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An optimization model for fuzzy nonlinear programming with Beale's conditions using trapezoidal membership functions

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Abstract

Non-linear Programming (NLP) is an optimization technique for determining the optimum solution to a broad range of research issues. Many times, the objective function is nonlinear, owing to various economic behaviors such as demand, cost, and many others. Since the appearance of Kuhn and Tucker's fundamental theoretical work, a general NLP problem can be resolved using many methods to find the optimum results. This article tackles the challenge of nonlinear programming (NLP) problems with uncertainty in inequality constraints. Traditional, "crisp" NLP methods might not be ideal when dealing with imprecise or subjective data. Here, we propose a fuzzy mathematical model that incorporates Beale's condition to handle such NLPPs. Furthermore, the model demonstrates how quadratic programming problems can be solved using membership functions(MF's). This leads to more realistic and robust solutions. The model unfolds in three stages: Mathematical Formulation: Establishing the fuzzy NLP framework with Beale's condition and membership functions. Computational Procedures: Outlining algorithms for solving fuzzy NLP problems based on MF's and a robust ranking index. Numerical Illustration: Applying the model to a specific case study and comparing results from both approaches. Through comprehensive analysis, we demonstrate the model's ability to find optimal solutions while considering vagueness and uncertainty in NLPPs. This opens door for more adaptable and realistic optimization in various problem domains.

Keywords: *Fuzzy Nonlinear Programming, Beale's Condition, Membership Functions, Uncertainty, Robust Optimization.*

MSC (2020): *90C30, 90C20, 90C70, 90C90.*

1. Introduction

The objective function and constraints were both linear in the decision variable in Linear Programming (LP). Whereas this linear relationship is represented in several practical applications, also the relationship between the parameters and objective function might have nonlinear relation in many real-time applications. The computational work involved in solving the problems which require the application of knowledge of differential calculus. LP also assumes that the cost of production or the contribution of profit units or problems need not differ over the implementation period and different production levels. As a result, the issue is simplified on the assumption. Though, the profit and resource requirements of competing aspirants in the real world will differ at different production levels. The standard approach appears to replace the estimated relation and view the specified problem as an inhomogeneous model of the ideal problem to approximate the nonlinear relationship. However, in such a case, the conclusion will never be appropriate for the specific situation or it may reveal options that do not provide the appropriate optimality. There is no renowned algorithm for solving a specified general NLPP effectively and efficiently. A model that fits the data well in one issue may not work well in another. These are one of the motivating factors why all NLPPs cannot be clustered together. To estimate the difficulty of the methodology, differentiate between the factors that make LPP more convincing and those that consider an NLPP more complicated. The method used to solve LPP is based upon the principle that optimal solutions are to be located at the extreme ends of the convex polyhedron. This indicates that we should restrict our search to vertexes, which can be achieved in a limited number of iterations. However, in NLPP, the optimum value could be everywhere along the feasible region's boundaries or anywhere even inside the feasible region. A linear relation between most of the model parameters seems to be very accurately adaptable to linear algebraic transformation, although non-linear relations tend to require extreme caution when mostly resulting in complicated issues. Because of the relationship in nonlinearity, there is a significant difference between local and global results. It further implies that any local optimum solution must be evaluated for optimality over the whole feasible region, rather than just at the extreme points, which is possible in an LPP. This furthermore suggests that simplex-type algorithms are unsuitable for solving NLPPs.

Fuzzy nonlinear programming (FNLP) problems arise when optimizing problems with nonlinear relationships between variables and uncertain, imprecise, or subjective data. Imagine trying to optimize something like resource allocation in a complex supply chain where demand is based on forecasts and market trends, rather than definitive numbers. In such cases, using traditional, "crisp" optimization techniques (where everything is precise) might not be ideal. That's where FNLP comes in. It utilizes fuzzy set theory, which allows for representing and quantifying degrees of membership to sets instead of just being a member or not. This enables incorporating uncertainty and ambiguity into the optimization process, leading to more realistic and robust solutions.Here are some key characteristics of FNLP problems:

- Nonlinear objective function: The relationship between variables and the optimal solution is not linear, adding complexity to the optimization process.
- Fuzzy parameters: Values like costs, demands, or resource availability are not precise but represented by fuzzy sets with membership degrees.
- Fuzzy constraints: Restrictions on the variables involve some level of vagueness or ambiguity, unlike the well-defined constraints in traditional optimization.

