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On extended biharmonic hypersurfaces with three curvatures

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Abstract

The subject of harmonic and biharmonic submanifolds, with important role in mathematical physics and differential geometry, arises from the variation problems of ordinary mean curvature vector field. Generally, harmonic submanifolds are biharmonic, but not vice versa. Of course, many examples of biharmonic hypersurfaces are harmonic. A well-known conjecture of Bang-Yen Chen on Euclidean spaces says that every biharmonic submanifold is harmonic. Although the conjecture has not been proven (in general case), it has been affirmed in many cases, and this has led to its spread to various types of submanifolds. Inspired by the conjecture, we study the Lorentz submanifolds of the Lorentz-Minkowski spaces. We consider an advanced version of the conjecture (namely, L_1-conjecture) on Lorentz hypersurfaces of the pseudo-Euclidean 5*-space* $\mathbf{L^5} := \mathbf{E^5_1}$ *(i.e. the Minkowski 5-space). We confirm the extended conjecture on Lorentz hypersurfaces with three principal curvatures.*

Keywords: *Minkowski space, L_1-biharmonic, isoparametric,* 1*-minimal.*

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1. Introduction

By a conjecture of Bang-Yen Chen (in 1987), the biharmonic submanifolds of Euclidean spaces have to be minimal. The conjecture has been confirmed in many cases (see for instance [1, 3, 4, 5, 6, 8, 9, 13, 18]). In applied mathematics, the biharmonic surfaces appear as solutions of strongly elliptic semilinear differential equations of order four ([7]). Also, the biharmonic Bezier surfaces play important roles in computational geometry. From physical point of view, the biharmonic surfaces play central roles in elastics and fluid mechanics.

In this paper, we study an extended version of Chen conjecture on time-like hypersurfaces in the Minkowski 5-space with constant mean curvature and three distinct principal curvatures. We show that, such a hypersurface is 1-minimal.

The paper is organized as follows. Section 2 is appropriated to prerequisites. In section 3, we study Lorentz hypersurfaces with at least three distinct principal curvatures in 5-dimensional Minkowski space satisfying the L_1-biharmonicity condition. We distinguish between diagonal and non-diagonal states for the second fundamental form (shape operator) of Lorentz hypersurfaces. Diagonal case is studied in Section 3. In non-diagonal case, the shape operator has three possible matrix forms, which will be explained in Section 4.

2. Preliminaries

First, we recall prerequisite concepts and notations from [2, 10, 11, 12, 14, 17]. By definition, the Minkowski 5-space $\mathbf{L^5}$ is obtained from Euclidean 5-space $\mathbf{E^5}$ by endowing with the following non-degenerate inner product $\langle \mathbf{v}, \mathbf{w} \rangle := -\mathbf{v_1 w_1} + \mathbf{\Sigma_{i=2}^5 v_i w_i}$, for every $\mathbf{v}, \mathbf{w} \in \mathbf{E^5}$. Every Lorentzian hypersurface M_1^4 of $\mathbf{L^5}$ is defined by an isometric immersion $\mathbf{x} : \mathbf{M_1^4} \to \mathbf{L^5}$ such that iduced metric on M_1^4 is Lorentzian. For each nonzero vector $\mathbf{v} \in \mathbf{L^5}$, the value of $\langle \mathbf{v}, \mathbf{v} \rangle$ can be a negative, zero or positive number and then, the vector $\mathbf{v}$ is said to be time-like, light-like or space-like, respectively. A given basis $\mathcal{B} := \{e_1, \cdots, e_4\}$ of the tangent space of M_1^4 is called *orthonormal* if $\langle e_i, e_j \rangle = \epsilon_i \delta_i^j$ (without Einstein convention) for $i, j = 1, \cdots, 4$, where $\epsilon_1 = -1$ and $\epsilon_i = 1$ for $i = 2, 3, 4$. As usual, δ_i^j stands for the Kronecker delta. $\mathcal{B}$ is called $pseudo-orthonormal$ if it satisfies $\langle e_1, e_1 \rangle = 0$, $\langle e_2, e_2 \rangle = 0$, $\langle e_2, e_1 \rangle = -1$ and $\langle e_j, e_i \rangle = \delta_i^j$, for $i = 1, 2, 3, 4$ and $j = 3, 4$.

