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Spectra of $(M, \mathcal{M})$-corona-join of graphs

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Abstract

In this paper, we introduce the $(M, \mathcal{M})$-corona-join of G and $\mathcal{H}_k$ constrained by vertex subsets $\mathcal{T}$, which is the union of two graphs: one is the M-generalized corona of a graph G and a family of graphs $\mathcal{H}_k$ constrained by vertex subset $\mathcal{T}$ of the graphs in $\mathcal{H}_k$, where M is a suitable matrix; and the other one is the $\mathcal{M}$-join of $\mathcal{H}_k$, where $\mathcal{M}$ is a collection of matrices. We determine the spectra of the adjacency, the Laplacian, the signless Laplacian and the normalized Laplacian matrices of some special cases of the $(M, \mathcal{M})$-corona-join of G and $\mathcal{H}_k$ constrained by vertex subsets $\mathcal{T}$. These results enable us to deduce the spectra of all the existing variants of extended corona of graphs. Further, by using this graph operation, we construct infinitely many graphs which are simultaneously cospectral with respect to the above mentioned four type of matrices.

Keywords: *Join of graphs, Corona of graphs, Adjacency spectrum, Laplacian spectrum, Signless Laplacian spectrum.*

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1. Introduction

The study of the spectra of various matrices associated to graphs is an active research topic in spectral graph theory as the knowledge of the spectra of these matrices reveal several structural properties of the graphs. In spectral graph theory, it is a common problem that "to what extent the spectrum of a graph constructed using graph operations can be described in terms of the spectrum of the constituting graph(s)?" Over the last five decades, researchers have paid much attention to the spectra of graphs obtained by using several graph operations such as the complement, the disjoint union, the join, the Cartesian product, the strong product, the Kronecker product, the NEPS and the rooted product, etc. We refer the reader to [8, 9, 14, 25, 26, 27, 30] and the references therein for more graph operations and the results on the spectra of these graphs.

Corona is another well-known graph operation. Let G be a graph with n vertices and H be a graph. The corona of G and H is obtained by taking one copy of G and n copies of H, and joining the i-th vertex of G to all the vertices of the i-th copy of H for $i = 1, 2, \ldots, n$. The spectral properties of corona of graphs were studied in [4]. Since then several variants of corona of graphs have been defined in the literature and their spectral properties were studied; see [1, 3, 5, 7, 10, 11, 19, 20, 21, 22, 23, 24, 28, 32] and the references therein. For the convenience of the reader, we give the definition of some of the variants of corona of graphs used in this paper. The neighborhood corona of graphs G and H is the graph obtained by taking a copy of G and n copies of H, and joining the vertices in the neighborhood of the i-th vertex of G to all the vertices of the i-th copy of H for $i = 1, 2, \ldots, n$ [18]. Adiga and Rakshith [2] defined the following: The *extended neighborhood corona* (resp. *extended corona*) *of two graphs* G_1 *and* G_2 is the graph obtained by taking the neighborhood corona (resp. corona) of G_1 and G_2, and joining each vertex of the i-th copy of G_2 to every vertex of the j-th copy of G_2, provided the vertices v_i and v_j are adjacent in G_1. Following this, Tajarrod and Sistani [31] defined the following: Let $V(G_2) = \{v_1, v_2, \ldots, v_n\}$ and let v_{i_k} denote the vertex corresponding to v_k in the i-th copy of G_2. Then the *identity-extended corona* (resp. *identity-extended neighborhood corona*) *of two graphs* G_1 *and* G_2 is the graph obtained by taking the corona (resp. neighborhood corona) of G_1 and G_2, and joining the vertex v_{i_k} of the i-th copy of G_2 to the vertex v_{j_k} of the j-th copy of G_2, provided the vertices v_i and v_j are adjacent in G_1. The *neighborhood extended corona* (resp. *neighborhood extended neighborhood corona*) *of two graphs* G_1 *and* G_2 is the graph obtained by taking the corona (resp. neighborhood corona) of G_1 and G_2, and joining the vertex v_{i_k} of the i-th copy of G_2 to the vertices v_{j_l} of the j-th copy of G_2, provided the vertices v_i and v_j are adjacent in G_1 and u_k and u_l are adjacent in G_2.

In [13, 29] the authors defined the M-generalized corona of graphs constrained by vertex subsets and showed that it generalizes all the variants of corona of graphs defined in the literature except the variants of corona of graphs defined in [2, 31]. Further, various spectra of the graphs constructed by this graph operation were obtained and the spectra of the variants of corona of graphs (except those mentioned above) were deduced. Recently, the same authors introduced the $\mathcal{M}$-join of the graphs [12] and obtained its various spectra.

In this paper, we define a graph operation by suitably combining the M-generalized corona of graphs and the $\mathcal{M}$-join of the graphs. We show that this construction includes all the variants of corona of graphs as its particular cases and we obtain the various spectra of the graphs formed by this graph operation.

