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A graph product and its applications in generating non-cospectral equienergetic graphs

Subin P. Joseph
Government Engineering College Wayanad, India
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Abstract

A new graph product is defined in this paper and several applications of this product are described. The adjacency matrix of the product graph is given and its complete spectrum in terms of the spectrum of constituent graphs is determined. Sequences of cospectral graphs can be generated from the known cospectral graphs using the new product. Several sequences of non-cospectral equienergtic graphs can also be generated as an application of the graph product defined.

Keyword: *Graph product Spectrum Cospectral graphs Equienergetic graphs.*

MSC [2008]: *05C50, 05C75, 05C76*

1. Introduction

Let $G = (V, E)$ be a graph with vertex set V and edge set E. The order of this graph is the positive integer $n = |V|$. The adjacency matrix of the graph is defined to be the n^{th} order square matrix A_G with $(i,j)^{th}$ entry 1 if the i^{th} and j^{th} vertex are adjacent, otherwise the entry is zero. The ordinary energy of a graph[7] G is defined to be the sum of the absolute values of all the eigenvalues of the the adjacency matrix A_G. It is denoted by

$$\mathcal{E}_G = \sum_{i=1}^{n} |\lambda_i|, \tag{1.1}$$

where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the eigenvalues of A_G.

Graphs are classified according to their energies. A graph G is called hypoenergetic graph [11] if $\mathcal{E}_G < n$. An orderenergetic graph [1] is defined to be a graph G with $\mathcal{E}_G = n$. Graphs with energy $\mathcal{E}_G = 2n - 2$ are called borderenergetic graphs [5]. If $\mathcal{E}_G > 2n - 2$, then G is called a called hyperenergetic graph[8]. If two graphs are having same spectrum for their adjacency matrix, then such graphs are called cospectral graphs. If two graphs with same order are having equal energy, then they are called equienergetic graphs[18]. Clearly cospectral graphs are equienergetic. Generating sequences of equienergetic non-cospectral graphs is an active area in spectral graph theory. Orderenergetic graphs and borderenergetic graphs are clearly equienergetic graphs. Studies and applications of such different types of graphs can be found in [2, 4, 9, 10, 11, 13, 14, 17] and references therein.

There are several graph operations defined in the literature and they are applied in different areas of graph theory. Several unary graph operations are used to study the energy of graphs[14]. Recently, a general unary graph operation is constructed in[13] and it is used to generate orderenergetic graphs and equienergetic graphs. There are several binary graph operations based on cartesian product of vertex sets such as strong graph products and tensor graph products[12] with applications in spectral graph theory. Corona graph product[3] is a binary graph operation with vertex set different from the cartesian product of component graphs. This graph product is having significant applications in several areas of graph theory including spectral graph theory[15].

We define a new graph product and some of its applications are discussed in this paper. It is shown that the new product is entirely different from the corona product of two graphs. The definition of the graph product is given in the next section. The complete spectrum of the new product graph is determined in the third section. Some applications of the product graphs are discussed in the fourth section. Two different methods for generating sequences of non-cospectral equienergetic graphs using this product are also discussed in this paper.

2. Definition of new graph product

Let G be a graph of order m with vertex set $\{u_1, u_2, \cdots, u_m\}$ and H be a graph of order n with vertex set $\{v_1, v_2, \cdots, v_n\}$. Also let G is having $|E(G)| = e_1$ number of edges and H is having $|E(H)| = e_2$ number of edges. We define a new graph product $G \otimes H$ of the two graphs G and H as follows.

1. The vertex set of $G \otimes H$ is given by

$$\begin{aligned}\{&a_{11}, a_{12}, \cdots, a_{1n}, a_{21}, a_{22}, \cdots, a_{2n}, \cdots, a_{m1}, a_{m2}, \cdots, a_{mn},\\ &b_{11}, b_{12}, \cdots, b_{1n}, b_{21}, b_{22}, \cdots, b_{2n}, \cdots, b_{m1}, b_{m2}, \cdots, b_{mn}\}.\end{aligned} \tag{2.1}$$

2. The edge set of $G \otimes H$ consists of the following three types of edges.
   - If the edge $\{u_i, u_j\} \in E(G)$, then the edges $\{a_{ik}, a_{jl}\}$; for $1 \leq k, l \leq n$ belong to $G \otimes H$.
   - If the edge $\{v_i, v_j\} \in E(H)$, then the edges $\{b_{ri}, b_{rj}\}$; for $1 \leq r \leq m$ belong to $G \otimes H$.
   - For $1 \leq i \leq m$, the edges $\{a_{ip}, b_{iq}\}$; for $1 \leq p, q \leq n$, belong to $G \otimes H$.