While FNLP adds complexity compared to "crisp" optimization, it offers more realistic and adaptable solutions in situations with inherent uncertainty and imprecision. Nonlinear programming typically describes rather more significant challenges than linear programming. As such, even when some of the restrictions are linear and the cost function is nonlinear, this scenario is always complex and difficult. For instance, the set of feasible solutions may or may not be convex, and the optimum solution may be lying inside the feasible set, on its boundary, or its vertex. For the most part, the scientific programming issue manages the ideal use or distribution of constrained assets to meet the ideal goal. The fuzzy NLPP is valuable in taking care of issues due to the uncertain, emotional nature of the problematic definition or has a precise arrangement. In this case, an objective function must improve while working within certain constraints. [[1]-[2]] introduced the fuzzy theory and fuzzy rule-based decision-making, and the right decision is used in decision problems to attain the optimum result [3]. The verdicts are fuzzy in maximum real-life situations, and initial attempts at the choices are essential to formulate a suitable model or cases to be analyzed. Likewise, we introduce fuzziness in our models of such situations to suggest means of processing fuzzy information [4]. In linear programming solution that satisfies both the constraints and the objective function has been named an optimum solution. Accordingly, such a problem has an objective function, as well as variables, which include constraints and coefficients all are described in the form of fuzzy. If the objective or limitations are nonlinear, at that point, we think of it as a nonlinear programming problem. In this model to tackle such an optimization problem, a fuzzy mathematical model has been proposed to address the NLP with inequality constraints in terms of fuzziness using Beale's condition to find an optimum solution. In addition, the model suggests how quadratic programming problems have been demonstrated. However, this model represents the mathematical formulation and is followed by the computational procedure with the numerical illustration. The procedure employs trapezoidal fuzzy MFs and its mathematical calculations to describe the illustrated numerical results of NLP. It has offered a fuzzy model to the general NLP, which helps to handle the vagueness and also clarified its optimum solution within the description of MFs [[5]-[6]]. Also, the above procedure was executed in the numerical illustration in two different cases: the first case was discussed with fuzziness, and the second case offered with a robust ranking. Lastly, the study of the optimum solution of the above two cases reveals the newness and cost-effectiveness of a fuzzy model, clarifying the vagueness, and giving much more optimum values.

2. Literature Review

The following key contributions shed light on the diverse approaches and applications of fuzzy NLP: Tang and Wang [7] explored non-symmetric frameworks for solving NLPPs with inequality constraints in a fuzzy context. They proposed a model using penalty coefficients expressed as fuzzy sets, allowing for more nuanced representation of resource limitations and constraints. Additionally, their model employed nonlinear membership functions, providing greater flexibility in capturing real-world complexities. Tang et al. [8] presented a hybrid optimization approach for a specific type of NLP problem. Their method combines a genetic algorithm with a genetic defect mechanism and a weighted gradient search. This approach leverages the strengths of both genetic algorithms, known for their exploration capabilities, and gradient search, effective in local optimization.Fung et al. [9] expanded the hybrid genetic algorithm and also focused on essential strategies for applying to NLPPs involving both the types of constraints. Focusing on the LUDE algorithm, Sarimveis and Nikolakopoulos [10] suggested an approach for constrained penalty weight based optimization problems. Syau and Lee [11] explained the methodology for fuzzy convex optimization with numerical illustrations of multiobjective programming. Chen [12] uses mathematical programming with fuzzy sets and Yager's ranking index for cost-based queueing decision problem. Qin et al. [13] outlined an interval parameter nonlinear system for managing stream water quality in a fuzzy environment. Fuzzy programming and interval procedures are combined in a common outline to suggest the fuzziness in both sides of the nonlinear constraints. Kassem [14] developed a method for determining the consistency of optimal results for multiobjective NLPPs. Ravi Sankar et al. [15] presented a new approach for optimizing the nonlinear objective function using a genetic algorithm which has both coefficient and constraints are in the form of fuzziness. Abd-El-Wahed et al. [16] presented a hybrid approach which incorporates two heuristic optimization techniques, they are particle swarm optimization and genetic algorithm. Jameel and Sadeghi [17] addressed fuzzy NLPPs with an adequate numerical examples and compared the crisp problem which demonstrate a more accurate solution. Ali H. et al. [18] has constructed the scheduling problem as a stochastic optimization problem, specifically NLP, and then modeled in a multiobjective optimization problem which offers suitable scheduling with various models. Bi-level preferential operation problems have been suggested by M. F. Khan et al. [19] through a methodology for estimating the reliability parameters by employing the nonlinear optimization with the Kuhn Tucker approach. Gupta et al. [20]. described a data driven mechanism based on fuzzy-based Lagrangian method have supported vector machines for readily accessible biomedical data interpretation. Lin et al. [21] proposed an NLP model for production inventory based on statistical data. Likewise, Lu et al. [22] has presented NLP approach for evaluating manufacturing process. Saberi Najafi et al. [23] investigated a nonlinear model for fully fuzzy LP which can be addressed under uncertainties with fully unrestricted variables and parameters. A few researchers have proposed various methodologies for tackling nonlinear issues such as quadratic programming, allocation problems in supply chain management, and many other issues considered in a fuzzy situation; it starts the best approach to take care of these to satisfy the stack holder's needs cost-effectively [[24]-[25]]. Palanivel K. and Amrit Das [26] proposed a new fuzzy optimization model through computational procedure which has an important role in acquiring the optimum result by utilizing the necessary and sufficient conditions of the Lagrangian multipliers method in terms of fuzziness. Veeramani et al.[27] discussed fuzzy NLP approach for optimizing multi-objective sum of linear and linear fractional programming problem. Xianfeng Ding et al. [28] studied quadratic programming whose parameters are all fuzzy numbers based on the A-PSO algorithm. Numerical examples are illustrated to compare and analyze the algorithm and its results, which enhances the efficiency and effectiveness of the proposed method. Sumati Mahajan et al. [29] presented the study of fuzzy fractional quadratic programming problem, which has discussed through a hybrid method that combines analytic and numerical approaches. Also methodology illustrates a transportation problem in tourism sector switching between both balanced and unbalanced cases. Shivani et al. [30] studied multiobjective NLP with rough interval parameters which originates for the management of solid wastes, the model aimed to optimize the cost of waste transportation, cost of waste treatment and the revenue generated from various treatment facilities. Sumati Maharajan et al. [31] developed a simplified novel goal programming method under intuitionistic fuzzy environment using both membership/non-membership functions. Additionlly, it helps to obtain a Pareto optimal solution to a multiobjective quadratic programming problem. Existing research provides diverse methodologies for fuzzy NLPP's, but a clear gap remains in identifying the approach that best balances stakeholder needs requires further exploration.

