned} & \left|\mathcal{A} + \left[I^w_{\frac{a+nb}{n+1}+} f(b) + I^w_{\frac{na+b}{n+1}-} f(a)\right]\right| \\ =\ & \tfrac{(b-a)^2}{(n+1)^2}\left|\int_0^1 w(t)\left[f'\left(\tfrac{t}{n+1}a + \tfrac{n+1-t}{n+1}b\right) - f'\left(\tfrac{n+1-t}{n+1}a + \tfrac{t}{n+1}b\right)\right]dt\right| \\ (2.7) \le\ & \tfrac{(b-a)^2}{(n+1)^2}\left[\int_0^1 w(t)\left|f'\left(\tfrac{t}{n+1}a + \tfrac{n+1-t}{n+1}b\right)\right|dt\right. \\ & \left. + \int_0^1 w(t)\left|f'\left(\tfrac{n+1-t}{n+1}a + \tfrac{t}{n+1}b\right)\right|dt\right] \\ =\ & \tfrac{(b-a)^2}{(n+1)^2}\left(|I_1| + |I_2|\right). \end{aligned}$$

Using the modified (h, s, m)-convexity of $|f'|$, we get

$$\begin{aligned} |I_1| \leq & \int_0^1 w(t)\left[h^s\left(\tfrac{t}{n+1}\right)|f'(a)| + m\left(1-h\left(\tfrac{t}{n+1}\right)\right)^s \left|f'\left(\tfrac{b}{m}\right)\right|\right] dt \\ = & |f'(a)| \int_0^1 w(t)h^s\left(\tfrac{t}{n+1}\right) dt + m\left|f'\left(\tfrac{b}{m}\right)\right| \int_0^1 w(t)\left(1-h\left(\tfrac{t}{n+1}\right)\right)^s dt. \end{aligned} \tag{2.8}$$

In the same way,

$$|I_2| \leq |f'(b)| \int_0^1 w(t)h^s\left(\frac{t}{n+1}\right) dt + m\left|f'\left(\frac{a}{m}\right)\right| \int_0^1 w(t)\left(1-h\left(\frac{t}{n+1}\right)\right)^s dt. \tag{2.9}$$

From (2.8) and (2.9), we easily obtain (2.6). The theorem is proved. □

Remark 2.12. *Considering $n = 0$, the following results can be derived from the above theorem:*

1. *Theorem 2.1, case $q = 1$ from [26], for m-convex functions, $h(t) = t$ and $s = 1$.*
2. *Theorem 2.2 of [11], obtained for convex functions, $m = 1$, using $w(t) = 1 - 2t$ and using I_1 only.*
3. *Theorem 2.4 of [16] for convex functions, $h(t) = t$ and $s = m = 1$, a known result for k-fractional integrals.*
4. *Theorem 2.3 of [32], with I_1, $w(t) = 1-2t$ and (α, m)-convex functions are considered.*
5. *Theorem 3 of [45], for convex functions, $m = 1$, a result for Riemann-Liouville fractional integrals.*
6. *Theorem 5 of [55] for s-convex functions, $m = 1$, and Riemann-Liouville fractional integrals.*

Considering $n = 1$, Theorem 2.2 of [24] can be obtained for convex functions.

An alternative to Theorem 2.1, regarding (h, m)-convexity, can be proved analogously.

Theorem 2.13. *Let $f:[0,\infty)\to[0,\infty)$ be differentiable on $\left[a,\frac{b}{m}\right]$ such that $|f'|$ is (h,m)-convex on $\left[a,\frac{b}{m}\right]$ $(0\le a<b<\infty)$. If $f'\in L^1\left[a,\frac{b}{m}\right]$, $h\in L^1[0,1]$ and $n\in\mathbf{N}_0$, then we have the following inequality:*

$$\begin{aligned} &\left|\mathcal{A}+\left[I^w_{\frac{a+nb}{n+1}+}f(b)+I^w_{\frac{na+b}{n+1}-}f(a)\right]\right| \\ \le\ & \tfrac{(b-a)^2}{(n+1)^2}(|f'(a)|+|f'(b)|)\textstyle\int_0^1 w(t)h\left(\frac{t}{n+1}\right)dt \\ &+\tfrac{(b-a)^2}{(n+1)^2}m\left(\left|f'\left(\frac{a}{m}\right)\right|+\left|f'\left(\frac{b}{m}\right)\right|\right)\textstyle\int_0^1 w(t)h\left(\frac{n+1-t}{n+1}\right)dt\end{aligned}$$

with $\mathcal{A}$ as before.