According to an orthonormal or pseudo-orthonormal basis chosen on the tangent bundle of M_1^4, there are two possible matrix forms $\mathcal{G}_1 := diag[-1,1,1,1]$ and

$\mathcal{G}_2 = diag[\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, 1, 1]$ for the (induced) Lorentz metric on M_1^4.

In the case $\mathcal{G}_1$ (with respect to an orthonormal basis), the fundamental form has two possible matrix forms

$\mathcal{F}_1 = diag[\lambda_1, \lambda_2, \lambda_3, \lambda_4]$ and $\mathcal{F}_2 = diag[\begin{bmatrix} \kappa & \lambda \\ -\lambda & \kappa \end{bmatrix}, \eta_1, \eta_2]$, where $\lambda \neq 0$.

(Note that, the matrix $\mathcal{F}_2$ has two conjugate complex eigenvalues $\kappa \pm i\lambda$).

In the case $\mathcal{G}_2$ (with respect to a pseudo-orthonormal basis), the fundamental form has two possible matrix forms

$\mathcal{F}_3 = diag[\begin{bmatrix} \kappa & 0 \\ 1 & \kappa \end{bmatrix}, \lambda_1, \lambda_2]$ and $\mathcal{F}_4 = diag[\begin{bmatrix} \kappa & 0 & 0 \\ 0 & \kappa & 1 \\ -1 & 0 & \kappa \end{bmatrix}, \lambda]$.

Remark 2.1. In the case $\mathcal{G}_2$, we substitute the pseudo-orthonormal basis $\mathcal{B} := \{e_1, e_2, e_3, e_4\}$ by a new orthonormal one $\tilde{\mathcal{B}} := \{\tilde{e_1}, \tilde{e_2}, e_3, e_4\}$, where $\tilde{e_1} := \frac{1}{2}(e_1 + e_2)$ and $\tilde{e_2} := \frac{1}{2}(e_1 - e_2)$. Then, we obtain new matrix forms $\tilde{\mathcal{F}_3} = diag[\begin{bmatrix} \kappa + \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \kappa - \frac{1}{2} \end{bmatrix}, \lambda_1, \lambda_2]$ and $\tilde{\mathcal{F}_4} = diag[\begin{bmatrix} \kappa & 0 & \frac{\sqrt{2}}{2} \\ 0 & \kappa & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & \kappa \end{bmatrix}, \lambda]$ (instead of $\mathcal{F}_3$ and $\mathcal{F}_4$, respectively)

Now, we define the principal curvatures of M_1^4, denoted by $\kappa_1, \kappa_2, \kappa_3, \kappa_4$ as follow.

In case $S = \mathcal{F}_1$, we put $\kappa_i := \lambda_i$, for $i = 1, \cdots, 4$, where λ_i's are the eigenvalues of $\mathcal{F}_1$.

In the case $S = \mathcal{F}_2$, we put $\kappa_1 = \kappa + i\lambda$, $\kappa_2 = \kappa - i\lambda$, and $\kappa_i := \eta_{i-2}$, for $i = 3, 4$.

In cases $S = \tilde{\mathcal{F}_3}$, we take $\kappa_i := \kappa$ for $i = 1, 2$, and $\kappa_i := \lambda_{i-2}$, for $i = 3, 4$.

In case $S = \tilde{\mathcal{F}_4}$, we take $\kappa_i := \kappa$ for $i = 1, 2, 3$, and $\kappa_4 := \lambda$.

The characteristic polynomial of S on M_1^4 is of the form
$Q(t) = \prod_{i=1}^4 (t - \kappa_i) = \sum_{j=0}^4 (-1)^j s_j t^{4-j}$, where, $s_0 := 1$,
$s_i := \sum_{1 \leq j_1 < \cdots < j_i \leq 4} \kappa_{j_1} \cdots \kappa_{j_i}$ for $i = 1, 2, 3, 4$.