3. Preliminaries

We review various important preliminary concepts and perspectives on fuzzy arithmetic in this section. Now it seems to address some definitions which are required:

3.1. Definition [5]

A trapezoidal fuzzy number M can be represented as M= $[m_1, m_2, m_3, m_4]$ with the following MF:

$$\mu_M(x) = \begin{cases} \frac{x-m_1}{m_2-m_1}, & m_1 \leq x \leq m_2 \\ 1, & m_2 \leq x \leq m_3 \\ \frac{x-m_4}{m_3-m_4}, & m_3 \leq x \leq m_4 \end{cases}$$

the trapezoidal MF $\mu_M(x)$ is illustrated in the figure 3.1.

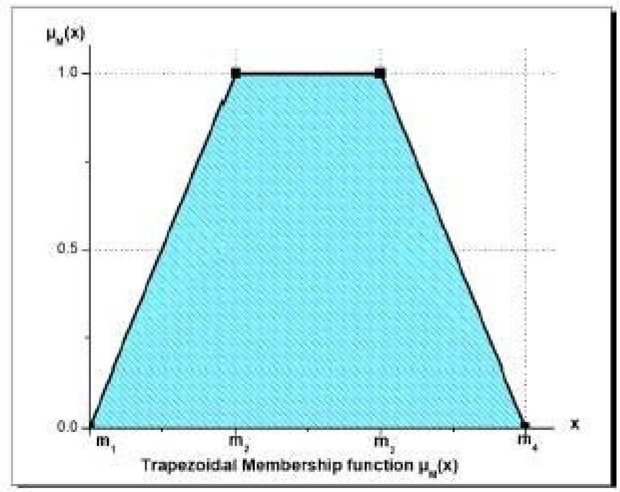

Figure 3.1: Trapezoidal Membership function $\mu M(x)$

3.2. Definition (α cut) [5]

Assumed a fuzzy set M in X and any real number α in [0, 1], then the α cut of M, denoted by ${}^{\alpha}M$ is the crisp set ${}^{\alpha}M = \{x \in X : \mu_M(x) \geq \alpha\}$. For illustration, let M be a fuzzy set whose membership function is given as above $\mu_M(x)$. To find α-cut of M, where $\alpha \in$ [0,1], let us set the reference functions of M to each left and right.

$$\alpha = \frac{x^{(1)} - m_1}{m_2 - m_1} \text{ and } \alpha = \frac{x^{(2)} - m_1}{m_3 - m_4}$$

Expressing x to α, where $x^{(1)} = (m_2 - m_1)\alpha + m_1$ and $x^{(2)} = m_4 + (m_3 - m_4)\alpha$ which provides the α-cut of M is ${}^{\alpha}M = [x^{(1)}, x^{(2)}] = [(m_2 - m_1)\alpha + m_1, m_4 + (m_3 - m_4)\alpha]$.

3.3. Definition (Robust ranking index) [5]

The robust ranking index satisfies compensation, homogeneity, and additive properties, and produces results that are controlled by human perception. If M is a fuzzy number, the robust ranking index is measured as $R(M) = \int_0^1 (0.5) * [M_\alpha^L, M_\alpha^u] d\alpha$, where $[M_\alpha^L, M_\alpha^u] = [(m_2 - m_1)\alpha + m_1, m_4 + (m_3 - m_4)\alpha]$ is the α-cut of the fuzzy number M. Here the robust ranking index R(M) offers the numerical significance of fuzzy M.