Clearly, the total number vertices in the product graph is $2mn$. The number of edges of first, second and third types are n^2e_1, me_2 and mn^2 respectively. So the total number of edges in the product graph is $n^2(m + e_1) + me_2$.

We illustrate the above construction of product graph using simple examples. Let $G = H = K_2$, then the product graph $G \otimes H$ is given in Figure 1. For the second example, let $G = K_2$ and $H = K_3$. Then the corresponding product graphs $G \otimes H$ and $H \otimes G$ are given in Figure 2. From the second example it is clear that this graph product is not commutative. It is also evident that the new operation in general is entirely different form the corona product of two graphs [3] or any other known graph product available in the literature. It can be easily verified that the new product is same as corona product only when $n = 1$.

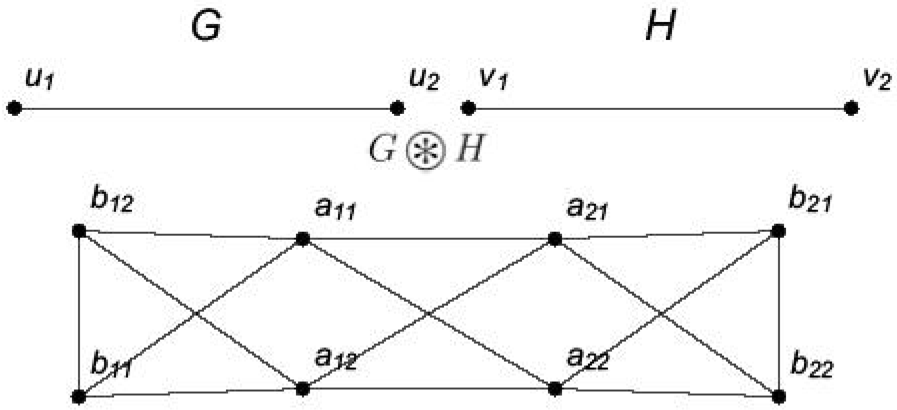

Figure 1: The product graph $G \otimes H$, where $G = H = K_2$.

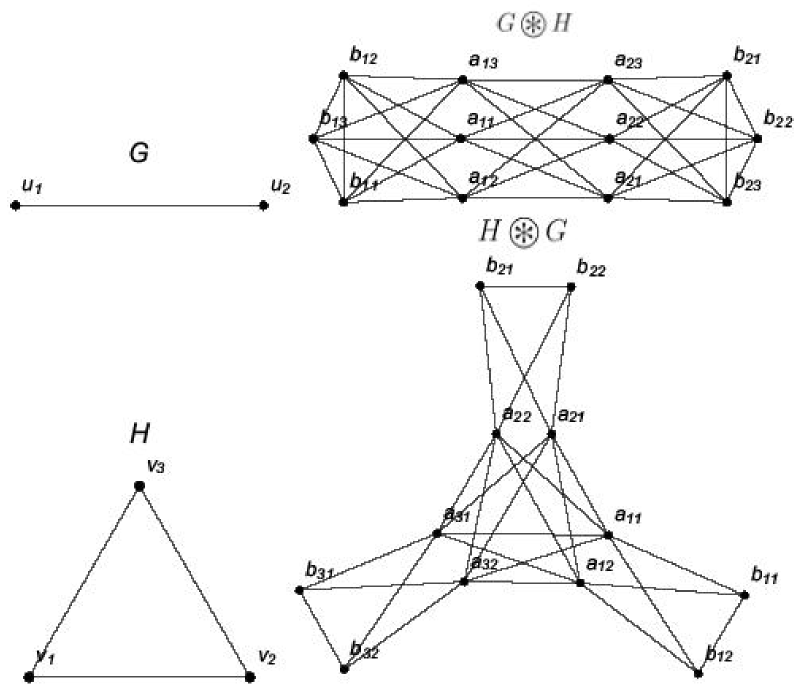

Figure 2: The product graph $G \otimes H$ and $H \otimes G$, where $G = K_2$ and $H = K_3$.