Now we recall some basic notions of spectral graph theory. We consider only finite, and simple graphs. For a graph G with vertex set $\{v_1, v_2, \ldots, v_n\}$ and the edge set $\{e_1, e_2, \ldots, e_m\}$, the adjacency matrix $A(G)$ of G is the $0{-}1$ matrix of size $n \times n$, whose (i, j)-th entry is 1 if and only if the vertices v_i and v_j are adjacent in G. The *incidence matrix $B(G)$ of G* is the $0 - 1$ matrix of size $n \times m$ whose (i, j)-th entry is 1 if and only if the vertex v_i is incident with the edge e_j. The degree matrix $D(G)$ of G is the $diag(d_1, d_2, \ldots, d_n)$, where d_i is the degree of v_i for $i = 1, 2, \ldots, n$ in G. The Laplacian matrix, the signless Laplacian matrix and the normalized Laplacian matrix of G are defined as $L(G) = D(G) - A(G)$, $Q(G) = D(G) + A(G)$ and $\widehat{L}(G) = I_n - D(G)^{-\frac{1}{2}} A(G) D(G)^{-\frac{1}{2}}$, respectively provided G has no isolated vertices for $\widehat{L}(G)$. The multi-set of eigenvalues of $A(G)$, $L(G)$, $Q(G)$ and $\hat{L}(G)$ are said to be the A-spectrum, L-spectrum, Q-spectrum and $\widehat{L}$-spectrum of G, respectively. Two graphs are said to be A-cospectral (resp. L-cospectral, Q-cospectral and $\hat{L}$-cospectral) if they have the same A-spectrum (resp. L-spectrum, Q-spectrum and $\hat{L}$-spectrum). The A-spectrum, L-spectrum and Q-spectrum of G are denoted by
$\lambda_1(G) \geq \lambda_2(G) \geq \ldots \geq \lambda_n(G); 0 = \mu_1(G) \leq \mu_2(G) \leq \ldots \leq \mu_n(G); \nu_1(G) \geq \nu_2(G) \geq \ldots \geq \nu_n(G)$, respectively. Two graphs are regular A-cospectral if and only if they are L-cospectral (resp. Q-cospectral and $\widehat{L}$-cospectral). In this case, we say that these two graphs are regular cospectral.

The complement graph of G is denoted by $\overline{G}$. The union of two graphs

G and H, denoted by $G \cup H$, is the graph having vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. The matrix of size $n \times m$ in which all the entries are 1 is denoted by $J_{n\times m}$.

The rest of the paper is arranged as follows: In section 2, we introduce the $(M, \mathcal{M})$-corona-join of G and $\mathcal{H}_k$ constrained by vertex subsets $\mathcal{T}$. In Section 3, first we determine the spectra of a matrix in a specific form. Then by using this result, we deduce the A-spectra, the L-spectra, the Q-spectra and the $\widehat{L}$-spectra of the new graph for some special cases. From these results, we deduce the existing results on the spectra of the variants of extended corona of graphs. Finally, we construct infinitely many simultaneously A-cospectral, L-cospectral, Q-cospectral and $\widehat{L}$-cospectral graphs by using this graph operation.

2. $(M, \mathcal{M})$-corona-join of graphs

Definition 2.1. *([13, 29]) Let G be a graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$. Let $\mathcal{H}_k = (H_1, H_2, \ldots, H_k)$ be a family of k graphs and let $\mathcal{T} = (T_1, T_2, \ldots, T_k)$ be a family of sets, where $T_j \subseteq V(H_j)$, $j = 1, 2, \ldots, k$. Let $M = [m_{ij}]$ be a $0-1$ matrix of size $n \times k$. Then the M-generalized corona of G and $\mathcal{H}_k$ constrained by $\mathcal{T}$, denoted by $G \widetilde{\otimes}_{[M:\mathcal{T}]} \mathcal{H}_k$, is the graph obtained by taking one copy of G, $H_1, H_2, \ldots, H_k$ and joining the vertex v_i to all the vertices in T_j if and only if $m_{ij} = 1$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, k$.*

Hedetniemi introduced the following graph operation.

Definition 2.2. *([15]) Let G and H be graphs, and $\pi \subseteq V(G) \times V(H)$ be a binary relation. Then the π-graph of G and H is the graph whose vertex set is $V(G) \cup V(H)$ and the edge set is $E(G) \cup E(H) \cup \pi$.*

Notice that the binary relation π can be viewed as a matrix $M = [m_{ij}]$, where $m_{ij} = 1$ or 0, if the i-th vertex of H_1 and the j-th vertex of H_2 are related or not, respectively, so the $\pi-$graph of G and H is the graph obtained by taking one copy of G and H, and joining the i-th vertex of G to the j-th vertex of H if and only if $m_{ij} = 1$. This graph is denoted by $G \vee_M H$ and is called the *M-join of G and H* [12]. The above definition is extended as follows:

Definition 2.3. *([12]) Let $\mathcal{H}_k$ be a sequence of k graphs $H_1, H_2, \ldots, H_k$ with $\mid V(H_i) \mid = n_i$ for $i = 1, 2, \ldots, k$ and let $\mathcal{M} = (M_{12}, M_{13}, \ldots, M_{1k}, M_{23}, M_{24}, \ldots, M_{2k}, \ldots, M_{(k-1)k})$, where M_{ij} is a $0-1$ matrix of size*

$n_i \times n_j$. *Then the $\mathcal{M}$-join of the graphs in $\mathcal{H}_k$, denoted by $\bigvee_{\mathcal{M}} \mathcal{H}_k$, is the graph* $\bigcup\limits_{i,j=1,i<j}^{k} \Big(H_i \vee_{M_{ij}} H_j\Big)$.

With the notions mentioned above, we define the following.

Definition 2.4. *The $(M,\mathcal{M})$-corona-join of G and $\mathcal{H}_k$ constrained by vertex subsets $\mathcal{T}$ is the graph obtained by taking the union of $G\widetilde{\otimes}_{[M:\mathcal{T}]}\mathcal{H}_k$ and the $\mathcal{M}$-join of $\mathcal{H}_k$. We denote it as $G\bigvee_{[M:\mathcal{M}:\mathcal{T}]}\mathcal{H}_k$.*

Example 2.1. *Consider the graphs G, H_1, H_2, H_3, H_4 as shown in Figure 1. Let $\mathcal{H}_4 = (H_1, H_2, H_3, H_4)$. Let $\mathcal{T} = (T_1, T_2, T_3, T_4)$, where $T_1 = \{u_3\}$, $T_2 = \{w_2, w_3\}$, $T_3 = \{x_3, x_4\}$, $T_4 = \{y_1, y_4\}$. The white hollow circle vertices represent the elements in $T_j, j = 1, 2, \ldots, 4$. Let*

$$M = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}, M_{12} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}, M_{14} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix},$$

$$M_{23} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, M_{34} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, M_{13} = \mathbf{0}_{3\times 4}, M_{24} = \mathbf{0}_{4\times 4}$$

and $\mathcal{M} = (M_{12}, M_{13}, M_{14}, M_{23}, M_{24}, M_{34})$. Then $G\widetilde{\otimes}_{[M:\mathcal{T}]}\mathcal{H}_4$, $\bigvee_{[M:\mathcal{M}:\mathcal{T}]}\mathcal{H}_4$ and $G\bigvee_{[M:\mathcal{M}:\mathcal{T}]}\mathcal{H}_4$ are shown in Figure 1.