Refinements of the previous results, can be obtained by imposing new additional conditions on $|f'|^q$.

Theorem 2.14. *Let $f:[0,\infty)\to[0,\infty)$ be differentiable on $\left[a,\frac{b}{m}\right]$ such that $|f'|^q$, $q>1$ is (h,s,m)-convex on $\left[a,\frac{b}{m}\right]$ $(0\le a<b<\infty)$. If $|f'|^q\in L^1\left[a,\frac{b}{m}\right]$, $h\in L^1[0,1]$ and $n\in\mathbf{N}_0$, then we have*

$$\begin{aligned} &\left|\mathcal{A}+\left[I^w_{\frac{a+nb}{n+1}+}f(b)+I^w_{\frac{na+b}{n+1}-}f(a)\right]\right| \\ \le\ & \tfrac{(b-a)^2}{(n+1)^2}B_q\left[\left(|f'(a)|^q C_1+m\left|f'\left(\frac{b}{m}\right)\right|^q C_2\right)^{\frac{1}{q}}\right. \\ &\qquad\left.+\left(|f'(b)|^q C_1+m\left|f'\left(\frac{a}{m}\right)\right|^q C_2\right)^{\frac{1}{q}}\right]\end{aligned}$$

with $\mathcal{A}$ as before, furthermore, $B_q=\left(\int_0^1 w^p(t)\,dt\right)^{\frac{1}{p}}$, $C_1=\int_0^1 h^s\left(\frac{t}{n+1}\right)dt$, $C_2=\int_0^1\left(1-h\left(\frac{t}{n+1}\right)\right)^s dt$ and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 2.6, we have inequality (2.7). Hölder's inequality implies

$$|I_1|\le\left(\int_0^1 w^p(t)\,dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\frac{t}{n+1}a+\frac{n+1-t}{n+1}b\right)\right|^q dt\right)^{\frac{1}{q}} \tag{2.10}$$

and

$$|I_2|\le\left(\int_0^1 w^p(t)\,dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\frac{n+1-t}{n+1}a+\frac{t}{n+1}b\right)\right|^q dt\right)^{\frac{1}{q}}. \tag{2.11}$$

By the (h,s,m)-convexity of $|f'|^q$, we obtain

$$\begin{aligned} & \int_0^1 \left| f'\left(\frac{t}{n+1}a + \frac{n+1-t}{n+1}b\right)\right|^q dt \\ \le\ & |f'(a)|^q \int_0^1 h^s\left(\frac{t}{n+1}\right)dt + m\left|f'\left(\frac{b}{m}\right)\right|^q \int_0^1 \left(1-h\left(\frac{t}{n+1}\right)\right)^s dt, \end{aligned} \tag{2.12}$$

and

$$\begin{aligned} & \int_0^1 \left| f'\left(\frac{n+1-t}{n+1}a + \frac{t}{n+1}b\right)\right|^q dt \\ \le\ & |f'(b)|^q \int_0^1 h^s\left(\frac{t}{n+1}\right)dt + m\left|f'\left(\frac{a}{m}\right)\right|^q \int_0^1 \left(1-h\left(\frac{t}{n+1}\right)\right)^s dt. \end{aligned} \tag{2.13}$$

Substituting (2.12) and (2.13) in (2.10) and (2.11) yields the required inequality. □

Remark 2.15. *Theorem 5 of [46] as particular case of the above result, if we take $n = 1$. Other known results from the literature that can be derived of the above theorem are the following: Theorem 2.2 of [26], Theorem 2.3 [11] and Theorem 1 of [39].*

An alternative to Theorem 2.14, regarding (h,m)-convexity, can be proved analogously.

