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Properties of nearly S-paracompact spaces

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Abstract

We study some basic properties of a nearly S-paracompact space and its characterizations under certain hypotheses about space. We es- tablish relationships between this class of spaces and other well-known spaces. Also, we analyze the invariance of nearly S-paracompactness under direct and inverse images of some types of functions.

Keywords: *Semi-open set, nearly S-paracompact space, semiregular space, pre-semi-open function, almost closed function.*

Subjclass [2020]: Primary 54A05, 54D20; Secondary 54F65, 54G05, 54C10.

1. Introduction

In 1969, Singal and Arya [21] introduced the class of nearly paracompact spaces and characterized them by the property that "every regular open covering of the space admits a locally finite open refinement". Later, in 2006, Al-Zoubi [3] introduced a weaker version of paracompactness called S-paracompact space, which are spaces in which every open cover of the space has a locally finite semi-open refinement. Recently, Sanabria et al. [17] have studied the invariance of nearly S-paracompact spaces under direct and inverse images of open, perfect and regular perfect functions and have analyzed the behavior of such spaces through the sum and topological product. In this paper, we study some basic properties of nearly S-paracompact spaces, which are defined by the property that every regular open cover of the space has a locally finite semi-open refinement. This class contains the S-paracompact and nearly paracompact spaces. In Section 3, we establish the relationships between nearly S-paracompact spaces and nearly paracompact spaces, S-paracompact spaces, almost paracompact spaces, nearly compact spaces, countably S-closed spaces and S-closed spaces. In Section 4, we study the invariance of nearly S-paracompact spaces under direct and inverse images of pre-semi-open, almost completely continuous, almost closed, almost continuous, regular open and s-continuous functions.

2. Preliminaries

Throughout this paper, (X, τ) always means a topological space on which no separation axioms are assumed unless explicitly stated. If A is a subset of (X, τ), then we denote the closure of A and the interior of A by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subset A of (X, τ) is said to be semi-open [11] if there exists $U \in \tau$ such that $U \subset A \subset \mathrm{Cl}(U)$. This is equivalent to say that $A \subset \mathrm{Cl}(\mathrm{Int}(A))$. A subset A of (X, τ) is called regular open (resp. α-open [13]) if $A = \mathrm{Int}(\mathrm{Cl}(A))$ (resp. $A \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$). The complement of a semi-open (resp. regular open) set is called a semi-closed (resp. regular closed) set. The semi-closure of A, denoted by $\mathrm{sCl}(A)$, is defined by the intersection of all semi-closed sets containing A. The collection of all semi-open (resp. regular open, α-open) sets of a topological space (X, τ) is denoted by $\mathrm{SO}(X, \tau)$ (resp. $\mathrm{RO}(X, \tau)$, τ^α). It is known that τ^α forms a topology on X such that $\tau \subset \tau^\alpha \subset \mathrm{SO}(X, \tau)$, $\mathrm{SO}(X, \tau^\alpha) = \mathrm{SO}(X, \tau)$ and $\mathrm{RC}(X, \tau^\alpha) = \mathrm{RC}(X, \tau)$ (see [9] and [13]). Also, it is well-known that $\mathrm{RO}(X, \tau)$ is a base for a topology τ_s on X such that $\tau_s \subset \tau$. The space

(X, τ_s) is called the semiregularization of (X, τ) and if $\tau = \tau_s$ then (X, τ) is said to be semiregular. A space (X, τ) is said to be almost regular [19], if for each regular closed set F and any point $p \notin F$, there exist disjoint open sets U and V such that $p \in U$ and $F \subset V$. Note that semiregularity and almost regularity are independent [19]. A space (X, τ) is said to be extremally disconnected (briefly e.d.) if the closure of every open set in (X, τ) is open. A collection $\mathcal{V}$ of subsets of a space (X, τ) is said to be locally finite (resp. s-locally finite [2]), if for each $x \in X$ there exists $U_x \in \tau$ (resp. $U_x \in \mathrm{SO}(X, \tau)$) containing x and U_x intersects at most finitely many members of $\mathcal{V}$. A space (X, τ) is said to be paracompact [7] (resp. S-paracompact [3]), if every open cover of X has a locally finite open (resp. semi-open) refinement which covers to X (we do not require a refinement to be a cover). A space (X, τ) is said to be almost paracompact [20] if every open cover $\mathcal{U}$ of X has a locally finite open refinement $\mathcal{V}$ such that the collection $\{\mathrm{Cl}(V) : V \in \mathcal{V}\}$ is a cover of X. A space (X, τ) is said to be nearly paracompact [21] if every regular open cover $\mathcal{U}$ of X has a locally finite open refinement which covers to X.