For $j = 1, \cdots, 4$, the *jth mean curvature* H_j of M_1^4 is defined by $H_j = \frac{1}{\binom{4}{j}} s_j$. When H_j is identically null, M_1^4 is called $(j-1)$-*minimal*. When M_1^4 has diagonal shape operator with constant eigenvalues it is called *isoparametric*. Having nondiagonal shape operator, M_1^4 is called *isoparametric* if its minimal polynomial is constant. By Theorem 4.10 in [12], if M_1^4 has complex principal curvatures, then it cannot be isoparametric.

The Newton operator on M_1^4 is defined by

$$P_0 = I, P_j = s_j I - S \circ P_{j-1}, (j = 1, 2, 3, 4), \tag{2.1}$$

where, I is the identity map. Also, its explicit formula is $P_j = \sum_{i=0}^{j}(-1)^i s_{j-i} S^i$ (where $S^0 = I$) (see [2, 15]).

When $S = \mathcal{F}_1$, we have $P_j = diag[\mu_{1;j}, \cdots, \mu_{4;j}]$, for $j = 1, 2, 3$.

In the case $S = \mathcal{F}_2$, we have

$P_1 = diag\left[\begin{bmatrix} \kappa+\eta_1+\eta_2 & -\lambda \\ \lambda & \kappa+\eta_1+\eta_2 \end{bmatrix}, 2\kappa+\eta_2, 2\kappa+\eta_1\right]$

and

$$P_2 = diag\left[\begin{bmatrix} \kappa(\eta_1+\eta_2)+\eta_1\eta_2 & -\lambda(\eta_1+\eta_2) \\ \lambda(\eta_1+\eta_2) & \kappa(\eta_1+\eta_2)+\eta_1\eta_2 \end{bmatrix},\right.$$

$\left.\kappa^2+\lambda^2+2\kappa\eta_2, \kappa^2+\lambda^2+2\kappa\eta_1.\right]$

When $S = \tilde{\mathcal{F}}_3$, we have $P_1 = diag[\begin{bmatrix} \lambda_1+\lambda_2+\kappa-\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \lambda_1+\lambda_2+\kappa+\frac{1}{2} \end{bmatrix},$

$2\kappa+\lambda_2, 2\kappa+\lambda_1]$

and

$$P_2 = diag[\begin{bmatrix} \lambda_1\lambda_2+(\kappa-\frac{1}{2})(\lambda_1+\lambda_2) & -\frac{1}{2}(\lambda_1+\lambda_2) \\ \frac{1}{2}(\lambda_1+\lambda_2) & \lambda_1\lambda_2+(\kappa+\frac{1}{2})(\lambda_1+\lambda_2) \end{bmatrix},$$

$\kappa(\kappa+2\lambda_2), \kappa(\kappa+2\lambda_1)]$.

If $S = \tilde{\mathcal{F}}_4$, we have $P_1 = diag[\begin{bmatrix} 2\kappa+\lambda & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 2\kappa+\lambda & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 2\kappa+\lambda \end{bmatrix}, 3\kappa]$

and

$$P_2 = diag[\begin{bmatrix} 2\kappa\lambda+\kappa^2-\frac{1}{2} & -\frac{1}{2} & -\frac{\sqrt{2}}{2}(\kappa+\lambda) \\ \frac{1}{2} & 2\kappa\lambda+\kappa^2+\frac{1}{2} & \frac{\sqrt{2}}{2}(\kappa+\lambda) \\ \frac{\sqrt{2}}{2}(\kappa+\lambda) & \frac{\sqrt{2}}{2}(\kappa+\lambda) & 2\kappa\lambda+\kappa^2 \end{bmatrix}, 3\kappa^2].$$

The following function on M_1^4 will be used frequently:

$\mu_{i;k} = \sum_{1\le j_1<\cdots<j_k\le 4; j_l\neq i} \kappa_{j_1}\cdots\kappa_{j_k}, (i=1,2,3,4; 1\le k\le 3)$.
In all cases we have the following important identities
Here, we recall some identities from [2, 15].

$$\begin{aligned} &\mu_{i,1} = 4H_1 - \lambda_i, \\ &\mu_{i,2} = 6H_2 - \lambda_i\mu_{i,1} = 6H_2 - 4\lambda_i H_1 + \lambda_i^2, (1\le i\le 4), \end{aligned} \tag{2.2}$$

$$\begin{aligned} &tr(P_1) = 12H_1, \\ &tr(P_2) = 12H_2, tr(P_1\circ S) = 12H_2, tr(P_2\circ S) = 12H_3, \end{aligned} \tag{2.3}$$

$$\begin{aligned} &trS^2 = 4(4H_1^2 - 3H_2), \\ &tr(P_1\circ S^2) = 12(2H_1H_2 - H_3), tr(P_2\circ S^2) = 4(4H_1H_3 - H_4). \end{aligned} \tag{2.4}$$

The *kth linearized operator* $L_k : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ is defined as $L_k(f) := tr(P_k \circ \nabla^2 f)$, where, $\langle \nabla^2 f(X), Y\rangle = \langle \nabla_X \nabla f, Y\rangle$ for each tangent vector fields X and Y (see [2, 10, 11, 15, 16]). In special case $k=1$, we have

$$L_1(f) = \sum_{i=1}^{4} \epsilon_i \mu_{i,1}(e_ie_if - \nabla_{e_i}e_if). \tag{2.5}$$

For a Lorentzian hypersurface $x : M_1^4 \to \mathbf{L^5}$ we have

$$(i) L_1 x = 12H_2\mathbf{n}, \mathbf{(ii) L_1 n = -6\nabla(H_2) - 12[2H_1H_2 - H_3]n}, \tag{2.6}$$

$$L_1^2 x = 12L_1(H_2\mathbf{n}) = \mathbf{24[P_2\nabla H_2 - 9H_2\nabla H_2] + 12[L_1H_2 - 12H_2(2H_1H_2 - H_3)]n}. \tag{2.7}$$

If x satisfies $L_1^2 x = 0$, then M_1^4 is said to be L_1-*biharmonic*. By equalities (2.6) and (2.7), from the condition $L_1(H_2\mathbf{n}) = \mathbf{0}$ (which is equivalent to L_1-biharmonicity), we obtain simpler conditions on M_1^4 to be L_1-biharmonic as:

$$(i) L_1H_2 = 12H_2(2H_1H_2 - H_3), (ii) P_2\nabla H_2 = 9H_2\nabla H_2. \tag{2.8}$$

3. Diagonal shape operator

In this section, we study L_1-biharmonic Lorentzian hypersurfaces in $\mathbf{L^5}$ with diagonal shape operator and three distinct principal curvatures. We confirm the modified conjecture on the mentioned hypersurfaces.

Proposition 3.1. Every L_1-biharmonic orientable Lorentzian hypersurface M_1^4 in $\mathbf{L^5}$ having diagonal shape operator, constant mean curvature and nonconstant 2nd mean curvature has a nonconstant principal curvature of multiplicity one.

Proof. We consider the open subset $\mathcal{U} \subset \mathcal{M}_\infty^\triangle$, on which we have $\nabla H_2 \neq 0$. By conditions (2.8)(ii), taking $e_1 := \frac{\nabla H_2}{||\nabla H_2||}$ we get $P_2 e_1 = 9H_2 e_1$ on $\mathcal{U}$. Without loss of generality, we can take a suitable orthonormal local basis $\{e_1, e_2, e_3, e_4\}$ for the tangent bundle of M, consisting of the eigenvectors of the shape operator S such that $Se_i = \lambda_i e_i$ and $P_2 e_i = \mu_{i,2} e_i$, (for $i = 1, 2, 3, 4$) and then