Notice that the construction of the graph $(M,\mathcal{M})$-corona-join of G and $\mathcal{H}_k$ constrained by $\mathcal{T}$ includes the construction of M-generalized corona of graphs as well as all the variants of extended corona of graphs: First one is obtained by taking $\mathcal{M}$ as the collection of zero matrices. For the second one, let G_1 and G_2 be graphs with n_1 and n_2 vertices respectively. Let $A(G) = [a_{ij}]$. Then for each pair M, M' as mentioned in Table 1, the graph $G\bigvee_{[M:\mathcal{M}:\mathcal{T}]}\mathcal{H}_k$, where

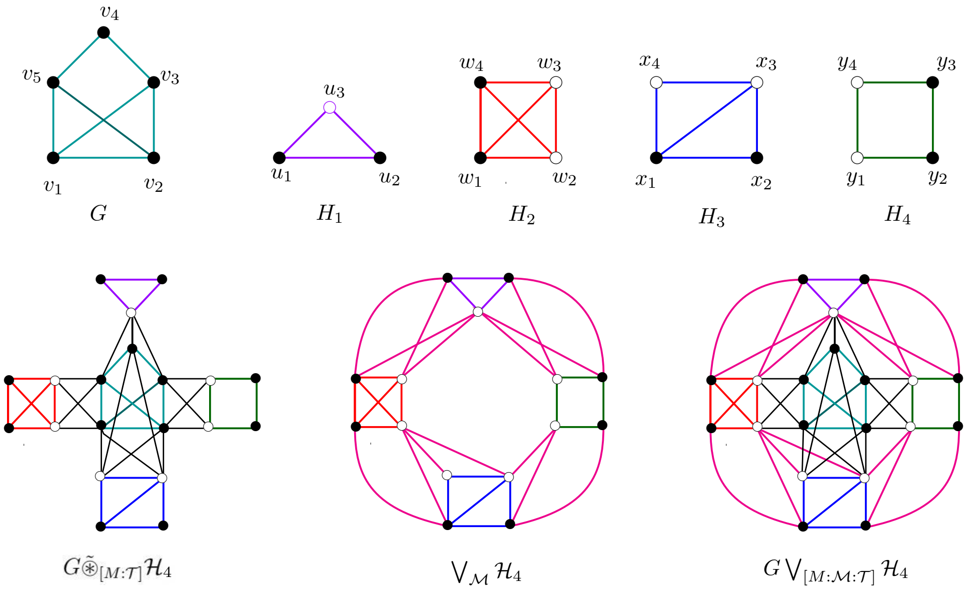

Figure 1: Example for $(M, \mathcal{M})$-corona-join of graphs constrained by vertex subsets

$\mathcal{M} = (a_{12}M', a_{13}M', \ldots, a_{(n_1-1)n_1}M')$; $\mathcal{H} = (\underbrace{G_2, G_2, \ldots, G_2}_{n_1\text{times}})$ and

$\mathcal{T} = (\underbrace{V(G_2), V(G_2), \ldots, V(G_2)}_{n_1\text{times}})$,

gives the existing variant of extended corona of graphs as mentioned in the same table.

S. No	Name of the corona operation	M	M'
1.	Extended neighborhood corona of G_1 and G_2	$A(G_1)$	J_{n_2}
2.	Extended corona of G_1 and G_2	I_{n_1}	J_{n_2}
3.	Identity extended corona of G_1 and G_2	I_{n_1}	I_{n_2}
4.	Identity extended neighborhood corona of G_1 and G_2	$A(G_1)$	I_{n_2}
5.	Neighborhood extended corona of G_1 and G_2	I_{n_1}	$A(G_2)$
6.	Neighborhood extended neighborhood corona of G_1 and G_2	$A(G_1)$	$A(G_2)$

Table 2.1: The existing variants of extended corona of graphs as particular cases of $G \bigvee_{[M:\mathcal{M}:\mathcal{T}]} \mathcal{H}_k$

3. Spectra of $G \bigvee_{[M:\mathcal{M}:\mathcal{T}]} \mathcal{H}_k$

The following result gives a characterization of commuting matrices through their eigenvectors.

Proposition 3.1. *([16, Proposition 2.3.2]) Let $A_1, A_2, \ldots, A_m$ be symmetric matrices of order n. Then the following are equivalent.*

1. *$A_iA_j = A_jA_i$, $\forall i, j \in \{1, 2, \ldots, m\}$;*
2. *There exists an orthonormal basis $\{x_1, x_2, \ldots, x_n\}$ of $\mathbf{R}^n$ such that $x_1, x_2, \ldots, x_n$ are eigenvectors of A_i, $\forall i = 1, 2, \ldots, m$.*

Definition 3.1. *([13, 29]) Let $A_1, A_2, \ldots, A_m \in M_n(\mathbf{C})$. Then $\lambda_1, \lambda_2, \ldots, \lambda_m \in \mathbf{C}$ are said to be co-eigenvalues of $A_1, A_2, \ldots, A_m$, if there exists a vector $X \in \mathbf{C}^n$ such that $A_iX = \lambda_iX$ for $i = 1, 2, \ldots, m$. In this case, we simply say that, λ_i's are co-eigenvalues of A_i's for $i = 1, 2, \ldots, m$.*

Theorem 3.1. *(Laplace Expansion Theorem)([17, 0.8.9]) Let $A \in M_n(F)$, let $k \in \{1, 2, \ldots, n\}$ be given, and let $\beta \subseteq \{1, 2, \ldots, n\}$ be any given index set of cardinality k. Then*

$$\begin{aligned} detA &= \textstyle\sum_\alpha (-1)^{p(\alpha,\beta)} detA[\alpha, \beta] detA[\alpha^c, \beta^c] \\ &= \textstyle\sum_\alpha (-1)^{p(\alpha,\beta)} detA[\beta, \alpha] detA[\beta^c, \alpha^c] \end{aligned}$$

in which the sums are over all index sets $\alpha \subseteq \{1, 2, \ldots, n\}$ of cardinality k, and $p(\alpha, \beta) = \sum_{i\in\alpha} i + \sum_{j\in\beta} j$.