Theorem 2.16. *Let $f : [0,\infty) \to [0,\infty)$ be differentiable on $\left[a, \frac{b}{m}\right]$ such that $|f'|^q$, $q > 1$ is (h,m)-convex on $\left[a, \frac{b}{m}\right]$ $(0 \le a < b < \infty)$. If $|f'|^q \in L^1\left[a, \frac{b}{m}\right]$, $h \in L^1[0,1]$ and $n \in \mathbf{N}_0$, then we have*

$$\begin{aligned} & \left|\mathcal{A} + \left[I^w_{\frac{a+nb}{n+1}+} f(b) + I^w_{\frac{na+b}{n+1}-} f(a)\right]\right| \\ \le\ & \frac{(b-a)^2}{(n+1)^2} B_q C \left[\left(|f'(a)|^q + m\left|f'\left(\frac{b}{m}\right)\right|^q\right)^{\frac{1}{q}} + \left(|f'(b)|^q + m\left|f'\left(\frac{a}{m}\right)\right|^q\right)^{\frac{1}{q}}\right] \end{aligned}$$

with $\mathcal{A}$, B_q as before, $C = \left(\int_0^1 h\left(\frac{t}{n+1}\right)dt\right)^{\frac{1}{q}}$.

Remark 2.17. *The above theorem complements Theorem 3.2 of [15].*

Another variant of Theorem 2.14 is the following.

Theorem 2.18. *Let $f:[0,\infty)\to[0,\infty)$ be differentiable on $\left[a,\frac{b}{m}\right]$ such that $|f'|^q$, $q>1$ is (h,s,m)-convex on $\left[a,\frac{b}{m}\right]$ $(0\le a<b<\infty)$. If $|f'|^q\in L^1\left[a,\frac{b}{m}\right]$, $h\in L^1[0,1]$ and $n\in\mathbf{N}_0$, then we have*

$$\begin{aligned}&\left|\mathcal{A}+\left[I^w_{\frac{a+nb}{n+1}+}f(b)+I^w_{\frac{na+b}{n+1}-}f(a)\right]\right|\\ \le\ & \tfrac{(b-a)^2}{(n+1)^2}D\Bigg[\left(|f'(a)|^q D_1+m\left|f'\left(\tfrac{b}{m}\right)\right|^q D_2\right)^{\frac1q}\\ &\qquad+\left(|f'(b)|^q D_1+m\left|f'\left(\tfrac{a}{m}\right)\right|^q D_2\right)^{\frac1q}\Bigg]\end{aligned}$$

with $\mathcal{A}$ as before, furthermore, $D=\left(\int_0^1 w(t)\,dt\right)^{\frac1p}$, $D_1=\int_0^1 w(t)h^s\left(\frac{t}{n+1}\right)dt$, $D_2=\int_0^1 w(t)\left(1-h\left(\frac{t}{n+1}\right)\right)^s dt$ and $\frac1p+\frac1q=1$.

Proof. From Lemma 2.6, we have inequality (2.7). Using well known power mean inequality, we obtain

$$|I_1|\le\left(\int_0^1 w(t)\,dt\right)^{\frac1p}\left(\int_0^1 w(t)\left|f'\left(\frac{t}{n+1}a+\frac{n+1-t}{n+1}b\right)\right|^q dt\right)^{\frac1q}\tag{2.14}$$

and

$$|I_2|\le\left(\int_0^1 w(t)\,dt\right)^{\frac1p}\left(\int_0^1 w(t)\left|f'\left(\frac{n+1-t}{n+1}a+\frac{t}{n+1}b\right)\right|^q dt\right)^{\frac1q}.\tag{2.15}$$

By the (h,s,m)-convexity of $|f'|^q$, we obtain

$$\begin{aligned}&\int_0^1 w(t)\left|f'\left(\tfrac{t}{n+1}a+\tfrac{n+1-t}{n+1}b\right)\right|^q dt\\ \le\ & |f'(a)|^q\int_0^1 w(t)h^s\left(\tfrac{t}{n+1}\right)dt+m\left|f'\left(\tfrac{b}{m}\right)\right|^q\int_0^1 w(t)\left(1-h\left(\tfrac{t}{n+1}\right)\right)^s dt\end{aligned}\tag{2.16}$$

and

$$\begin{aligned} & \int_0^1 w(t) \left| f'\left(\tfrac{n+1-t}{n+1} a + \tfrac{t}{n+1} b\right)\right|^q dt \\ \leq \; & \left|f'(b)\right|^q \int_0^1 w(t) h^s\left(\tfrac{t}{n+1}\right) dt + m \left|f'\left(\tfrac{a}{m}\right)\right|^q \int_0^1 w(t)\left(1-h\left(\tfrac{t}{n+1}\right)\right)^s dt. \end{aligned} \tag{2.17}$$