3.4. Remarks

- If the ranking of $R(M) > R(N)$ then M is called fuzzy maximum, then N.
- If the ranking of $R(M) < R(N)$ then M is called fuzzy minimum, then N.

4. The model for fuzzy nonlinear programming

Research emphasis on fuzzy optimization issues in the area of nonlinear programming is mainly limited. However, there is little interest in nonlinear programming, including fuzzy quadratic programming. Besides this, there are many kinds of fuzzy nonlinear problems in many real issues, especially in complex industrial systems. Research emphasis on problems of fuzzy optimization in the field of nonlinear programming is generally limited. However, there is little interest in nonlinear programming to address the vagueness of the issues. Besides this, in many real issues, there are many kinds of fuzzy nonlinear problems that occur, especially in complex manufacturing systems. It cannot be represented and solved by traditional models. However, research on modeling and enhancing approaches for nonlinear programming in a fuzzy background is not only crucial in the concept of fuzzy optimization, but it is also significant and of extensive importance in the application to the issues. Thus, the fuzzy optimization model has been proposed in three stages, namely the formulation of the problem, computational procedure, and lastly, numerical illustration followed by a comparative analysis.

4.1. Formulation of the NLPP with inequality constraints in terms of fuzziness [26]

Fuzzy NLPP has well-defined as the problem of finding a fuzzy vector $[(x_1^{(k)}), (x_2^{(k)}), \ldots, (x_n^{(k)})]$, for all $k = 1, 2, 3, 4$, where $(x_i^{(k)}), i = 1, 2, 3, \ldots, n\&$ for all $k = 1, 2, 3, 4$ is a trapezoidal fuzzy membership function, which optimizes (maximize / minimize) the objective function Z, which is a real-valued function of 'n' fuzzy variables defined by

$$[Z^{(k)}] = f((x_1^{(k)}), (x_2^{(k)}), \ldots, (x_n^{(k)})), for\ all\ k = 1, 2, 3, 4. \quad (4.1)$$

Under the constraints,

$g^j([(x_1^{(k)}), (x_2^{(k)}), \ldots, (x_n^{(k)})])\ \{\leq, \geq, or =\}(b_j^{(k)}); for\ all\ k = 1, 2, 3, 4\ \&\ j = 1, 2, \ldots, m.$

Where $g^{j,}s$ are 'm' real- valued function of 'n' fuzzy variables and $b^{j'}s$ are 'm' fuzzy constants, and $(x_i^{(k)}) \geq 0$, i=1,2,3, ...,n and for all k=1,2,3,4.

Moreover, as stated before, the problem can be restated as

Maximize $[Z^{(k)}] = f((x_1^{(k)}), (x_2^{(k)}), \ldots, (x_n^{(k)})), for\ all\ k = 1, 2, 3, 4.$

Under the constraints,

$$g^j([(x_1^{(k)}), (x_2^{(k)}), \ldots, (x_n^{(k)})]) \leq, (b_j^{(k)}); \tag{4.2}$$

for all k=1,2,3,4 & j=1,2, ..., m. where $g^{j,}s$ are 'm' real- valued function of 'n' fuzzy variables and $b^{j'}s$ are 'm' fuzzy constants, and $(x_i^{(k)}) \geq 0$, i=1,2,3, ...,n and for all k=1,2,3,4.

$$(x_i^{(k)}) \geq 0, i = 1, 2, 3, \ldots, n\ and\ for\ all\ k = 1, 2, 3, 4. \tag{4.3}$$

The fuzzy vector that satisfies conditions (4.3) and (4.4) is a feasible solution to the fuzzy NLP.

$$[X^{(k)}] = [(x_1^{(k)}), (x_2^{(k)}), \ldots, (x_n^{(k)})],$$

for all k = 1,2,3,4.

4.2. Computational Procedure

Using Beale's conditions, find an optimum solution to a fuzzified NLPP. Now the problem becomes:

Maximize $[Z^{(k)}] = f((x_1^{(k)}), (x_2^{(k)}), \ldots, (x_n^{(k)})), for\ all\ k = 1, 2, 3, 4.$

Subject to the constraints $[(x_1^k, x_2^k), \ldots, (x_n^{(k)})] \leq C$, and $(x_i^{(k)}) \geq 0$, i=1,2,3, ...,n and for all $j = 1, 2, 3, 4$ and x_i's are real-valued ńfuzzy variables. Now introduce the new variables S_i (slack variables) such that

$(x_i^{(k)}) + S_i^2 = C$, $i = 1, 2, 3,$ …,n and for all $k = 1, 2, 3, 4$.