We define the following notations.

$$\begin{aligned}
\mathcal{R}_{n\times m}(s) &:= \left\{[m_{ij}] \in M_{n\times m}(\mathbf{R}) : \sum_{j=1}^{m} m_{ij} = s \text{ for } i = 1,2,\ldots,n\right\};\\
\mathcal{C}_{n\times m}(c) &:= \left\{[m_{ij}] \in M_{n\times m}(\mathbf{R}) : \sum_{i=1}^{n} m_{ij} = c \text{ for } j = 1,2,\ldots,m\right\};\\
\mathcal{RC}_{n\times m}(s,c) &:= \left\{M \in M_{n\times m}(\mathbf{R}) : M \in \mathcal{R}_{n\times m}(s) \cap \mathcal{C}_{n\times m}(c)\right\}.
\end{aligned}$$

First we determine the spectrum of a matrix having some specific properties.

Theorem 3.2. *Let $A_1 \in M_{n_1\times n_1}(\mathbf{R})$ and $N \in M_{n_1\times k}(\mathbf{R})$, with $N = [m_{ij}]$, $i = 1,2,\ldots,n_1$; $j = 1,2,\ldots,k$. Let $M_{ij} \in \mathcal{R}_{n_2\times n_2}(r_{ij})$ be a symmetric matrix for $i,j = 1,2,\ldots,k$; $i \le j$, which commutes with all the others. Consider the matrix*

$$A = \begin{bmatrix} A_1 & N \otimes J_{1\times n_2} \\ N^T \otimes J_{n_2\times 1} & A_2 \end{bmatrix},$$

where

$$A_2 = \begin{bmatrix} M_{11} & M_{12} & \cdots & M_{1k} \\ M_{12} & M_{22} & \cdots & M_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ M_{1k} & M_{2k} & \cdots & M_{kk} \end{bmatrix}.$$

Then the spectrum of A is

(1) the eigenvalues of the matrix

$$E = \begin{bmatrix} A_1 & \sqrt{n_2}N \\ \sqrt{n_2}N^T & R \end{bmatrix},$$

where $R = \begin{bmatrix} r_{11} & r_{12} & \ldots & r_{1k} \\ r_{12} & r_{22} & \ldots & r_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ r_{1k} & r_{2k} & \ldots & r_{kk} \end{bmatrix}$,

(2) the eigenvalues of the matrix

$$E_t = \begin{bmatrix} \lambda_t(M_{11}) & \lambda_t(M_{12}) & \ldots & \lambda_t(M_{1k}) \\ \lambda_t(M_{12}) & \lambda_t(M_{22}) & \ldots & \lambda_t(M_{2k}) \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_t(M_{1k}) & \lambda_t(M_{2k}) & \ldots & \lambda_t(M_{kk}) \end{bmatrix},$$

for $t = 2, 3, \ldots, n_2$*, where* $\lambda_t(M_{11}), \lambda_t(M_{12}), \ldots, \lambda_t(M_{kk})$ *are co-eigenvalues of* M_{11}*,* M_{12}*,* $\ldots$*,* M_{kk}*.*

Proof. Since M_{ij}s are symmetric matrices which commute with each other and $M_{ij} \in \mathcal{R}_{n_2} \times n_2(r_{ij})$ for $i, j = 1, 2, \ldots, k$, by Proposition 3.1, there exists an orthonormal basis $\{X_1(= \frac{1}{\sqrt{n_2}} J_{1\times n_2}),\ X_2,\ \ldots,\ X_{n_2}\}$ of $\mathbf{R}^{n_2}$ such that X_t is an eigenvector of M_{ij} corresponding to the eigenvalue $\lambda_t(M_{ij})$ for $t = 1, 2, \ldots, n_2$ and $i, j = 1, 2, \ldots, k$. Now consider the square matrix Q of order n_2, whose t-th column is X_t for $t = 1, 2, \ldots, n_2$. Then we have, $Q^T M_{ij} Q = D_{ij} = diag(r_{ij}, \lambda_2(M_{ij}), \ldots, \lambda_{n_2}(M_{ij}))$ for $i, j = 1, 2 \ldots, k$. Let

$$\begin{aligned} A' &= \begin{bmatrix} I_{n_1} & \mathbf{0} \\ \mathbf{0} & I_k \otimes Q^T \end{bmatrix} A \begin{bmatrix} I_{n_1} & \mathbf{0} \\ \mathbf{0} & I_k \otimes Q \end{bmatrix} \\ &= \begin{bmatrix} A_1 & B_1 & B_2 & \ldots & B_k \\ B_1^T & D_{11} & D_{12} & \cdots & D_{1k} \\ B_2^T & D_{12} & D_{22} & \cdots & D_{2k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B_k^T & D_{1k} & D_{2k} & \cdots & D_{kk} \end{bmatrix}, \end{aligned}$$

where

$$B_i = \begin{bmatrix} m_{1i} & 0 & \ldots & 0 \\ m_{2i} & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ m_{n_1 i} & 0 & \ldots & 0 \end{bmatrix}$$

for $i = 1, 2, \ldots, k$.

The eigenvalues of A and A' are the same, since they are similar matrices. So, it is enough to determine the eigenvalues of A'.