If we put (2.16) and (2.17) in (2.14) and in (2.15), respectively, it allows us to obtain the required inequality. In this way the proof is complete. □

Remark 2.19. *Theorem 2.3 of [26] can be obtained as particular case of the previous theorem. For $n = 1$, Theorem 2.18 complements Theorem 6 of [46].*

Analogously to Theorem 2.18, for (h, m)-convexity, we can obtain the following.

Theorem 2.20. *Let $f : [0, \infty) \to [0, \infty)$ be differentiable on $\left[a, \frac{b}{m}\right]$ such that $|f'|^q$, $q > 1$ is (h, m)-convex on $\left[a, \frac{b}{m}\right]$ $(0 \leq a < b < \infty)$. If $|f'|^q \in L^1\left[a, \frac{b}{m}\right]$, $h \in L^1[0, 1]$ and $n \in \mathbf{N}_0$, then we have*

$$\begin{aligned} & \left|\mathcal{A} + \left[I^w_{\frac{a+nb}{n+1}+} f(b) + I^w_{\frac{na+b}{n+1}-} f(a)\right]\right| \\ \leq \; & \frac{(b-a)^2}{(n+1)^2} D \left[\left(\left|f'(a)\right|^q D_1' + m\left|f'\left(\tfrac{b}{m}\right)\right|^q D_2'\right)^{\frac{1}{q}} \right. \\ & \left. + \left(\left|f'(b)\right|^q D_1' + m\left|f'\left(\tfrac{a}{m}\right)\right|^q D_2'\right)^{\frac{1}{q}}\right] \end{aligned}$$

with $\mathcal{A}$, D as before, $D_1' = \int_0^1 w(t) h\left(\frac{t}{n+1}\right) dt$ and $D_2' = \int_0^1 w(t) h\left(\frac{n+1-t}{n+1}\right) dt$.

Remark 2.21. *The above theorem complements Theorem 3.6 of [15].*

3. Conclusions

In this work we have obtained several integral inequalities, which contain several results, reported in the literature, including fractional operators. The scope of our results has been shown throughout the work.

The generality of the results obtained, taking into account the Remark 1.8 and the operators of the Definition 1.10, allow us to derive new results for various classes of functions, either convex, h-convex, m-convex and s-convex functions, defined in a closed interval of non-negative values of real numbers. It is clear that the problem of extending these results to other types of general convex functions remains open.

References

[1] P. Agarwal, M. Jleli, and M. Tomar, "Certain Hermite-Hadamard type inequalities via generalized k-fractional integrals", *J. Inequal. Appl.,* vol. 2017, no. 55, 2017. doi: 10.1186/s13660-017-1318-y

[2] A. Akkurt, M. E. Yildirim, and H. Yildirim, "On some integral inequalities for (k, h)-Riemann-Liouville fractional integral", *New Trends in Mathematical Sciences,* vol. 4, no. 1, pp. 138-146, 2016. doi: 10.20852/ntmsci.2016217824

[3] M. A. Ali, J. E. Nápoles V., A. Kashuri, and Z. Zhang, "Fractional non conformable Hermite-Hadamard inequalities for generalized φ-convex functions", *Fasc. Math.,* vol. 64, pp. 5-16, 2020. doi: 10.21008/j.0044-4413.2020.0007

[4] M. Alomari and M. Darus, "Some Ostrowski Type Inequalities for Quasi-Convex Functions with Applications to Special Means", R*GMIA Res. Rep. Coll.,* vol. 13, no. 2, art. 3, 2010. https://bit.ly/47lvvei

[5] M. U. Awan, M. A. Noor, F. Safdar, A. Islam, M. V. Mihai, and K. I. Noor, "Hermite-Hadamard type inequalities with applications", *Miskolc Math. Notes,* vol. 21, no. 2, pp. 593-614, 2020. doi: 10.18514/MMN.2020.2837

[6] S. Bermudo, P. Kórus, and J. E. Nápoles Valdés, "On q-Hermite-Hadamard inequalities for general convex functions", *Acta Math. Hungar.,* vol. 162, pp. 364-374, 2020. doi: 10.1007/s10474-020-01025-6

[7] M. Bessenyei and Z. Páles, "On generalized higher-order convexity and Hermite-Hadamard-type inequalities", *Acta Sci. Math. (Szeged),* vol. 70, no. 1, pp. 13-24, 2004.