$$\mu_{1,2} = 9H_2. \tag{3.1}$$

By the polar decomposition $\nabla H_2 = \sum_{i=1}^{4} e_i(H_2) e_i$, we get

$$e_1(H_2) \neq 0, e_2(H_2) = e_3(H_2) = e_4(H_2) = 0. \tag{3.2}$$

By (2.2) and (3.1) we have

$$H_2 = \frac{1}{3}\lambda_1(\lambda_1 - 4H). \tag{3.3}$$

Then, having assumed H to be constant, from (3.2) we get

$$e_1(\lambda_1) \neq 0, e_2(\lambda_1) = e_3(\lambda_1) = e_4(\lambda_1) = 0, \tag{3.4}$$

which gives that λ_1 is non-constant. Now, putting $\nabla_{e_i} e_j = \sum_{k=1}^{4} \omega_{ij}^k e_k$ (for $i, j = 1, 2, 3, 4$), the identity $e_k < e_i, e_j >= 0$ gives $\epsilon_j \omega_{ki}^j = -\epsilon_i \omega_{kj}^i$ (for $i, j, k = 1, 2, 3, 4$). Furthermore, for distinct $i, j, k = 1, 2, 3, 4$, the Codazzi equation implies

$$e_i(\lambda_j) = (\lambda_i - \lambda_j)\omega_{ji}^j, (\lambda_i - \lambda_j)\omega_{ki}^j = (\lambda_k - \lambda_j)\omega_{ik}^j. \tag{3.5}$$

Since by (3.4) we have $e_1(\lambda_1) \neq 0$, we claim $\lambda_j \neq \lambda_1$ for $j = 2, 3, 4$. Because, assuming $\lambda_j = \lambda_1$ for some integer $j \neq 1$, we have $e_1(\lambda_j) =$

$e_1(\lambda_1) \neq 0$. On the other hand, from (3.5) we obtain $0 = (\lambda_1 - \lambda_j)\omega_{j1}^j = e_1(\lambda_j) = e_1(\lambda_1)$. So, we got a contradiction. □

One can find the similar ordinary version of Proposition 3.1 in [8] and [18].

Proposition 3.2. Let M_1^4 be a L_1-biharmonic Lorentzian hypersurface in L^5 with diagonal shape operator, which has exactly three distinct principal curvatures, constant mean curvature and non-constant second mean curvature. Then, there exists a locally moving orthonormal tangent frame $\{e_1, e_2, e_3, e_4\}$ of principal vectors of M_1^4 with associated principal curvatures $\lambda_1, \lambda_2 = \lambda_3, \lambda_4$, which satisfy the following equalities:

$$
\begin{array}{l}
(i)\nabla_{e_1}e_1 = 0, \nabla_{e_2}e_1 = \alpha e_2, \nabla_{e_3}e_1 = \alpha e_3, \nabla_{e_4}e_1 = -\beta e_4,\\
(ii)\nabla_{e_2}e_2 = -\alpha e_1 + \omega_{22}^3 e_3 + \gamma e_4, \nabla_{e_i}e_2 = \omega_{i2}^3 e_3 for i = 1, 3, 4;\\
(iii)\nabla_{e_3}e_3 = -\alpha e_1 - \omega_{32}^3 e_3 + \gamma e_4, \nabla_{e_i}e_3 = -\omega_{i2}^3 e_2 for i = 1, 2, 4,\\
(iv)\nabla_{e_1}e_4 = 0, \nabla_{e_2}e_4 = -\gamma e_2, \nabla_{e_3}e_4 = -\gamma e_3, \nabla_{e_4}e_4 = \beta e_1,
\end{array}
\tag{3.6}
$$

where $\alpha := \frac{e_1(\lambda_2)}{\lambda_1 - \lambda_2}$, $\beta := \frac{e_1(\lambda_1 + 2\lambda_2)}{\lambda_1 - \lambda_4}$, $\gamma := \frac{e_4(\lambda_2)}{\lambda_2 - \lambda_4}$.