Therefore the problem can be restated as
Maximize

$$[Z^{(k)}] = f((x_1^{(k)}), (x_2^{(k)}), \ldots, (x_n^{(k)})), for\ all\ k = 1, 2, 3, 4.$$

Subject to the constraints,

$$(4.4) \quad (x_i^{(k)}) + S_i^2 = C, i = 1, 2, 3, , \ldots, n\ and\ for\ all\ k = 1, 2, 3, 4.$$

Now, solve the problem using the above Beale's conditions and choose arbitrarily m and n-m basic and non-basic variables respectively. Let us consider x_b and x_{nb} as basic and non-basic variables.

Iteration 1: Assume that, $x_b = (x_1^{(k)})$ and $x_{nb} = (x_2^{(k)}, \ldots x_n^{(k)})$ for all $k = 1, 2, 3, 4$.

Express the basic feasible variables $(x_1^{(k)})$ for all $k = 1, 2, 3, 4$ in terms of the non-basic variables.

$$[(x_2^{(k)}, x_3^{(k)}, \ldots x_n^{(k)})],$$

for all $k = 1, 2, 3, 4$.

$$(4.5) \qquad (x_1^{(k)}) = (Constant - (x_2^{(k)} - x_3^{(k)} - \ldots - x_n^{(k)}))$$

for all $k = 1, 2, 3, 4$. Express the objective functions

Maximize

$$[Z^{(k)}] = f((x_1^{(k)}), (x_2^{(k)}), \ldots, (x_n^{(k)})), for\ all\ k = 1, 2, 3, 4.$$

in terms of the non-basic variables $[(x_2^{(k)}), \ldots x_n^{(k)})]$, for all $k = 1, 2, 3, 4$.
Therefore, Beale's conditions for maximization concerning each non-basic variable at zero values can be stated as

$$(4.6) \qquad \frac{\partial f(x)}{\partial x_i} = Constant$$

$\frac{\partial f(x)}{\partial x_i}$ at $(x_i = 0)$ = Constant; where $= 2, 3, \ldots n$.

Let us choose the non-basic variables that provide the significant improvements and replace the remaining non-basic variables which are zero in the Equation. (4.6), solve the equations and find the right solution.

Iteration 2: Assume that, $x_b = (x_2^{(k)})$ & $x_{nb} = [(x_1^{(k)}, x_3^{(k)}, \ldots x_n^{(k)})]$, for all $k = 1, 2, 3, 4$. Express the basic variables $(x_2^{(k)})$ for all $k = 1, 2, 3, 4$ in terms of the non-basic variables. $[(x_1^{(k)}, x_3^{(k)}, \ldots x_n^{(k)})]$, for all $k = 1, 2, 3, 4$.

$$(x_2^{(k)}) = (Constant - (x_1^{(k)} - x_3^{(k)} - \ldots - x_n^{(k)})) \tag{4.7}$$

for all $k = 1, 2, 3, 4$. Express the objective functions

Maximize

$$[Z^{(k)}] = f((x_1^{(k)}), (x_2^{(k)}), \ldots, (x_n^{(k)})), for\ all\ k = 1, 2, 3, 4.$$

in terms of the non-basic variables $[(x_1^{(k)}), (x_3^{(k)}) \ldots x_n^{(k)})]$, for all $k = 1, 2, 3, 4$. Therefore, Beale's conditions for maximization concerning each non-basic variable at zero values can be stated as

Let us choose the non-basic variables that provide the significant improvements and replace the remaining non-basic variables which are zero in the Equation (4.7), solve the equations and find the right solution. Hence, the above iterations 1 and 2, which give the same solution, are the optimal solution to the considered problem.

5. Numerical illustrations

This section addresses two cases of numerical illustrations that can simplify the models for solving the problem of fuzzy NLP using trapezoidal membership functions and its mathematical calculations. The fuzzy model explains the procedure employing the membership function approach in case 1 and, the same problem was investigated with the help of the robust ranking approach in case 2.

Let us consider the NLP in terms of fuzziness using Beale's method: the fuzzified form of the considered NLPP can be stated as below:

Maximize

$$[0, 1, 3, 4](x_1^{(k)}) + [1, 2, 4, 5](x_2^{(k)}) - [-1, 0, 2, 3](x_1^{(k)}),$$

for all $k = 1, 2, 3, 4$.

Subject to the constraints,

$$[-1, 0, 2, 3](x_1^{(k)}) + [0, 1, 3, 4](x_2^{(k)}) + [-1, 0, 2, 3](x_3^{(k)}) = [2, 3, 5, 6],$$

for all $k = 1, 2, 3, 4$.

$$(x_1^{(k)}), (x_2^{(k)}), (x_3^{(k)}) \geq 0, \tag{5.1}$$

for all $k = 1, 2, 3, 4$.

5.1. Case (i): NLP with fuzzy membership functions

The above NLPP has been optimized with fuzziness by employing Beale's condition, as discussed earlier. There is only one equation in the numerical illustration under consideration. As a result, the number of basic variables is one. Take one of the variables to be a basic variable and the other two to be non-basic variables.