Take $\beta = \{1, 2, \ldots, n_1, n_1 + 1, n_1 + n_2 + 1, n_1 + 2n_2 + 1, \ldots, n_1 + (k-1)n_2 + 1\}$ and let $E = A'[\beta, \beta]$. Then

$$xI_{n_1+k} - E \;\;=\;\; \begin{bmatrix} xI_{n_1} - A_1 & -\sqrt{n_2}N \\ -\sqrt{n_2}N^T & xI_k - R \end{bmatrix}$$

and $(xI_{n_1+kn_2} - A')[\beta^c, \beta^c] = xI_{k(n_2-1)} - D'$, where

$$D' = \begin{bmatrix} D'_{11} & D'_{12} & \cdots & D'_{1k} \\ D'_{21} & D'_{22} & \cdots & D'_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ D'_{k1} & D'_{k2} & \cdots & D'_{kk} \end{bmatrix},$$

with D'_{ij} being the matrix obtained by deleting the first row and the first column of D_{ij}.

Also, notice that $det((xI_{n_1+kn_2} - A')[\alpha,\beta]) = 0$ for $\alpha \subset \{1,2,\dots,n_1+kn_2\}$ with $\alpha \neq \beta$ and α has n_1+n_2 elements. So by the Laplace Expansion Theorem,

$$\begin{aligned} P_{A'}(x) &= det((xI_{n_1+kn_2} - A')[\beta,\beta]) \times det((xI_{n_1+kn_2} - A')[\beta^c,\beta^c]) \\ &= P_E(x)P_{D'}(x). \end{aligned}$$

By using the Laplace Expansion Theorem repeatedly $n_2 - 1$ times, we can obtain

$$P_{D'}(x) = \prod_{t=2}^{n_2} P_{E_t}(x)$$

and hence the result follows. □

In the next result, we obtain the spectra of $G \bigvee_{[M:\mathcal{M}:\mathcal{T}]} \mathcal{H}_k$ for some G, $\mathcal{H}_k$, M and $\mathcal{M}$.

Theorem 3.3. *Let G_i be a graph with n_i vertices and m_i edges for $i = 1,2$. Let $M \in \mathcal{RC}_{n_1\times k}(\alpha,\beta)$ be a 0-1 matrix and let $M' \in \mathcal{R}_{n_2} \times n_2(r')$ be a symmetric matrix which commutes with $A(G_2)$. Let $\mathcal{H}_k = (G_2,G_2,\dots,G_2)$ and $\mathcal{T} = (V(G_2),V(G_2),\dots,V(G_2))$. Let H be an r-regular graph with k vertices and $A(H) = [h_{ij}]$. Let $\mathcal{M} = (h_{12}M', h_{13}M', \dots, h_{1k}M', h_{23}M', h_{24}M', \dots, h_{2k}M', \dots, h_{(k-1)k}M')$. Let Γ be $G_1 \bigvee_{[M:\mathcal{M}:\mathcal{T}]} \mathcal{H}_k$. Then the A-spectrum, the L-spectrum, the Q-spectrum and the $\widehat{L}$-spectrum of Γ are*

(1) the eigenvalues of the matrix $E = \begin{bmatrix} A & c\sqrt{n_2}M \\ c\sqrt{n_2}M^T & B \end{bmatrix}$;

(2) $\theta_t + \rho\lambda_s(H)\lambda_t(M')$ *for* $t = 2,3,\dots,n_2$; $s = 1,2,\dots,k$,

where

$$A = \begin{cases} A(G_1) & \text{for the } A\text{-spectrum of } \Gamma; \\ L(G_1) + \alpha n_2 I_{n_1} & \text{for the } L\text{-spectrum of } \Gamma; \\ Q(G_1) + \alpha n_2 I_{n_1} & \text{for the } Q\text{-spectrum of } \Gamma; \\ \dfrac{1}{c_1}[L(G_1) + \alpha n_2 I_{n_1}] & \text{for the } \widehat{L}\text{-spectrum of } \Gamma, \end{cases}$$

$$c = \begin{cases} 1 & \text{for the } A\text{-spectrum, the } Q\text{-spectrum of } \Gamma; \\ -1 & \text{for the } L\text{-spectrum of } \Gamma; \\ -\dfrac{1}{\sqrt{c_1 c_2}} & \text{for the } \widehat{L}\text{-spectrum of } \Gamma, \end{cases}$$

$$B = \begin{cases} r_2 I_k + r' A(H) & \text{for the } A\text{-spectrum of } \Gamma, \\ & \text{provided } G_2 \text{ is } r_2 \text{ regular;} \\ (\beta + rr')I_k - r'A(H) & \text{for the } L\text{-spectrum of } \Gamma; \\ (2r_2 + \beta + rr')I_k + r'A(H) & \text{for the } Q\text{-spectrum of } \Gamma, \\ & \text{provided } G_2 \text{ is } r_2 \text{ regular;} \\ \dfrac{1}{c_2}\left[(\beta + rr')I_k - r'A(H)\right] & \text{for the } \widehat{L}\text{-spectrum of } \Gamma; \end{cases}$$

$$\theta_t = \begin{cases} \lambda_t(G_2) & \text{for the } A\text{-spectrum of } \Gamma; \\ \mu_t(G_2) + \beta + rr' & \text{for the } L\text{-spectrum of } \Gamma; \\ \nu_t(G_2) + \beta + rr' & \text{for the } Q\text{-spectrum of } \Gamma; \\ \dfrac{1}{c_2}(\mu_t(G_2) + \beta + rr') & \text{for the } \widehat{L}\text{-spectrum of } \Gamma, \end{cases}$$

$$\rho = \begin{cases} 1 & \text{for the } A\text{-spectrum, the } Q\text{-spectrum of } \Gamma; \\ -1 & \text{for the } L\text{-spectrum of } \Gamma; \\ -\dfrac{1}{c_2} & \text{for the } \widehat{L}\text{-spectrum of } \Gamma, \end{cases}$$

$c_1 = r_1 + \alpha n_2$; $c_2 = r_2 + \beta + rr'$, provided that, G_i is r_i-regular ($r_i > 1$) for $i = 1, 2$.