3. Spectrum of the new graph product

The main result in this paper is the derivation of the adjacency spectrum of the new graph product. So it is necessary to find out the adjacency matrix of the graph $G \otimes H$ in terms of the adjacency matrices of the component graphs G and H. We prescribe the ordering of the vertices of $G \otimes H$ as given in the definition of vertex set (2.1). Then, it is not hard to verify that the adjacency matrix of the graph product $G \otimes H$ is of order $2mn$ and is given by

$$A_{G \otimes H} = \begin{pmatrix} A_G \otimes J_n & I_m \otimes J_n \\ I_m \otimes J_n & I_m \otimes A_H \end{pmatrix}, \tag{3.1}$$

where A_G and A_H are the adjacency matrices of the component graphs G and H respectively, I_m denotes the identity matrix of order m, J_n denotes the all one square matrix of order n and $A \otimes B$ denotes the Kronecker product[6] of two matrices A and B. Our aim is to determine the spectra of $G \otimes H$ using the spectra of the component graphs G and H and using the coronal [15] of the graph H.

Definition 3.1. *[15] Let H be a graph with order n. Then the coronal of the graph H is defined to be the sum of the entries in the matrix $(\lambda I_n - A_H)^{-1}$. It is denoted by the symbol $\chi_H(\lambda)$ and is given by the formula*

$$\chi_H(\lambda) = \mathbf{1}_n^T \left(\lambda I_n - A_H\right)^{-1} \mathbf{1}_n, \tag{3.2}$$

where $\mathbf{1}_n$ is the length n column vector where all elements are 1's.

Clearly,

$$\begin{aligned} \chi_H(\lambda) &= \tfrac{\mathbf{1}_n^T Adj(\lambda I_n - A_H)\mathbf{1}_n}{Det(\lambda I_n - A_H)} \\ &= \tfrac{\phi_H(\lambda)}{f_H(\lambda)}, \end{aligned} \tag{3.3}$$

where $\phi_H(\lambda)$ is a polynomial of degree $n-1$ in λ and $f_H(\lambda)$ is the characteristic polynomial of the adjacency matrix of the graph H. If the greatest common divisor of these two polynomials is not a constant, then the above polynomial ratio can be further simplified. Let the simplified form be

$$\chi_H(\lambda) = \frac{Q_{d-1}(\lambda)}{P_d(\lambda)}, \tag{3.4}$$

where $Q_{d-1}(\lambda)$ and $P_d(\lambda)$ are polynomials of degrees $d-1$ and d respectively and

$$GCD\left(\phi_H\left(\lambda\right), f_H\left(\lambda\right)\right) = R_{n-d}\left(\lambda\right)$$

is of degree $n-d$.

We now state some results from the theory of matrix algebra[6, 16, 19] which is needed in the proof of the main theorem.

Lemma 3.1. *If A and D are square matrices(need not be same order) and B and C are matrices with compatible orders, then the determinant of the following block matrix is given by*

$$Det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = Det\left(D\right) Det\left(A - BD^{-1}C\right), \tag{3.5}$$

provided D^{-1} exists.

Lemma 3.2. *Let A is an r^{th} order square matrix with eigenvalues $\{\alpha_i\}, 1 \leq i \leq r$ and B is an s^{th} order square matrix with eigenvalues $\{\beta_i\}, 1 \leq i \leq s$, then the eigenvalues of the square matrix $A \otimes B$ of order rs is given by all possible products $\{\alpha_i \beta_j\}$, for $1 \leq i \leq r$ and $1 \leq j \leq s$, which is rs in number.*

Lemma 3.3. *Let A, B and C be matrices with orders m, n, and p respectively, then,*

i. $A \otimes (B + C) = A \otimes B + A \otimes C$
ii. $Det\left(A \otimes B\right) = Det(A)^n Det(B)^m$
iii. $\left(A \otimes B\right)^{-1} = A^{-1} \otimes B^{-1}$.