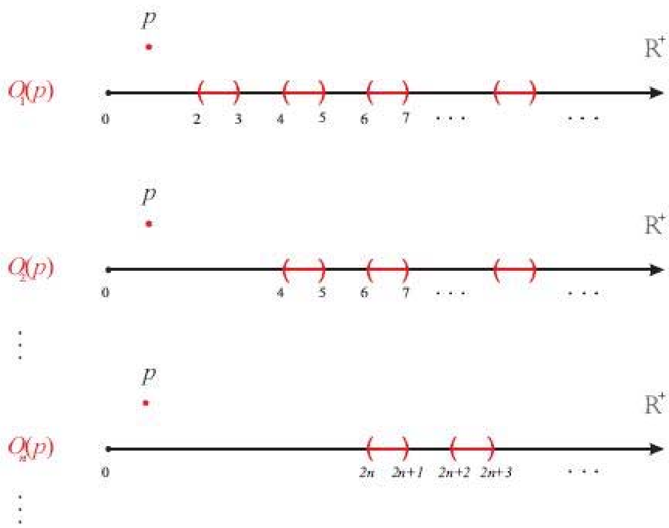
3. Nearly S-paracompact spaces

In this section, we study some basic properties of nearly S-paracompact spaces and establish relations between this class of spaces and other known spaces.

Definition 3.1. A space (X, τ) is said to be nearly S-paracompact [17], if every regular open cover $\mathcal{U}$ of X has a locally finite semi-open refinement $\mathcal{V}$ such that $X = \bigcup\{V : V \in \mathcal{V}\}$.

Clearly, every S-paracompact space is nearly S-paracompact and every nearly paracompact space is nearly S-paracompact, but the converse is not necessarily true as we can see in the following two examples.

Example 3.2. There exists a nearly S-paracompact space which is not nearly paracompact. *Let $X = \mathbf{R}^+ \cup \{p\}$, where $\mathbf{R}^+ = [0, +\infty)$ and $p \notin \mathbf{R}^+$. Topologize X as follows: $\mathbf{R}^+$ has the usual topology and is an open subspace of X; a basic neighborhood of $p \notin X$ takes the form* $O_n(p) = \{p\} \cup \bigcup_{i=n}^{\infty} (2i, 2i+1)$, *where $n \in \mathbf{N}$.*



Basic neighborhoods of the point $p \notin \mathbf{R}^+$.

It was shown in [12, Example 2.3] that X is an Hausdorff S-paracompact space and hence nearly S-paracompact, but it is not regular. Now, by [3, Corollary 2.3], it follows that X is semiregular and since X is not regular, by [19, Theorem 3.1], we have that it is not almost-regular. Consequently, by [21, Theorem 2.1], we conclude that X is not nearly paracompact.

Example 3.3. There exists a nearly S-paracompact space which is not S-paracompact. *Consider the half-disc topology (see [24, Example 78]). It is known that this topology is Hausdorff, but it is neither semiregular nor paracompact, and since its semiregularization is the open upper half-plane with the Euclidean topology which is paracompact, then we conclude that the half-disc topology is nearly paracompact and hence nearly S-paracompact. On the other hand, by [3, Corollary 2.3], we conclude that the half-disc topology is not S-paracompact.*

Remark 3.4. *Note that the two examples above show the independence between the notions of nearly paracompact and S-paracompact spaces.*

It is well known that every nearly paracompact (resp. S-paracompact) space is almost paracompact [21] (resp. [12]). Thus, it is natural that we ask the following question: There is some relationship between the notions of nearly S-paracompact and almost paracompact spaces?