In particular,

(a) if $k = n_1$ and, $A(G_1)$, M and $A(H)$ commute with each other, then the eigenvalues of E are

$$\frac{1}{2}\left(\lambda_i(A) + \lambda_i(B) \pm \sqrt{(\lambda_i(A) - \lambda_i(B))^2 - 4c^2 n_2 \lambda_i(M)^2}\right),$$

where $\lambda_i(A)$, $\lambda_i(B)$ and $\lambda_i(M)$ are co-eigenvalues of A, B and M for $i = 1, 2, \ldots, n_1$.

(b) if G_1 is regular and $M = J_{n_1 \times k}$, then the eigenvalues of E are

1. $\frac{1}{2}\left(\lambda_1(A) + \lambda_1(B) \pm \sqrt{(\lambda_1(A) - \lambda_1(B))^2 - 4c^2 k n_1 n_2}\right)$.

2. $\lambda_t(A)$ for $t = 2, 3, \ldots, n_1$;
3. $\lambda_t(B)$ for $t = 2, 3, \ldots, k$.

(c) *if G_1 is r_1-regular, $k = m_1$, $M = B(G_1)$ and $B = t_1 I_{m_1} + t_2 J_{m_1} + t_3 B(G_1)^T B(G_1)$, then the eigenvalues of E are*

1. t_1 with multiplicity $m_1 - n_1$;
2. $\frac{1}{2}\left(\lambda_t(A) + \eta_t \pm \sqrt{(\lambda_1(A) - \eta_t)^2 - 4c^2 n_2 \nu_t(G_1)}\right)$ for $t = 2, 3, \ldots, n_1$,

where

$$\eta_t = \begin{cases} t_1 + m_1 t_2 + 2r_1 t_3, & \text{for } t = 1, \\ t_1 + t_3 \nu_t(G_1), & \text{for } t = 2, 3, \ldots, n_1. \end{cases}$$

Proof. It can be verified that

$$A(\Gamma) = \begin{bmatrix} A(G_1) & M \otimes J_{1\times n_2} \\ M^T \otimes J_{n_2\times 1} & I_k \otimes A(G_2) + A(H) \otimes M' \end{bmatrix},$$

$$L(\Gamma) = \begin{bmatrix} L(G_1) + \alpha n_2 I_{n_1} & -M \otimes J_{1\times n_2} \\ -M^T \otimes J_{n_2\times 1} & I_k \otimes [L(G_2) + (\beta + rr')I_{n_2}] - A(H) \otimes M' \end{bmatrix},$$

$$Q(\Gamma) = \begin{bmatrix} Q(G_1) + \alpha n_2 I_{n_1} & M \otimes J_{1\times n_2} \\ M^T \otimes J_{n_2\times 1} & I_k \otimes [Q(G_2) + (\beta + rr')I_{n_2}] + A(H) \otimes M' \end{bmatrix},$$

$$\widehat{L}(\Gamma) = \begin{bmatrix} \frac{1}{c_1}\left(L(G_1) + \alpha n_2 I_{n_1}\right) & -\frac{1}{\sqrt{c_1 c_2}}\left(M \otimes J_{1\times n_2}\right) \\ -\frac{1}{\sqrt{c_1 c_2}}\left(M^T \otimes J_{n_2\times 1}\right) & \frac{1}{c_2}\left(I_k \otimes [L(G_2) + (\beta + rr')I_{n_2}] - A(H) \otimes M'\right) \end{bmatrix}.$$

Taking $A_1 = A$ and

$$M_{ii} = \begin{cases} A(G_2) & \text{for the } A\text{-spectrum of } \Gamma; \\ L(G_2) + (\beta + rr')I_{n_2} & \text{for the } L\text{-spectrum of } \Gamma; \\ Q(G_2) + (\beta + rr')I_{n_2} & \text{for the } Q\text{-spectrum of } \Gamma; \\ \dfrac{1}{c_2}[L(G_2) + (\beta + rr')I_{n_2}] & \text{for the } \widehat{L}\text{-spectrum of } \Gamma, \end{cases}$$

$$M_{ij} = \rho h_{ij} M'$$

for $i, j = 1, 2, \ldots, k$ and $i < j$ in Theorem 3.2, we obtain that $R = B$ and $E_t = \lambda_t(M_{11})I_k + \rho\lambda_t(M')A(H)$ for $t = 2, 3, \ldots, n_2$, and so the eigenvalues of E_t are $\theta_t + \rho\lambda_t(M')\lambda_s(H)$ for $t = 2, 3, \ldots, n_2$; $s = 1, 2, \ldots, k$. So the result follows.

(a) If $k = n_1$ and, $A(G_1)$, M and $A(H)$ commute with each other, then

$$E = \begin{bmatrix} A & c\sqrt{n_2}M \\ c\sqrt{n_2}M^T & B \end{bmatrix},$$

where A, M and B commute with each other. By Proposition 3.1, there exists a orthonormal matrix P such that

$$\begin{aligned} D_1 &= P^T A P = (\lambda_1(A), \lambda_2(A), \ldots, \lambda_{n_1}(A)), \\ D_2 &= P^T M P = (\lambda_1(M), \lambda_2(M), \ldots, \lambda_{n_1}(M)), \\ D_3 &= P^T B P = (\lambda_1(B), \lambda_2(B), \ldots, \lambda_{n_1}(B)). \end{aligned}$$

Let

$$\begin{aligned} D' &= \begin{bmatrix} P^T & \mathbf{0} \\ \mathbf{0} & P^T \end{bmatrix} \begin{bmatrix} A & c\sqrt{n_2}M \\ c\sqrt{n_2}M^T & B \end{bmatrix} \begin{bmatrix} P & \mathbf{0} \\ \mathbf{0} & P \end{bmatrix} \\ &= \begin{bmatrix} D_1 & c\sqrt{n_2}D_2 \\ c\sqrt{n_2}D_2 & D_3 \end{bmatrix}. \end{aligned}$$