Lemma 3.4. *If A, B, C and D are matrices, then* $\left(A \otimes B\right)\left(C \otimes D\right) = \left(AC\right) \otimes \left(BD\right), provided that the matrix products given above are possible.$

Lemma 3.5. *If $A = [a_{ij}]$ is a q^{th} order square matrix, then*

$$J_{pq} A J_{qp} = \left(\sum_{i=1}^{q}\sum_{j=1}^{q} a_{ij}\right) J_p, \tag{3.6}$$

where J_{pq} denotes the all one matrix of order $p \times q$ and $J_p = J_{pp}$.

Lemma 3.6. *Eigenvalues of the all one square matrix J_n are n and zero with multiplicity one and $(n-1)$ respectively. In other words,*

$$Det\left(\lambda I_n - J_n\right) = \lambda^{n-1}\left(\lambda - n\right). \tag{3.7}$$

Lemma 3.7. *For each* $i = 1, 2, 3, \cdots m$, *let* λ_i *be the eigenvalues of a matrix* A, *then for any real number* c, *the eigenvalues of* $A + cI_m$ *are* $\lambda_i + c$ *for* $i = 1, 2, 3, \cdots m$. $\lambda_i + c$; $1 \leq i \leq m$.

The following theorem gives the spectra of the new graph product in terms of the spectra of its component graphs and the coronal of the second graph.

Theorem 3.1. *Let* G *and* H *be two graphs of order* m *and* n *respectively. Then the characteristic polynomial of the graph product* $G \otimes H$ *is given by*

$$(3.8)\ f_{G\otimes H}(\lambda) = \lambda^{m(n-1)} R_{n-d}(\lambda)^m \prod_{i=1}^{m} \left(\lambda P_d(\lambda) - n\left(\gamma_i P_d(\lambda) + Q_{d-1}(\lambda)\right)\right),$$

where for $i = 1, 2, 3, \cdots m$, γ_i *are the eigenvalues of the graph* G *and* R_{n-d}, P_d *and* Q_{d-1} *are given by equation (3.4).*

Proof. Let A_G and A_H are the adjacency matrices of the graphs G and H respectively. Then, from equation (3.1) the characteristic polynomial of $G \otimes H$ is given by

$$\begin{aligned} f_{G\otimes H}(\lambda) &= Det\left(\lambda I_{2mn} - A_{G\otimes H}\right) \\ &= Det \begin{matrix} \lambda I_{mn} - A_G \otimes J_n & -I_m \otimes J_n \\ -I_m \otimes J_n & \lambda I_{mn} - I_m \otimes A_H \end{matrix} \\ &= Det\left(\lambda I_{mn} - I_m \otimes A_H\right) Det\Big[\left(\lambda I_{mn} - A_G \otimes J_n\right) \\ &- \left(I_m \otimes J_n\right)\left(\lambda I_{mn} - I_m \otimes A_H\right)^{-1}\left(I_m \otimes J_n\right)\Big], \end{aligned} \tag{3.9}$$

where we have used lemma 3.1 in the last equation.

Now consider,

$$\begin{aligned} &\left(I_m \otimes J_n\right)\left(\lambda I_{mn} - I_m \otimes A_H\right)^{-1}\left(I_m \otimes J_n\right) \\ &= \left(I_m \otimes J_n\right)\left[I_m \otimes \left(\lambda I_n - A_H\right)\right]^{-1}\left(I_m \otimes J_n\right) \\ &= \left(I_m \otimes J_n\right)\left[I_m \otimes \left(\lambda I_n - A_H\right)^{-1}\right]\left(I_m \otimes J_n\right) \\ &= \left[I_m \otimes J_n\left(\lambda I_n - A_H\right)^{-1}\right]\left(I_m \otimes J_n\right) \\ &= \left[I_m \otimes J_n\left(\lambda I_n - A_H\right)^{-1} J_n\right] \\ &= \left[I_m \otimes \chi_H(\lambda) J_n\right], \end{aligned}$$

where we have used lemma 3.3 in the second equation, lemma 3.4 in the third and fourth equations and lemma 3.5 in the fifth equation. Substituting this in equation (3.9) we get,