Theorem 3.5. *If (X, τ) is a nearly S-paracompact space, then it is almost paracompact.*

Proof. Let $\mathcal{U} = \{U_\gamma : \gamma \in \Gamma\}$ be an open cover of X. Then, the collection $\{\mathrm{Int}(\mathrm{Cl}(U_\gamma)) : \gamma \in \Gamma\}$ is a regular open cover of X. Since (X, τ) is nearly S-paracompact, then by [18, Lemma 1.3] (case $\mathcal{I} = \{\emptyset\}$), $\{\mathrm{Int}(\mathrm{Cl}(U_\gamma)) : \gamma \in \Gamma\}$ has a locally finite precise semi-open refinement $\mathcal{V} = \{V_\gamma : \gamma \in \Gamma\}$ which is a cover of X. Now, for each $\gamma \in \Gamma$ there exists an open subset O_γ of X such that $O_\gamma \subset V_\gamma \subset \mathrm{Cl}(O_\gamma)$, since V_γ is semi-open. Hence, for each $\gamma \in \Gamma$, $O_\gamma \subset V_\gamma \subset \mathrm{Int}(\mathrm{Cl}(U_\gamma)) \subset \mathrm{Cl}(U_\gamma)$. For each $\gamma \in \Gamma$, let $H_\gamma = O_\gamma - [\mathrm{Cl}(U_\gamma) - U_\gamma]$. Then $\{H_\gamma : \gamma \in \Gamma\}$ is a locally finite collection of open sets which refines $\mathcal{U}$ and the collection $\{\mathrm{Cl}(H_\gamma) : \gamma \in \Gamma\}$ is a cover of X. Hence (X, τ) is almost paracompact. □

The converse of Theorem 3.5 is not necessarily true as we can see in the following example.

Example 3.6. There exists an almost paracompact space which is not nearly S-paracompact. *Let $X = \mathbf{R}^+ \cup \{p\} \cup \{q\}$, where $\mathbf{R}^+ = [0, +\infty)$, $p \notin \mathbf{R}^+$, $q \notin \mathbf{R}^+$ and $p \neq q$. Topologize X as follows: $\mathbf{R}^+$ has the usual topology and is an open subspace of X; a basic neighborhood of $p \notin X$ takes the form $O_n(p) = \{p\} \cup \bigcup_{i=n}^{\infty} (2i, 2i+1)$, where $n \in \mathbf{N}$; a basic neighborhood of $q \notin X$ takes the form $O_m(q) = \{q\} \cup \bigcup_{i=m}^{\infty} (2i-1, 2i)$, where $m \in \mathbf{N}$. It was shown in [12, Example 2.4] that X is an Hausdorff almost paracompact space. Now, let us consider the regular open cover $\mathcal{W} = \{[0,1)\} \cup \bigcup_{i=1}^{\infty} \left\{\left(i - \frac{1}{3}, i + \frac{1}{3}\right)\right\} \cup \{O_1(p)\} \cup \{O_1(q)\}$ of X. Then, by definition of the topology on X, every semi-open refinement of $\mathcal{W}$ is not locally finite at the points p or q. This shows that X is not nearly S-paracompact.*

Theorem 3.7. *A semi-regular space (X, τ) is nearly S-paracompact if and only if it is S-paracompact.*

Proof. Let $\mathcal{U}$ be any open cover of X. For each $x \in X$ there exists $U_x \in \mathcal{U}$ such that $x \in U_x$ and, since (X, τ) is semi-regular, there exists a regular open set V_x such that $x \in V_x \subset U_x$. Now, the collection $\mathcal{V} = \{V_x : x \in X\}$ is a regular open cover of X and hence has a locally finite semi-open refinement $\mathcal{W}$ which covers to X. Then, $\mathcal{W}$ is a locally finite semi-open refinement of $\mathcal{U}$, it follows that (X, τ) is S-paracompact. □

Theorem 3.8. *Let (X, τ) be an almost regular space. Then (X, τ) is nearly S-paracompact if and only if every regular open cover $\mathcal{U}$ of X has a locally finite regular closed refinement $\mathcal{V}$ which covers to X.*