Proof. Similar to the proof of Proposition 3.1, taking a suitable local basis $\{e_1, e_2, e_3, e_4\}$ for TM, one can see that the equalities $(3.1) - (3.5)$ occur and λ_1 is of multiplicity one. Also, direct calculations give $[e_2, e_3](\lambda_1) = [e_3, e_4](\lambda_1) = [e_2, e_4](\lambda_1) = 0$, which yields

$$
\omega_{23}^1 = \omega_{32}^1, \omega_{34}^1 = \omega_{43}^1, \omega_{24}^1 = \omega_{42}^1. \tag{3.7}
$$

Now, having assumed M_1^4 to has three distinct principal curvatures, (without loss of generality) we can take $\lambda_2 = \lambda_3$, and then $\lambda_4 = 4H_1 - \lambda_1 - 2\lambda_2$. Hence, applying equalities (3.5) for distinct positive integers i, j and k less than 5, we get $e_2(\lambda_2) = e_3(\lambda_2) = 0$ and then,

$$
\begin{array}{l}
(i)\omega_{11}^1 = \omega_{12}^1 = \omega_{13}^1 = \omega_{14}^1 = \omega_{31}^2 = \omega_{21}^3 = \omega_{34}^2 = \omega_{24}^3 = \omega_{42}^4 = \omega_{43}^4 = 0,\\
(ii)\omega_{21}^2 = \omega_{31}^3 = \frac{e_1(\lambda_2)}{\lambda_1 - \lambda_2}, \omega_{41}^4 = \frac{-e_1(\lambda_1 + 2\lambda_2)}{\lambda_1 - \lambda_4}, \omega_{24}^2 = \omega_{34}^3 = \frac{-e_4(\lambda_2)}{\lambda_2 - \lambda_4},\\
(iii)(\lambda_1 - \lambda_4)\omega_{24}^1 = (\lambda_1 - \lambda_2)\omega_{42}^1, (\lambda_1 - \lambda_4)\omega_{34}^1 = (\lambda_1 - \lambda_2)\omega_{43}^1.
\end{array}
\tag{3.8}
$$

From (3.7) and (3.8) we get $\omega_{24}^1 = \omega_{42}^1, = \omega_{34}^1 = \omega_{43}^1 = \omega_{12}^4 = \omega_{13}^4 = 0$. Therefore, all items of the proposition obtain from the above results. □

Proposition 3.3. Let M_1^4 be a L_1-biharmonic orientable Lorentzian hypersurface in $\mathbf{L^5}$ with diagonal shape operator, which has three distinct principal curvatures, constant mean curvature and non-constant second mean curvature. Then, there exists an orthonormal (local) tangent frame $\{e_1, e_2, e_3, e_4\}$ of principal vectors of M_1^4 with associated principal curvatures $\lambda_1, \lambda_2 = \lambda_3, \lambda_4$, satisfying $e_4(\lambda_2) = 0$ and

$$e_1(\lambda_2)e_1(\lambda_1 + 2\lambda_2) = \frac{1}{2}\lambda_2(\lambda_1 - \lambda_2)(\lambda_4 - \lambda_1)(2\lambda_1 + 4\lambda_2 + \lambda_4). \tag{3.9}$$

Proof. From Gauss curvature tensor $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$, by substituting X, Y and Z by different choices from e_1, e_2, e_3 and e_4, using the results of Proposition 3.2, we get the following equalities:

$$\begin{array}{l} (i) e_1(\alpha) + \alpha^2 = -\lambda_1\lambda_2, \beta^2 - e_1(\beta) = -\lambda_1\lambda_4; \\ (ii) e_1\left(\frac{e_4(\lambda_2)}{\lambda_2-\lambda_4}\right) + \alpha\frac{e_4(\lambda_2)}{\lambda_2-\lambda_4} = 0; \\ (iii) e_4(\alpha) - (\alpha+\beta)\frac{e_4(\lambda_2)}{\lambda_2-\lambda_4} = 0; \\ (iv) e_4\left(\frac{e_4(\lambda_2)}{\lambda_2-\lambda_4}\right) + \alpha\beta - \left(\frac{e_4(\lambda_2)}{\lambda_2-\lambda_4}\right)^2 = \lambda_2\lambda_4. \end{array} \tag{3.10}$$