Iteration 1: Assume that, Assume that, $x_b = (x_1^{(k)})$ and $x_{nb} = (x_2^{(k)}, x_3^{(k)})$ for all $k = 1, 2, 3, 4$.

From Eqn. (5.1)

$$[-1, 0, 2, 3](x_1^{(k)}) + [0, 1, 3, 4](x_2^{(k)}) + [-1, 0, 2, 3](x_3^{(k)}) = [2, 3, 5, 6],$$

for all $k = 1, 2, 3, 4$. Express the basic variables $(x_1^{(k)})$ for all $k = 1, 2, 3, 4$ in terms of the non-basic variables $[x_2^{(k)}, x_3^{(k)}]$, for all $k = 1, 2, 3, 4$.

$$(x_1^{(k)}) = [2, 3, 5, 6] - [0, 1, 3, 4](x_2^{(k)}) - [-1, 0, 2, 3](x_3^{(k)}) \tag{5.2}$$

Expressing

$$Q(x) = [0, 1, 3, 4](x_1^{(k)}) + [1, 2, 4, 5](x_2^{(k)}) - [-1, 0, 2, 3](x_1^{(k)})^2$$

for all $k = 1, 2, 3, 4$.

In terms of the non-basic variables $[(x_2^{(k)}), (x_3^{(k)})]$, for all $k = 1, 2, 3, 4$.

$$\begin{aligned} Q(x) = {} & [0, 1, 3, 4]([2, 3, 5, 6] - [0, 1, 3, 4](x_2^{(k)}) - [-1, 0, 2, 3](x_3^{(k)})) + \\ & [1, 2, 4, 5](x_2^{(k)}) - [-1, 0, 2, 3]([2, 3, 5, 6] - [0, 1, 3, 4](x_2^{(k)}) - [-1, 0, 2, 3](x_3^{(k)}))^2 \end{aligned} \tag{5.3}$$

for all $k = 1, 2, 3, 4$.

$$\frac{\partial Q(x)}{\partial x_2} = -[0,1,3,4][0,1,3,4] + [1,2,4,5] - 2([2,3,5,6] - [0,1,3,4](x_2^{(k)})$$
$$[-1,0,2,3](x_3^{(k)}))(-[0,1,3,4]) \tag{5.4}$$

$$[\frac{\partial Q(x)}{\partial x_2}]_{(x_2=0,x_s=0)} = [\text{-12, -3, 15, 24}].$$

As a result, increasing x_2 will lead to greater development in the objective function than increasing x_3. Then it will be considered, x_3=0.

Substituting $(x_3^{(k)}) = [-2,-1,1,2]$, for all $k = 1, 2, 3, 4$, in Eqn (5.3), which results $(x_2^{(k)}) = [0.125, 1, 2.75, 3.625]$, for all $k = 1, 2, 3, 4$.

By substituting, we get $(x_2^{(k)}) = [0.125, 1, 2.75, 3.625]\&(x_2^{(k)}) = [0.125, 1, 2.75, 3.625]$, for all $k = 1, 2, 3, 4$. in Eqn (5.2). which implies that, $(x_1^{(k)}) = [-5.5, -2.625, 3.125, 6]$, for all $k = 1, 2, 3, 4$.

Hence the optimum solution is,

$(x_1^{(k)}) = [-5.5, -2.625, 3.125, 6], (x_2^{(k)}) = [0.125, 1, 2.75, 3.625], \&(x_3^{(k)}) = [-2, -1, 1, 2]$, for all $k = 1, 2, 3, 4$.

Iteration 2: Assume that, Assume that, $x_b = (x_2^{(k)})$ and $x_{nb} = (x_1^{(k)}, x_3^{(k)})$ for all $k = 1, 2, 3, 4$.
From Eqn. (5.1)

$$[-1,0,2,3](x_1^{(k)}) + [0,1,3,4](x_2^{(k)}) + [-1,0,2,3](x_3^{(k)}) = [2,3,5,6],$$

for all $k = 1, 2, 3, 4$. Express the basic variables $(x_2^{(k)})$ for all $k = 1, 2, 3, 4$ in terms of the non-basic variables $[x_1^{(k)}, x_3^{(k)}]$, for all $k = 1, 2, 3, 4$.

$$(x_2^{(k)}) = [1, 1.5, 2.5, 3] - [-0.5, 0.1, 1, 1.5](x_1^{(k)}) - [-0.5, 0.1, 1, 1.5](x_3^{(k)}), \tag{5.5}$$

for all $k = 1, 2, 3, 4$. Expressing

$$Q(x) = [0,1,3,4](x_1^{(k)}) + [1,2,4,5](x_2^{(k)}) - [-1,0,2,3](x_1^{(k)})^2$$

for all $k = 1, 2, 3, 4$.