Since E and D' are similar matrices, by using the Laplace Expansion Theorem repeatedly, we obtain that

$$\det(xI_{n_1} - D') = \prod_{i=1}^{n_1} \left(x^2 - [\lambda_i(A) + \lambda_i(B)]\,x + \lambda_i(A)\lambda_i(B) - c^2 n_2 \lambda_i(M)^2 \right)$$

(b) If $M = J_{n_1 \times k}$, then

$$E = \begin{bmatrix} A & c\sqrt{n_2}J_{n_1 \times k} \\ c\sqrt{n_2}J_{k \times n_1} & B \end{bmatrix}.$$

Since G_1 is regular, $\frac{1}{\sqrt{n_1}}J_{n_1 \times 1}$ is an eigenvector of A. Let $\{X_1 \left(= \frac{1}{\sqrt{n_1}}J_{n_1 \times 1}\right), X_2, \ldots, X_{n_1}\}$ and $\left\{Y_1 \left(= \frac{1}{\sqrt{k}}J_{k \times 1}\right), Y_2, \ldots, Y_k\right\}$ be sets of orthonormal eigenvectors of A and B, respectively. Let P be the matrix whose i-th column is X_i for $i = 1, 2, \ldots, n_1$ and let Q be the matrix whose i-th column is Y_i for $i = 1, 2, \ldots, k$.

Let

$$\begin{aligned} D' &= \begin{bmatrix} P^T & \mathbf{0} \\ \mathbf{0} & Q^T \end{bmatrix} \begin{bmatrix} A & c\sqrt{n_2}J_{n_1 \times k} \\ c\sqrt{n_2}J_{k \times n_1} & B \end{bmatrix} \begin{bmatrix} P & \mathbf{0} \\ \mathbf{0} & Q \end{bmatrix} \\ &= \begin{bmatrix} D_1 & D_2 \\ D_2^T & D_3 \end{bmatrix}, \end{aligned}$$

where $D_1 = diag(\lambda_1(A), \lambda_2(A), \ldots, \lambda_{n_1}(A))$; D_2 is the matrix of size $n_1 \times k$ whose first entry is $c\sqrt{kn_1n_2}$ and the other entries are 0; $D_3 = diag(\lambda_1(B), \lambda_2(B), \ldots, \lambda_k(B))$.

Since E and D' are similar matrices, by using the Laplace Expansion Theorem, by taking $\beta = \{1, n_1 + 1\}$, we obtain that

$$\det(xI_{n_1+k} - E) = \begin{vmatrix} x - \lambda_1(A) & c\sqrt{kn_1n_2} \\ c\sqrt{kn_1n_2} & x - \lambda_1(B) \end{vmatrix} \times \begin{vmatrix} xI_{n_1} - D_1' & \mathbf{0} \\ \mathbf{0} & xI_k - D_3' \end{vmatrix},$$

where $D_1' = diag(\lambda_2(A), \lambda_3(A), \ldots, \lambda_{n_1}(A))$;

$D_3' = diag(\lambda_2(B), \lambda_3(B), \ldots, \lambda_k(B))$.

From this we obtain the proof of this part.

(c) If $M = B(G_1)$ and $B = t_1 I_{m_1} + t_2 J_{m_1} + t_3 B(G_1)^T B(G_1)$, then

$$E = \begin{bmatrix} A & c\sqrt{n_2}B(G_1) \\ c\sqrt{n_2}B(G_1)^T & B \end{bmatrix}.$$

Let $\left\{X_1 \left(= \frac{1}{\sqrt{n_1}}J_{n_1 \times 1}\right), X_2, \ldots, X_{n_1}\right\}$ be a set of orthonormal eigenvectors of $B(G_1)B(G_1)^T$, where X_i is an eigenvector of $B(G_1)B(G_1)^T$ corresponding to $\nu_i(G_1)$ for $i = 1, 2, \ldots, n_1$.

Let $rank(B(G_1)) = t$. Then $\nu_i(G_1) \neq 0$ for $i = 1, 2, \ldots, t$ and $\nu_i(G_1) = 0$ for $i = t{+}1, t{+}2, \ldots, n_1$. Notice that $||B(G_1)^T X_i|| \neq 0$ for $i = 1, 2, \ldots, t$. For if $||B(G_1)^T X_i|| = 0$ for some i, then $B(G_1)^T X_i = 0$. This implies that $B(G_1)B(G_1)^T X_i = 0$ and so, $\nu_i(G_1) = 0$, a contradiction.

Now, let $Y_i = \dfrac{B(G_1)^T X_i}{||B(G_1)^T X_i||}$ for $i = 1, 2, \ldots, t$. Notice that Y_i is an eigenvector of $B(G_1)^T B(G_1)$ corresponding to the eigenvalue $\nu_i(G_1)$. Also, Y_i and Y_j are orthogonal, since X_i and X_j are so for $i, j = 1, 2, \ldots, t$; $i \neq j$.

Let $\{Y_{t+1}, Y_{t+2}, \ldots, Y_{m_1}\}$ be a set of orthogonal eigenvectors of $B(G_1)^T B(G_1)$ corresponding to the eigenvalue 0. Then for $i = 1, 2, \ldots, t$; $j = t+1, t+2, \ldots, m_1$, Y_i, Y_j are orthogonal, since they correspond to the distinct eigenvalues of the symmetric matrix $B(G_1)^T B(G_1)$.

Now let P be the matrix whose i-th column is X_i for $i = 1, 2, \ldots, n_1$ and let Q be the matrix whose i-th column is Y_i for $i = 1, 2, \ldots, m_1$.