[8] W. W. Breckner, “Stetigkeitsaussagen für eine Klasse verallgemeinerter konvexer funktionen in topologischen linearen Räumen”, *Publ. Inst. Math.*, vol. 23, pp. 13-20, 1978.

[9] A. M. Bruckner and E. Ostrow, “Some function classes related to the class of convex functions”, *Pacific J. Math.*, vol. 12, no. 4, pp. 1203-1215, 1962.

[10] R. Díaz and E. Pariguan, “On hypergeometric functions and Pochhammer k-symbol”, *Divulg. Mat.*, vol. 15, no. 2, pp. 179-192, 2007.

[11] S. S. Dragomir and R. P. Agarwal, “Two inequalities for differentiable mappings and applications to special means of real numbers and trapezoidal formula”, *Appl. Math. Lett.*, vol. 11, no. 5, pp. 91-95, 1998. doi: 10.1016/S0893-9659(98)00086-X

[12] S. S. Dragomir and S. Fitzpatrick, “The Hadamard’s inequality for s-convex functions in the second sense”, *Demonstratio Math.*, vol. 32, no. 4, pp. 687-696, 1999. doi: 10.1515/dema-1999-0403

[13] S. S. Dragomir and C. E. M. Pearce, *Selected Topics on Hermite Hadamard Inequalities and Applications*, RGMIA Monographs, Victoria University, 2000. https://bit.ly/3uoY3oL

[14] S. S. Dragomir, J. Pecaric, and L. E. Persson, “Some inequalities of Hadamard type”, *Soochow J. Math.*, vol. 21, pp. 335-341, 1995.

[15] G. Farid, A. U. Rehman, and Q. U. Ain, “k-fractional integral inequalities of Hadamard type for (h−m)−convex functions”, *Computational Methods for Differential Equations*, vol. 8, no. 1, pp. 119-140, 2020. doi: 10.22034/cmde.2019.9462

[16] G. Farid, A. U. Rehman, and M. Zahra, “On Hadamard inequalities for k-fractional integrals”, *Nonlinear Funct. Anal. Appl.*, vol. 21, no. 3, pp. 463-478, 2016.

[17] P. M. Guzmán, J. E. Nápoles Valdés, and Y. S. Gasimov, “Integral inequalities within the framework of generalized fractional integrals”, *Fractional Differential Calculus*, vol. 11, no. 1, pp. 69-84, 2021. doi: 10.7153/fdc-2021-11-05

[18] J. Hadamard, “Étude sur les propriétés des fonctions entiéres et en particulier d’une fonction considérée par Riemann”, *J. Math. Pures Appl.* , vol. 9, pp. 171-215, 1893.

[19] C. Hermite, "Sur deux limites d'une intégrale définie", *Mathesis*, vol. 3, no. 82, 1883.

[20] J. E. Hernández Hernández, "On Some New Integral Inequalities Related With The Hermite-Hadamard Inequality via h-Convex Functions", *MAYFEB J. Math.*, vol. 4, pp. 1-12, 2017.

[21] R. Hussain, A. Ali, G. Gulshan, A. Latif, and K. Rauf, "Hermite-Hadamard Type Inequalities for k-Riemann Liouville Fractional Integrals Via Two Kinds of Convexity", *Aust. J. Math. Anal. Appl.*, vol. 13, no. 1, art. 17, pp. 1-12, 2016.

[22] D. A. Ion, "Some estimates on the Hermite-Hadamard inequality through quasi-convex functions", *Annals of University of Craiova, Math. Comp. Sci. Ser.*, vol. 34, pp. 82-87, 2007.

[23] M. A. Khan and Y. Khurshid, "Hermite-Hadamard's inequalities for η-convex functions via conformable fractional integrals and related inequalities", *Acta Math. Univ. Comenianae*, vol. 90, no. 2, pp. 157-169, 2021.