In terms of the non-basic variables $[(x_2^{(k)}), (x_3^{(k)})]$, for all $k = 1, 2, 3, 4$.

$$\begin{aligned} Q(x) = \quad & [0,1,3,4](x_1^{(k)}) + [1,2,4,5]([1,1.5,2.5,3] - [-0.5,0.1,1,1.5](x_1^{(k)}) \\ & -[-0.5,0.1,1,1.5](x_3^{(k)})) - [-1,0,2,3](x_1^{(k)})^2 \end{aligned} \tag{5.6}$$

for all $k = 1, 2, 3, 4$.

$$\frac{\partial Q(x)}{\partial x_1} = -[0,1,3,4] - [-0.5,0.1,1,1.5] - [-2,0,4,6](x_1^{(k)}) \tag{5.7}$$

for all $k = 1, 2, 3, 4$.

$$[\frac{\partial Q(x)}{\partial x_1}]_{(x_2=0, x_3=0)} = [-0.5, 1, 2, 2.5].$$

As a result, increasing x_1 will lead to greater development in the objective function than increasing x_3. Then it will be considered, x_3=0.

From Eqn (5.7), which results in $(x_1^{(k)})$ =[-1.25,-0.5,1,1.75], for all $k = 1, 2, 3, 4$.

By substituting, we get

$(x_1^{(k)})$ = [-1.25,-0.5,1,1.75] & $(x_3^{(k)})$ = [-2,-1,1,2], for all $k = 1, 2, 3, 4$. in Eqn (5.6). which implies that, $(x_2^{(k)})$ = [0.625,1.25,2.5,3.125], for all $k = 1, 2, 3, 4$.

Hence the optimum solution is,
$(x_1^{(k)})$ = [-1.25,-0.5,1,1.75], $(x_2^{(k)})$ = [0.625,1.25,2.5,3.125], & $(x_3^{(k)})$ = [-2,-1,1,2], for all $k = 1, 2, 3, 4$.

Therefore, Q(x)=[1.6875,3.875,8.25,10.437]

The optimum solutions are the same in both the iteration, so the above solution becomes an optimum solution.

5.2. Case (ii): The robust ranking approach for NLP with fuzzy membership functions [5]

Let us solve the above NLPP by employing a robust ranking approach. Further, the ranking index of R[0,1,3,4] and its fuzzy membership function are as follows.

$$\mu_{R[0,1,3,4]}(x) = \begin{cases} x, & 0 \le x \le 1 \\ 1, & 1 \le x \le 3 \\ -x+4, & 3 \le x \le 4 \\ 0, & otherwise \end{cases} \tag{5.8}$$

The confidence interval for each degree α& the trapezoidal structures will be described in the following way by the functions of α.
Here $\alpha = x_1^{(\alpha)} \& \alpha = -x_2^{(\alpha)} + 4$

Therefore,

$$[x^{(1)}, x^{(2)}] = [M_\alpha^L, M_\alpha^U] = [(m_2 - m_1)\alpha + m_1, m_4 + (m_3 - m_4)\alpha] = [\alpha, -\alpha + 4]$$

$$R(M) = R[0,1,3,4] = \int_0^1 (0.5) * [M_\alpha^L, M_\alpha^U] d\alpha = \int_0^1 (0.5)(4) d\alpha = 2$$

Similarly, the ranking index R[1,2,4,5] is as follows:

$$\mu_{R[1,2,4,5]}(x) = \begin{cases} x-1, & 1 \le x \le 2 \\ 1, & 2 \le x \le 4 \\ -x+5, & 4 \le x \le 5 \\ 0, & otherwise \end{cases} \tag{5.9}$$

The confidence interval for each degree α& the trapezoidal structures will be described in the following way by the functions of α.

Therefore,

$$[x_{(1)}, x_{(2)}] = [M_\alpha^L, M_\alpha^U] = [\alpha + 1, -\alpha + 5]$$

$$R(M) = R[1,2,4,5] = \int_0^1 (0.5) * [M_\alpha^L, M_\alpha^U] d\alpha = 3$$

Furthermore, for all other fuzzy numbers, the ranking index has been determined as follows:

$$R(M) = R[-1, 0, 2, 3] = \int_0^1 (0.5) * [M_\alpha^L, M_\alpha^U] d\alpha = 1$$

$$R(M) = R[2, 3, 5, 6] = \int_0^1 (0.5) * [M_\alpha^L, M_\alpha^U] d\alpha = 4$$

By employing the above approach, the fuzzy nonlinear programming problem is reduced to the standard crisp problems, which is as follows.