$$\begin{aligned}
f_{G\otimes H}(\lambda) &= Det\left(\lambda I_{mn} - I_m \otimes A_H\right) Det\Big[\left(\lambda I_{mn} - A_G \otimes J_n\right) - \left(I_m \otimes \chi_H(\lambda) J_n\right)\Big] \\
&= Det(I_m)^n Det\left(\lambda I_n - A_H\right)^m Det\Big[\lambda I_{mn} - \left(A_G + \chi_H(\lambda) I_m\right) \otimes J_n\Big] \\
&= f_H(\lambda)^m \prod\left(\lambda - \alpha_i\beta_j\right), \text{where the product is taken over all the} \\
&\quad \text{eigenvalues } \alpha_i\text{'s and } \beta_j\text{'s of } A_G + \chi_H(\lambda) I_m \text{ and } J_n \text{ respectively} \\
&= \lambda^{m(n-1)} f_H(\lambda)^m \textstyle\prod_{i=1}^m \left(\lambda - n\alpha_i\right) \\
&= \lambda^{m(n-1)} f_H(\lambda)^m \textstyle\prod_{i=1}^m \left[\lambda - n\left(\gamma_i + \chi_H(\lambda)\right)\right] \\
&= \lambda^{m(n-1)} \left(R_{n-d}(\lambda) P_d(\lambda)\right)^m \textstyle\prod_{i=1}^m \left[\lambda - n\left(\gamma_i + \frac{Q_{d-1}(\lambda)}{P_d(\lambda)}\right)\right] \\
&= \lambda^{m(n-1)} R_{n-d}(\lambda)^m \textstyle\prod_{i=1}^m \Big[\lambda P_d(\lambda) - n\Big(\gamma_i P_d(\lambda) + Q_{d-1}(\lambda)\Big)\Big],
\end{aligned}$$

where we have used lemma 3.3 in the second equation, lemma 3.2 in the third equation, lemma 3.6 in the fourth equation, lemma 3.7 in the fifth equation and equation (3.4) in the sixth equation. Here, the first factor is an $m(n-1)$ degree polynomial, second factor is an $m(n-d)$ degree polynomial and each factor in the final product is a $d+1$ degree polynomial. So, the degree of the polynomial $f_{G\otimes H}(\lambda)$ is $m(n-1)+m(n-d)+m(d+1) = 2mn$ and its $2mn$ roots gives the spectrum of the graph product $G \otimes H$. □

4. Applications of the new graph product

In this section we discuss some applications of the graph product $G \otimes H$ and its spectrum given in theorem 3.1. It is possible to generate sequences of cospectral graphs and non-cospectral equienergetic graphs using the new graph product. The following theorem gives methods for generating cospectral graphs from the known cospectral graphs in two different ways.

Theorem 4.1. *Let G_1 and G_2 be two cospectral graphs, then for any graph H, the graph product $G_1 \otimes H$ and $G_2 \otimes H$ are cospectral graphs. Also, let G be any graph and let H_1 and H_2 be cospectral graphs with same coronals, then the graphs $G \otimes H_1$ and $G \otimes H_2$ are cospectral graphs.*

Proof. Since the graphs G_1 and G_2 are cospectral graphs, both are having same eigenvalues γ_i with equal multiplicities for each values. Then, from equations (3.4) and (3.8), it follows that the graphs $G_1 \otimes H$ and $G_2 \otimes H$ are also having same eigenvalues and hence they are cospectral graphs.

Now, consider the second method. Suppose H_1 and H_2 are cospectral graphs with same coronals. In this case, from equation (3.4) it follows that the polynomials $R_{n-d}(\lambda)$, $P_d(\lambda)$ and $Q_d(\lambda)$ for both H_1 and H_2 are same. Then, from equation (3.8) it follows that $G \otimes H_1$ and $G \otimes H_2$ are cospectral graphs. □

In the next theorem we describe a method for constructing non-cospectral equienergetic graphs using the new graph product.

Theorem 4.2. *Let H_1 and H_2 be non-cospectral equienergetic graphs with same coronals, then for any arbitrary graph G, the graphs $G \otimes H_1$ and $G \otimes H_2$ are also non-cospectral equienergetic graphs.*