[24] U. S. Kirmaci, "Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula", *Appl. Math. Comput.*, vol. 147, pp. 137-146, 2004. doi: 10.1016/S0096-3003(02)00657-4

[25] M. Klaričić, E. Neuman, J. Pečiarić, and V. Simić, "Hermite-Hadamard's inequalities for multivariate g-convex functions", *Math. Inequal. Appl.*, vol. 8, no. 2, pp. 305-316, 2005. doi: 10.7153/mia-08-28

[26] M. Klaričić Bakula, M. E. Ozdemir, and J. Pečiarić, "Hadamard Type Inequalities for m-Convex and (α, m)-Convex Functions", J. Inequal. Pure Appl. Math. (JIPAM), vol. 9, No. 4, art. 96, 2008.

[27] P. Kórus and J. E. Nápoles Valdés, "Some Hermite-Hadamard inequalities involving weighted integral operators via (h, s, m)-convex functions", *TWMS Journal of Applied and Engineering Mathematics*, vol. 13, no. 4, pp. 1461-1471, 2023.

[28] M. Matloka, "On some integral inequalities for (h, m)-convex functions", *Mathematical Economics*, vol. 9, no. 16, pp. 55-70, 2013.

[29] V. G. Mihesan, "A generalization of the convexity", *Seminar on Functional Equations, Approx. and Convex., Cluj-Napoca (Romania)*, 1993.

[30] M. S. Moslehian, "Matrix Hermite-Hadamard type inequalities", *Houston J. Math.*, vol. 39, no. 1, pp. 177-189, 2013. doi: 10.48550/arXiv.1203.5300

[31] S. Mubeen and G. M. Habibullah, "k-Fractional Integrals and Application", *Int. J. Contemp. Math. Sci.*, vol. 7, no. 2, pp. 89-94, 2012.

[32] M. Muddassar, M. I. Bhatti, and W. Irshad, "Generalisation of integral inequalities of Hermite-Hadamard type through convexity", *Bull. Aust. Math. Soc.*, vol. 88, no. 2, pp. 320-330, 2013. doi: 10.1017/S0004972712000937

[33] J. E. Nápoles Valdés, F. Rabossi, and A. D. Samaniego, "Convex functions: Ariadne's thread or Charlotte's Spiderweb?", *Advanced Mathematical Models & Applications*, vol. 5, no. 2, pp. 176-191, 2020.

[34] J. E. Nápoles Valdés, J. M. Rodríguez, and J. M. Sigarreta, "On Hermite-Hadamard type inequalities for non-conformable integral operators", *Symmetry*, vol. 11, no. 9, 1108, 2019. doi:10.3390/sym11091108

[35] M. A. Noor, K. I. Noor, and M. U. Awan, "Generalized fractional Hermite-Hadamard inequalities", *Miskolc Math. Notes*, vol. 21, no. 2, pp. 1001-1011, 2020. doi: 10.18514/MMN.2020.1143

[36] M. E. Özdemir, S. S. Dragomir, and C. Yildiz, "The Hadamard Inequality For Convex Function Via Fractional Integrals", *Acta Math. Sci.*, vol. 33B, no. 5, pp. 1293-1299, 2013. doi: 10.1016/S0252-9602(13)60081-8

[37] M. E. Özdemir, H. Kavurmaci, and M. Avci, "Ostrowski type inequalities for convex functions", *Tamkang J. Math.*, vol. 45, no. 4, pp. 335-340, 2014. doi: 10.5556/j.tkjm.45.2014.1143

[38] J. Park, "Some Hermite-Hadamard type inequalities for MT-convex functions via classical and Riemann-Liouville fractional integrals", *Appl. Math. Sci.*, vol. 9, no. 101, pp. 5011-5026, 2015. doi: 10.12988/ams.2015.56425

[39] C. E. M. Pearce and J. Pečiarić, "Inequalities for differentiable mappings with application to special means and quadrature formulae", *Appl. Math. Lett.*, vol. 13, no. 2, pp. 51-55, 2000. doi: 10.1016/S0893-9659(99)00164-0

[40] F. Qi and B.-N. Guo, "Integral representations and complete monotonicity of remainders of the Binet and Stirling formulas for the gamma function", *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat.*, vol. 111, no. 2, pp. 425-434, 2017. doi: 10.1007/s13398-016-0302-6

[41] E. D. Rainville, *Special Functions.* New York: Macmillan Co., 1960.