Maximize Q(x) $= 2x_1 + 3x_2 - x_1^2$

Subject to the constraints: $x_1 + 2x_2 \leq 4$ and $x_1, x_2 \geq 0$

Now apply the existing conventional approach to the nonlinear programming problem by using Beale's conditions and obtained the optimum solution for the above is

$$x_1 = 0.25, x_2 = 1.875, x_3 = 0 \text{ \& } Maximum\ Q(x) = 6.0625.$$

5.3. Comparison Analysis

The table 1 below provides a comparison of the optimal solution obtained from the existing, fuzzy model and robust ranking approach for the fuzzy nonlinear programming problem preferred in the numerical illustration above. From the results shown in the table, it is evident that the same results are given regardless of what existing or fuzzy membership and ranking approaches do. It shows the newness of the proposed model and also the decision-maker may use this kind of model to clear the vagueness of any suitable problem to achieve the best optimum value. Based on the above result, it has been recommended to use either of the models given instead of the existing model, namely the fuzzy membership function model or robust ranking approach, which is ideal.

Table 1: Optimum solution comparison of existing and proposed models.

The existing model is based on the conventional approach			
x_1	x_2	x_3	Max Q(x)
0.25	1.875	0	6.0625
The proposed model is based on the conventional approach in terms of fuzziness (i=1,2,3,4)			
$(x_1^{(k)})$ =[-1.25,-0.5,1,1.75]	$(x_2^{(k)})$ =[0.625,1.25,2.5,3.125]	$(x_3^{(k)})$ =[-2,-1,1,2]	[1.69,3.88,8.25,10.44]
The proposed model is based on the robust ranking approach.			
$(x_1^{(k)})$ =[-1.25,-0.5,1,1.75] =R[-1.25,-0.5,1,1.75] =0.25	$(x_2^{(k)})$ =[0.625,1.25,2.5,3.125] =R[0.625,1.25,2.5,3.125] =1.875	$(x_3^{(k)})$ =[-2,-1,1,2] =R[-2,-1,1,2] =0	[1.69,3.88,8.25,10.44] =R[1.69,3.88,8.25,10.44] =6.0625

6. Results and discussion

Employing the proposed model illustrations that the optimum value of the fuzzy nonlinear programming problem is [1.69,3.88,8.25,10.44], which might be a fresh attempt to clear the vagueness. The optimum solution for the fuzzified nonlinear programming problems will be continuously greater than 1.69 and less than 10.44, and the most likely outcome will be somewhere in the range of 3.88 and 8.25. The varieties in cost with significant probability have appeared in the following Figure 6.1. Also, obtained fuzzy optimum solutions xij might be empirically comprehended.

- Decision-makers (DM) choice will be the total fuzzified NLPP values will be greater than 1.69 and less than 10.44.
- According to the DM, in favor of the whole fuzzified NLPP values are tends to be bigger than or sufficient to 3.88 and smaller than or adequate to 8.25
- Hence, the percentage of the favors of the DM for the residual values of the whole fuzzified NLPP result is frequently attained as follows:

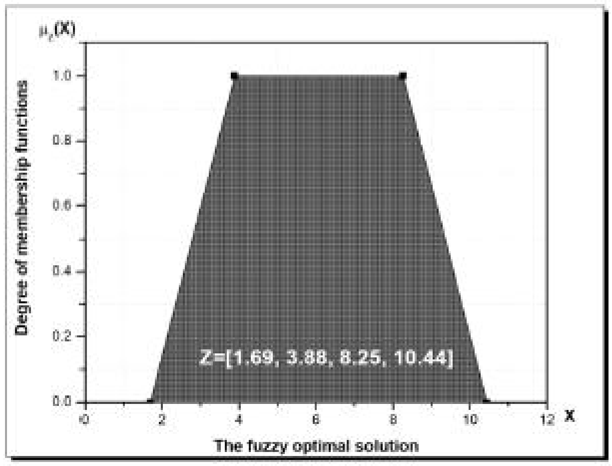

Figure 6.1: Fuzzy Optimal Solution of Trapezoidal FMF

Let x represent the fuzzified NLPP optimum result, and then the percentage of best in the DM, where

$$\mu_{max}(x) = \begin{cases} \frac{x-1.69}{3.88-1.69} & \text{for } 1.69 \leq x \leq 3.88 \\ 1 & \text{for} 3.88 \leq x \leq 8.25 \\ \frac{x-10.44}{8.25-10.44} & \text{for } 8.25 \leq x \leq 10.44 \\ 0 & otherwise \end{cases} \tag{6.1}$$

7. Conclusion

The fuzzy version of the problem has been addressed using Beale's conditions through fuzziness with the aid of a numerical illustration. Besides, which clarifies by solving two numerical illustrations, one is using membership functions and another one approaches by robust rankings. The membership function provides a significant role in the creation of a model in a fuzzy context. This model offers an ideal approach to handle the problems of NLP. Moreover, the optimal solution has been signified through fuzziness with the result and discussion. A representation of trapezoidal membership functions is also used to explain the approach. Furthermore, the comparison analysis could be a novel approach to addressing NLP under uncertainties. The model tries to overcome decision-makers uncertainties and subjective experiences, and it can aid in the resolution of decision-making problems. This similar model can be recommended in future studies of fuzzy optimization models as well other types of nonlinear optimization problems.

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