Proof. Since H_1 and H_2 are having same coronals, the polynomials $P_d(\lambda)$ and $Q_d(\lambda)$ for both H_1 and H_2 given by equation (3.4) are equal. Let

$$f_{H_1}(\lambda) = R_{n-d}(\lambda)\, P_d(\lambda) \tag{4.1}$$

and

$$f_{H_2}(\lambda) = R'_{n-d}(\lambda)\, P_d(\lambda) \tag{4.2}$$

Clearly $R'_{n-d}(\lambda) \neq R_{n-d}(\lambda)$, as the graphs H_1 and H_2 are non-cospectral. Then the characteristic polynomial of $G \otimes H_1$ is

$$f_{G\otimes H_1}(\lambda) = \lambda^{m(n-1)} R_{n-d}(\lambda)^m \prod_{i=1}^{m} \left(\lambda P_d(\lambda) - n\left(\gamma_i P_d(\lambda) + Q_{d-1}(\lambda)\right)\right) \tag{4.3}$$

and the characteristic polynomial of $G \otimes H_2$ is

$$f_{G\otimes H_2}(\lambda) = \lambda^{m(n-1)} R'_{n-d}(\lambda)^m \prod_{i=1}^{m} \left(\lambda P_d(\lambda) - n\left(\gamma_i P_d(\lambda) + Q_{d-1}(\lambda)\right)\right). \tag{4.4}$$

Let the roots of the polynomial $P_d(\lambda)$ be $\delta_1, \delta_2, \cdots \delta_d$ and let the roots of $R_{n-d}(\lambda)$ and $R_{n-d}{}'(\lambda)$ be $\alpha_1, \alpha_2, \cdots \alpha_{n-d}$ and $\beta_1, \beta_2, \cdots \beta_{n-d}$ respectively. Since the graphs H_1 and H_2 are non-cospectral equienergetic, we have,

$$\sum_{i=1}^{d} |\delta_i| + \sum_{i=1}^{n-d} |\alpha_i| = \mathcal{E}_{H_1} = \mathcal{E}_{H_2} = \sum_{i=1}^{d} |\delta_i| + \sum_{i=1}^{n-d} |\beta_i|. \tag{4.5}$$

Hence

$$\sum_{i=1}^{n-d} |\alpha_i| = \sum_{i=1}^{n-d} |\beta_i|. \tag{4.6}$$

The product factor in equations (4.3) and (4.4) are same and it is an $m(d+1)$ degree polynomial. Let its roots be η_i, for $i = 1, 2, \cdots, m(d+1)$. From the characteristic polynomial (4.3), the energy of the graph $G \otimes H_1$ is obtained as

$$\mathcal{E}_{G\otimes H_1} = m \sum_{i=1}^{n-d} |\alpha_i| + \sum_{i=1}^{m(d+1)} |\eta_i| \tag{4.7}$$

and from the characteristic polynomial (4.4), the energy of the graph $G \otimes H_2$ is obtained as

$$\mathcal{E}_{G\otimes H_2} = m \sum_{i=1}^{n-d} |\beta_i| + \sum_{i=1}^{m(d+1)} |\eta_i| \, . \tag{4.8}$$

Then, from the above two equations and the equation (4.6), we get $\mathcal{E}_{G\otimes H_1} = \mathcal{E}_{G\otimes H_2}$. Since the graphs H_1 and H_2 are non-cospectral graphs, it follows that $G \otimes H_1$ and $G \otimes H_2$ are non-cospectral equienergetic graphs. □

The above theorem can be used to generate families of non-cospectral equienergetic graphs from a given pair of non-cospectral equienergetic graphs with same coronals. The following corollary gives the existence of such non trivial non-cospectral equienergetic graphs with same coronals.

Corollary 4.1. *Let H_1 and H_2 be non-cospectral equienergetic r-regular graphs, then for any arbitrary graph G, the graphs $G \otimes H_1$ and $G \otimes H_2$ are also non-cospectral equienergetic graphs.*

Proof. Suppose that H_1 and H_2 are non-cospectral equienergetic r-regular graphs. It is known that the coronal of any two r-regular graphs are equal[15]. Hence, by theorem 4.2, the graphs $G \otimes H_1$ and $G \otimes H_2$ are non-cospectral equienergetic graphs. □

5. Concluding remarks

We have defined a new graph product and its characteristic polynomial is determined in terms of the characteristic polynomials of constituent graphs. As an application of the new product, methods of generating families of non-cospectral equienergetic graphs are discussed in this paper. Future research directions include finding other structural and spectral properties of new graph product and its applications in different areas of science and social sciences.

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Subin P. Joseph
Department of Mathematics,
Government Engineering College Wayanad,
Thalapuzha, Mananthavady,
Kerala,
India-670644
India
e-mail: subinpj@gecwyd.ac.in