Proof. The sufficiency follows directly from the fact that every regular closed set is semi-open. To show necessity, let $\mathcal{U}$ be a regular open cover of X. For each $x \in X$ there exists $U_x \in \mathcal{U}$ such that $x \in U_x$ and, since (X, τ) is an almost regular space, then there exists a regular open set W_x such that $x \in W_x \subset \mathrm{Cl}(W_x) \subset U_x$. Thus, the collection $\mathcal{W} = \{W_x : x \in X\}$ is a regular open cover of X and, by hypothesis, $\mathcal{W}$ has a locally finite semi-open refinement $\mathcal{W}' = \{G_\gamma : \gamma \in \Gamma\}$ which covers to X. Observe that $\mathcal{V} = \{\mathrm{Cl}(G_\gamma) : \gamma \in \Gamma\}$ is a locally finite collection of regular closed sets. Since $\mathcal{W}'$ refines $\mathcal{W}$, for each $\gamma \in \Gamma$ there exists $W_x \in \mathcal{W}$ such that $G_\gamma \subseteq W_x$, and so, for each $\gamma \in \Gamma$ we have $G_\gamma \subset W_x \subset \mathrm{Cl}(W_x) \subset U_x$ for some $U_x \in \mathcal{U}$. Therefore, for each $\gamma \in \Gamma$, $\mathrm{Cl}(G_\gamma) \subset \mathrm{Cl}(W_x) \subset U_x$ for some $U_x \in \mathcal{U}$. Thus, the collection $\mathcal{V} = \{\mathrm{Cl}(G_\gamma) : \gamma \in \Gamma\}$ is a refinement of $\mathcal{U}$. Finally, note that $X = \bigcup\{G_\gamma : \gamma \in \Gamma\} \subset \bigcup\{\mathrm{Cl}(G_\gamma) : \gamma \in \Gamma\}$, it follows that the collection $\mathcal{V}$ is a cover of X. □

Corollary 3.9. *For an almost-regular space (X, τ), the following are equivalent:*

1. *(X, τ) is nearly S-paracompact.*
2. *(X, τ) is nearly paracompact.*
3. *(X, τ) is almost paracompact.*
4. *Every regular open cover $\mathcal{U}$ of X has a locally finite regular closed refinement $\mathcal{V}$ which covers to X.*

Proof. This is an immediate consequence of Theorems 3.5, 3.8 and [21, Theorem 1.5]. □

Corollary 3.10. *For an regular space (X, τ), the following are equivalent:*

1. *(X, τ) is nearly S-paracompact.*
2. *(X, τ) is paracompact.*
3. *(X, τ) is nearly paracompact.*
4. *(X, τ) is almost paracompact.*
5. *(X, τ) is S-paracompact.*
6. *Every regular open cover $\mathcal{U}$ of X has a locally finite regular closed refinement $\mathcal{V}$ which covers to X.*

Proof. Since every regular space is almost regular, the equivalence between (1), (3), (4) and (6) follows from Corollary 3.9. The equivalence between (2), (4) and (5) follows from the results shown in [20] and [12]. □

Theorem 3.11. *Let (X,τ) be a space and consider the following statements:*

1. *(X,τ) is Hausdorff nearly S-paracompact.*
2. *For every regular closed subset A of X and every $x \notin A$, there exist disjoint sets $U \in \mathrm{RO}(X,\tau)$ and $V \in \mathrm{SO}(X,\tau)$ such that $x \in U$ and $A \subset V$.*
3. *For every $G \in \mathrm{RO}(X,\tau)$ and every $x \in G$, there exists $U \in \mathrm{RO}(X,\tau)$ such that $x \in U \subset \mathrm{sCl}(U) \subset G$.*

Then, the following implications hold (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3).