Let

$$\begin{aligned} D' &= \begin{bmatrix} P^T & \mathbf{0} \\ \mathbf{0} & Q^T \end{bmatrix} \begin{bmatrix} A & c\sqrt{n_2}B(G_1) \\ c\sqrt{n_2}B(G_1)^T & B \end{bmatrix} \begin{bmatrix} P & \mathbf{0} \\ \mathbf{0} & Q \end{bmatrix} \\ &= \begin{bmatrix} D_1 & c\sqrt{n_2}D_2 \\ c\sqrt{n_2}D_3 & D_4 \end{bmatrix}, \end{aligned}$$

where

$D_1 = diag(\lambda_1(A), \lambda_2(A), \ldots, \lambda_{n_1}(A))$; $D_2 = \begin{bmatrix} D & O \end{bmatrix}$; $D_3 = \begin{bmatrix} I_{n_1} \\ O^T \end{bmatrix}$;

$$D_4 = \begin{bmatrix} D_1' & \mathbf{0} \\ \mathbf{0} & t_1 I_{m_1 - n_1} \end{bmatrix},$$

with $D = diag(2r_1, \nu_2(G_1), \ldots, \nu_{n_1}(G_1))$, $D_1' = diag(\eta_1, \eta_2, \ldots, \eta_{n_1})$ and O is the zero matrix of size $n_1 \times (m_1 - n_1)$.

Since E and D' are similar matrices, by applying the Laplace Expansion Theorem repeatedly, we obtain the result.

□

Corollary 3.1. *The A-spectrum, the L-spectrum, the Q-spectrum and the $\widehat{L}$-spectrum of $G_1 \bigvee_{[M:\mathcal{M}:\mathcal{T}]} \mathcal{H}_k$, where*

(1) $M' \in \{I_{n_2},\ A(G_2),\ A(G_2) + I_{n_2}\}$ can be obtained from Theorem 3.3 by substituting the eigenvalues of M;

(2) $M' \in \{J_{n_2} - I_{n_2}, A(\overline{G}_2), A(\overline{G}_2) + I_{n_2}\}$ *can be obtained from Theorem 3.3 by substituting the eigenvalues of* M*, provided* G_2 *is regular;*

(3) $M \in \{I_{n_1}, J_{n_1} - I_{n_1}, A(G_1), A(G_1) + I_{n_1}, A(\overline{G}_1), A(\overline{G}_1) + I_{n_1}\}$ *and* $H \in \{G_1, \overline{G}_1, \overline{K}_{n_1}, K_{n_1}\}$ *can be obtained from Theorem 3.3(i) by substituting the eigenvalues of* M*;*

(4) $H \in \{\mathcal{L}(G_1), \overline{\mathcal{L}(G_1)}, \overline{K}_{m_1}, K_{m_1}\}$ *can be obtained from Theorem 3.3(ii) by substituting the eigenvalues of* M*.*

Note 3.1. The existing results on the A-spectrum, the L-spectrum and the Q-spectrum of the graphs mentioned in Table 1 (cf. [2, Theorems 3.1, 3.2, 3.3], [2, Theorems 4.1, 4.2, 4.3], [31, Theorems 3.1, 3.2, 4.1, 4.2]) can be deduced by substituting the corresponding values and matrices in Theorem 3.3.

In the following result, we obtain infinitely many simultaneously A-cospectral, L-cospectral, Q-cospectral and $\widehat{L}$-cospectral graphs by using Theorem 3.3.

Corollary 3.2. *Let* G_i *be an* r_i*-regular graph with* n_i *vertices and* $m_i(= \frac{1}{2}n_i r_i)$ *edges for* $i = 1, 2$*. Let* G'_i *be a regular graph such that* G_i *and* G'_i *are cospectral for* $i = 1, 2$ *and let* H' *be a regular graph with* $A(H') = [h'_{ij}]$ *such that* H *and* H' *are cospectral. Then the following pairs of graphs are simultaneously* A*-cospectral,* L*-cospectral,* Q*-cospectral and* $\widehat{L}$*-cospectral.*

(1) $G_1 \bigvee_{[M:\mathcal{M}:\mathcal{T}]} \mathcal{H}_k$ *and* $G_1 \bigvee_{[M:\mathcal{M}:\mathcal{T}]} \mathcal{H}'_k$*, where* $\mathcal{H}'_k = (G'_2, G'_2, \dots, G'_2)$*.*

(2) $G_1 \bigvee_{[M:\mathcal{M}:\mathcal{T}]} \mathcal{H}_k$ *and* $G'_1 \bigvee_{[M:\mathcal{M}:\mathcal{T}]} \mathcal{H}_k$*, where* $k = n_2$*;* $A(G_1)$*,* M *and* $A(H)$ *are such that they commute with each other and* G'_1 *is a graph such that* $A(G'_1)$*,* M *and* $A(H)$ *commute with each other.*

(3) $G_1 \bigvee_{[M:\mathcal{M}:\mathcal{T}]} \mathcal{H}_k$ *and* $G'_1 \bigvee_{[M:\mathcal{M}:\mathcal{T}]} \mathcal{H}_k$*, where* $M = J_{n_1 \times k}$*.*

(4) $G_1 \bigvee_{[M:\mathcal{M}:\mathcal{T}]} \mathcal{H}_k$ *and* $G_1 \bigvee_{[M:\mathcal{M}':\mathcal{T}]} \mathcal{H}_k$*, where* $M = J_{n_1 \times k}$ *and* $\mathcal{M}' = (h'_{12}M', h'_{13}M', \dots, h'_{1k}M', h'_{23}M', h'_{24}M', \dots, h'_{2k}M', \dots, h'_{(k-1)k}M')$*.*

(5) $G_1 \bigvee_{[M:\mathcal{M}:\mathcal{T}]} \mathcal{H}_k$ *and* $G'_1 \bigvee_{[M:\mathcal{M}:\mathcal{T}]} \mathcal{H}_k$*, where* $k = m_1$*,* $M = B(G_1)$*.*

(6) $G_1 \bigvee_{[M:\mathcal{M}:\mathcal{T}]} \mathcal{H}_k$ *and* $G'_1 \bigvee_{[N:\mathcal{M}:\mathcal{T}]} \mathcal{H}_k$*, where* $k = m_1$*,* $M = B(G_1)$ *and* $N = B(G'_1)$*.*

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