[42] M. Rostamian Delavar, S. S. Dragomir, and M. De La Sen, "Estimation type results related to Fejér inequality with applications", *J. Inequal. Appl.*, vol. 2018, no. 85, 2018. doi: 10.1186/s13660-018-1677-z

[43] M. Z. Sarikaya, "On new Hermite Hadamard Fejér Type integral inequalities", *Stud. Univ. Babeş-Bolyai Math.*, vol. 57, no. 3, pp. 377-386, 2012.

[44] M. Z. Sarikaya, A. Saglam, and H. Yildirim, "New inequalities of Hermite-Hadamard type for functions whose second derivatives absolute values are convex and quasi-convex", *International Journal of Open Problems in Computer Science and Mathematics*, vol. 5, no. 3, 2012. doi: 10.12816/0006114

[45] M. Z. Sarikaya, E. Set, H. Yaldiz, and N. Basak, "Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities", *Math. Comput. Modelling,* vol. 57, no. 9-10, pp. 2403-2407, 2013. doi: 10.1016/j.mcm.2011.12.048

[46] M. Z. Sarikaya and H. Yildirim, "On Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals", *Miskolc Math. Notes,* vol. 17, no. 2, pp. 1049-1059, 2017. doi: 10.18514/MMN.2017.1197

[47] E. Set, J. Choi, and A. Gözpinar, "Hermite-Hadamard Type Inequalities Involving Nonlocal Conformable Fractional Integrals", *Malaysian Journal of Mathematical Sciences,* vol. 15, no. 1, pp. 33-43, 2021.

[48] E. Set and A. Gözpinar, "A study on Hermite-Hadamard type inequalities for s-convex functions via conformable fractional integrals", *Stud. Univ. Babeş-Bolyai Math.,* vol. 62, no. 3, pp. 309-323, 2017. doi: 10.24193/subbmath.2017.3.04

[49] E. Set, A. Gözpinar, and A. Ekinci, "Hermite-Hadamard type inequalities via conformable fractional integrals", *Acta Math. Univ. Comenianae,* vol. 86, no. 2, pp. 309-320, 2016.

[50] G. Toader, "Some generalizations of the convexity", *Proceedings of the Colloquium on Approximation and Optimization,* University ClujNapoca, pp. 329-338, 1985.

[51] B.-Y. Xi, D.-D. Gao, and F. Qi, "Integral inequalities of Hermite-Hadamard type for (α, s)-convex and (α, s, m)-convex functions", *Ital. J. Pure Appl. Math.,* vol. 44, pp. 499-510, 2020.

[52] B.-Y. Xi, Y. Wang, and F. Qi, "Some integral inequalities of Hermite-Hadamard type for extended (s, m)-convex functions", *Transylv. J. Math. Mechanics,* vol. 5, no. 1, pp. 69-84, 2013.

[53] Z.-H. Yang and J.-F. Tian, "Monotonicity and inequalities for the gamma function", *J. Inequal. Appl.,* vol. 2017, 317, 2017. doi: 10.1186/s13660-017-1591-9

[54] Z.-H. Yang and J.-F. Tian, "Monotonicity and sharp inequalities related to gamma function", *J. Math. Inequal.,* vol. 12, no. 1, pp. 1-22, 2018. doi: 10.7153/jmi-2018-12-01

[55] C. Yildiz, M. E. Özdemir, and H. K. Önalan, "Fractional Integral Inequalities via s-Convex Functions", *Turkish Journal of Analysis and Number Theory,* vol. 5, no. 1, pp. 18-22, 2017. doi: 10.12691/tjant-5-1-4

[56] C. Zhu, M. Feckan, and J. Wang, "Factional integral inequalities for differential convex mappings and applications to special means and a midpoint formula", *Journal of Applied Mathematics, Statistics and Informatics,* vol. 8, no. 2, pp. 21-28, 2012. doi: 10.2478/v10294-012-0011-5

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