Proof. (1)$\Rightarrow$(2): Let x be any point of X and A be a regular closed subset of X such that $x \notin A$. Since X is a Hausdorff space, for each $y \in A$ there exists an open set W_y such that $y \in W_y$ and $x \notin \mathrm{Cl}(W_y)$. Observe that the collection $\mathcal{W} = \{\mathrm{Int}(\mathrm{Cl}(W_y) : y \in A\} \cup \{X - A\}$ is a regular open cover of X. Since (X,τ) is nearly S-paracompact, by [18, Lemma 1.3] (case $\mathcal{I} = \{\emptyset\}$), $\mathcal{W}$ has a locally finite precise semi-open refinement $\mathcal{M} = \{M_y : y \in A\} \cup \{G\}$ such that $M_y \subseteq \mathrm{Int}(\mathrm{Cl}(W_y))$ for each $y \in A$, $G \subset X - A$ and $\mathcal{M}$ is a cover of X. Note that if $y \in A$, then $\mathrm{Cl}(M_y) \subset \mathrm{Cl}(W_y)$, hence $x \notin \mathrm{Cl}(M_y)$. Let $V = \bigcup\{M_y : y \in A\}$, then V is a semi-open set such that $A \subset V$. Since $\mathcal{M}$ is locally finite, we have $\mathrm{Cl}(V) = \mathrm{Cl}(\bigcup\{M_y : y \in A\}) = \bigcup\{\mathrm{Cl}(M_y) : y \in A\}$. Therefore, if $U = X - \mathrm{Cl}(V)$ then U is a regular open set such that $x \in U$ and $U \cap V = (X - \mathrm{Cl}(V)) \cap V = \emptyset$.
(2)$\Rightarrow$(3): Let $G \in \mathrm{RO}(X,\tau)$ and $x \in G$, then $A = X - G$ is a regular closed set and $x \notin A$. By (2), there exist disjoint sets $U \in \mathrm{RO}(X,\tau)$ and $V \in \mathrm{SO}(X,\tau)$ such that $x \in U$ and $A \subset V$. Hence $x \in U \subset \mathrm{sCl}(U) \subset G$.
(3)$\Rightarrow$(2): Let A be a regular closed subset of X and let $x \in X$ such that $x \notin A$, then $G = X - A$ is a regular open subset of X with $x \in G$. By (3), there exists $U \in \mathrm{RO}(X,\tau)$ such that $x \in U$ and $\mathrm{sCl}(U) \subset G$. Thus, $V = X - \mathrm{sCl}(U)$ is a semi-open subset of X such that $U \cap V = \emptyset$ and $A \subset V$. □

Corollary 3.12. *If (X,τ) is an e.d. Hausdorff nearly S-paracompact space, then (X,τ) is almost regular. Consequently, (X,τ) is nearly paracompact.*

Proof. Let $U \in \mathrm{RO}(X, \tau)$ and $x \in U$. By Theorem 3.11, there exists $V \in \mathrm{RO}(X, \tau)$ such that $x \in V$ and $\mathrm{sCl}(V) \subset U$. Since X is e.d., by [16, Lemma 4.1], we have $\mathrm{sCl}(V) = \mathrm{Cl}(V)$ and so, there exists $V \in \mathrm{RO}(X, \tau)$ such that $x \in V$ and $\mathrm{Cl}(V) \subset U$. Now by [19, Theorem 2.2], it follows that (X, τ) is an almost regular space. From Corollary 3.12 it follows that (X, τ) is nearly paracompact. □

Recall that a space (X, τ) is said to be S-closed [25] (resp. countably S-closed [8]), if every semi-open (resp. countable semi-open) cover of X has a finite subfamily such that the closures of whose members cover X. Al-Zoubi [2] has shown that, in a countably S-closed space, every s-locally finite collection of semi-open sets is finite. According to [22], a space (X, τ) is said to be nearly compact, if every regular open cover of X has a finite subcover. Clearly, every nearly compact space is nearly paracompact and hence nearly S-paracompact. In the following theorem, we shows that if the space is countably S-closed, then the converse also is true.

Theorem 3.13. *If (X, τ) is a nearly S-paracompact countably S-closed space, then (X, τ) is nearly compact.*

Proof. Let $\mathcal{U} = \{U_\gamma : \gamma \in \Gamma\}$ be a regular open cover of X. Since (X, τ) is nearly S-paracompact, $\mathcal{U}$ has a locally finite semi-open refinement $\mathcal{V} = \{V_\mu : \mu \in \Delta\}$ which covers to X. Then, $\mathcal{V}$ is a s-locally finite collection of semi-open sets and since (X, τ) is countably S-closed, we have by [2, Theorem 2.7], that $\mathcal{V}$ is finite. Without loss of generality, assume $\mathcal{V} = \{V_{\mu_i} : i = 1, 2, \ldots, n\}$. Now, since $\mathcal{V}$ refines $\mathcal{U}$, then for each $i = 1, 2, \ldots, n$, there exist $U_{\gamma(i)} \in \mathcal{U}$ such that $V_{\mu_i} \subset U_{\gamma(i)}$. Thus, $X = \bigcup\{V_\mu : \mu \in \Delta\} = \bigcup\{V_{\mu_i} : i = 1, \ldots, n\} \subset \bigcup\{U_{\gamma(i)} : i = 1, \ldots, n\}$, it follows that (X, τ) is nearly compact. □

Corollary 3.14. *Let (X, τ) an e.d. space. The following are equivalent:*

1. *(X, τ) is nearly S-paracompact and S-closed.*
2. *(X, τ) is nearly compact.*

Proof. (1)⇒(2): Since every S-closed space is countably S-closed, Theorem 3.13 ensures that (X, τ) is nearly compact.
(2)⇒(1): Let $\mathcal{U} = \{U_\gamma : \gamma \in \Gamma\}$ a semi-open cover of X. Then, for every $\gamma \in \Gamma$ there exists an open set V_γ such that $V_\gamma \subset U_\gamma \subset \mathrm{Cl}(V_\gamma)$ and, as (X, τ) is e.d., $\mathrm{Cl}(V_\gamma) \in \tau$ for every $\gamma \in \Gamma$. Hence, $X = \bigcup\limits_{\gamma \in \Gamma} U_\gamma \subset \bigcup\limits_{\gamma \in \Gamma} \mathrm{Cl}(V_\gamma)$

and so, the collection $\mathcal{V} = \{\mathrm{Cl}(V_\gamma) : \gamma \in \Gamma\}$ is an open cover of X. Now, since (X, τ) is nearly compact, there exists a finite subcollection $\{\mathrm{Cl}(V_{\gamma_i}) : i = 1, 2, \ldots, n\} \subset \mathcal{V}$ such that $X = \bigcup_{i=1}^{n} \mathrm{Cl}(V_{\gamma_i}) = \bigcup_{i=1}^{n} \mathrm{Cl}(U_{\gamma_i})$. This shows that (X, τ) is S-closed. Finally, since every nearly compact space is nearly paracompact, it follows that (X, τ) is nearly S-paracompact. □

Theorem 3.15. *(X, τ^α) is nearly S-paracompact if and only if (X, τ) is nearly S-paracompact.*

Proof. Follows from the facts that $\mathrm{RO}(X, \tau^\alpha) = \mathrm{RO}(X, \tau)$ and $\mathrm{SO}(X, \tau^\alpha) = \mathrm{SO}(X, \tau)$. □

Given a space (X, τ), we denote by τ_{SO} the topology on X which has $\mathrm{SO}(X, \tau)$ as a subbase. It is well known that the collection $\mathrm{SO}(X, \tau)$ is a topology on X if and only if (X, τ) is e.d. [13], in this case $\tau_{\mathrm{SO}} = \mathrm{SO}(X, \tau)$.

Corollary 3.16. *Let (X, τ) an e.d. space. If (X, τ_{SO}) is nearly S-paracompact if and only if (X, τ) is nearly S-paracompact.*

Proof. Follows from Teorema 3.15 and the fact that in an e.d. space is satisfied that $\tau^\alpha = \mathrm{SO}(X, \tau) = \tau_{\mathrm{SO}}$. □

4. Invariance under direct and inverse images

Next, we analyze the invariance of nearly S-paracompact spaces under direct and inverse images of some types of functions. For this purpose, we consider that X and Y are sets that are endowed with respective topologies. Recall that a function $f : X \to Y$ is called completely continuous [4] (resp. almost completely continuous [10]) if $f^{-1}(O)$ is a regular open subset of X for every open (resp. regular open) subset O de Y. Also, $f : X \to Y$ is called pre-semi-open [6] (resp. almost closed [23]) if $f(O)$ is a semi-open (resp. closed) subset of Y for every semi-open (resp. regular closed)subset O of X.

Lemma 4.1. *[15] Let $f : X \to Y$ be almost closed surjection with N-closed point inverse. If $\{O_\omega : \omega \in \Omega\}$ is a locally finite open cover of X, then $\{f(O_\omega) : \omega \in \Omega\}$ is a locally finite cover of Y.*

Theorem 4.2. *Let $f : X \to Y$ be a pre-semi-open, almost completely continuous and almost closed surjective function with N-closed point inverse. If X is nearly S-paracompact, then Y is also nearly S-paracompact.*

Proof. Suppose that X is nearly S-paracompact and let $\mathcal{O} = \{O_\omega : \omega \in \Omega\}$ be a regular open cover of Y. Since f is almost completely continuous, $f^{-1}(\mathcal{O}) = \{f^{-1}(O_\omega) : \omega \in \Omega\}$ is a regular open cover of X. Thus, there exists a locally finite semi-open refinement $\mathcal{V} = \{V_\gamma : \gamma \in \Gamma\}$ of $\mathcal{U}$ which covers X. Since f is pre-semi-open and surjective, $f(\mathcal{V}) = \{f(V_\gamma) : \gamma \in \Gamma\}$ is a semi-open refinement of $\mathcal{O}$ which covers Y. Finally, by Lemma 4.1, we get that $f(\mathcal{V})$ is locally finite in Y and the proof ends. $\square$

As compact sets are N-closed and closed functions are almost closed, we have the following consequence of Theorem 4.2.

Corollary 4.3. *Let $f : X \to Y$ be a pre-semi-open, completely continuous and closed surjective function with compact point inverse. If X is nearly S-paracompact, then Y is nearly S-paracompact.*

According to [14], we say that a function $f : X \to Y$ is almost continuous if $f^{-1}(O)$ is an open subset of X for every regular open subset O of Y.

Theorem 4.4. *Let $f : X \to Y$ be a pre-semi-open, almost continuous and almost closed surjective function with N-closed point inverse. If X is paracompact, then Y is nearly S-paracompact.*

Proof. The proof is similar to that of Theorem 4.2. $\square$

Recall that a function $f : X \to Y$ is said to be s-continuous [5] or strongly semi-continuous [1] if $f^{-1}(O)$ is an open subset of X for every semi-open subset O of Y. Also, $f : X \to Y$ is called regular open [17] if $f(O)$ is a regular open subset of Y for every open subset O of X.

Theorem 4.5. *Let $f : X \to Y$ be a regular open, s-continuous bijective function. If Y is nearly S-paracompact, then X is nearly paracompact and hence, nearly S-paracompact.*

Proof. Let $\mathcal{U} = \{U_\gamma : \gamma \in \Gamma\}$ be a regular open cover of X. Since f is regular open and surjective, $\mathcal{U}' = \{f(U_\gamma) : \gamma \in \Gamma\}$ is a regular open cover of Y. Due to the nearly S-paracompactness of Y, this regular open cover has a locally finite precise semi-open refinement $\mathcal{V} = \{V_\gamma : \gamma \in \Gamma\}$

which covers Y. Then, the collection $\mathcal{V}' = \{f^{-1}(V_\gamma) : \gamma \in \Gamma\}$ is a open cover of X, because f is s-continuous. Let $f^{-1}(V_\gamma) \in \mathcal{V}'$. Since $\mathcal{V}$ refines $\mathcal{U}'$, there exists $f(U_\gamma) \in \mathcal{U}'$ such that $V_\gamma \subseteq f(U_\gamma)$, which implies that $f^{-1}(V_\gamma) \subseteq f^{-1}(f(U_\gamma)) = U_\gamma$, because f is injective. Thus, $\mathcal{V}'$ refines $\mathcal{U}$. It remains to be seen that $\mathcal{V}'$ is locally finite in X. If $x \in X$, then $y = f(x) \in Y$ and, as the collection $\mathcal{V}$ is locally finite in Y, there exists an open set $O_y \subset Y$ such that $y \in O_y$ and $\{\gamma \in \Gamma : V_\gamma \cap O_y \neq \emptyset\}$ is a finite set. Note that $O_x = f^{-1}(O_y)$ is an open subset of X such that $x \in O_x$ and, $O_x \cap f^{-1}(V_\gamma) = f^{-1}(O_y) \cap f^{-1}(V_\gamma) = f^{-1}(O_y \cap V_\gamma) \neq \emptyset$ if and only if $V_\gamma \cap O_y \neq \emptyset$, so O_x can intercept only a finite number of elements of $\mathcal{V}'$, so $\mathcal{V}'$ is locally finite. This shows that X is nearly paracompact. □

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