Now, from (2.5) and (2.8), applying Proposition (3.2) we obtain

$$\begin{array}{l} (\lambda_1 - 4H_1)e_1e_1(H_2) - (2(\lambda_2 - 4H_1)\alpha + (\lambda_1 + 2\lambda_2)\beta)e_1(H_2) \\ = 12H_2(2H_1H_2 - H_3), \end{array} \tag{3.11}$$

where $\alpha := \frac{e_1(\lambda_2)}{\lambda_1-\lambda_2}$ and $\beta := \frac{e_1(\lambda_1+2\lambda_2)}{\lambda_1-\lambda_4}$.

On the other hand, from (3.2) and (3.6), we obtain

$$e_ie_1(H_{k+1}) = 0, \tag{3.12}$$

for $i = 2, 3, 4$. Also, we differentiate β and α by e_4 which gives

$$(\lambda_1 - \lambda_2)e_4(\alpha) - \alpha e_4(\lambda_2) = e_4e_1(\lambda_2) = \frac{1}{2}(\lambda_1 - \lambda_4)e_4(\beta) + \beta e_4(\lambda_2),$$

then

$$\frac{1}{2}(\lambda_1 - \lambda_4)e_4(\beta) = (\lambda_1 - \lambda_2)e_4(\alpha) - (\alpha + \beta)e_4(\lambda_2),$$

which, by substituting the value of $e_4(\alpha)$ from (3.10), gives

$$e_4(\beta) = \frac{-8e_4(\lambda_2)(\alpha + \beta)(\lambda_2 - H_1)}{(\lambda_1 - \lambda_4)(\lambda_2 - \lambda_4)}.$$

Again, differentiating (3.11)along e_4 and using (3.12), (3.10) and the last value of $e_4(\beta)$, we get $e_4(\lambda_2) = 0$ or

$$\frac{4(\alpha+\beta)[-H_1(8\lambda_1+12\lambda_2)+{\lambda_1}^2+3\lambda_1\lambda_2+16H_1^2]e_1(H_2)}{\lambda_4-\lambda_1}+6H_2(\lambda_2-\lambda_4)^2=0. \tag{3.13}$$

Finally, we claim that $e_4(\lambda_2) = 0$.

Indeed, if the claim be false, then we have

$$\frac{4(\alpha+\beta)\gamma e_1(H_2)}{\lambda_1-\lambda_4}=6H_2(\lambda_2-\lambda_4)^2, \tag{3.14}$$

where $\gamma = -8H_1\lambda_1 + {\lambda_1}^2 + 3\lambda_1\lambda_2 - 12H_1\lambda_2 + 16H_1^2$. Differentiating (3.14) along e_4, we get

$$\begin{aligned}&\tfrac{2(\alpha+\beta)[6\gamma(\lambda_2-H_1)+(3\lambda_1-12H_1)(\lambda_1+\lambda_2-2H_1)(\lambda_1+3\lambda_2-4H_1)]e_1(H_2)}{(\lambda_1+\lambda_2-2H_1)^2}\\&=36H_2(4H_1+\lambda_1+3\lambda_2)^2.\end{aligned} \tag{3.15}$$

Eliminating $e_1(H_2)$ from (3.14) and (3.15), we obtain

$$\gamma(2\lambda_1-2H_1)=(\lambda_1-4H_1)(\lambda_1+\lambda_2-2H_1)(-4H_1+\lambda_1+3\lambda_2). \tag{3.16}$$

Also, we differentiate (3.16) along e_4 which gives $4H_1 = \lambda_1$. This is impossible since λ_1 is nonconstant. So, $e_4(\lambda_2) = 0$. Hence, the latest equality in (3.10) gives the main result. □

Theorem 3.4. Let $x : M_1^4 \to \mathbf{L^5}$ be a L_1-biharmonic Lorentzian hypersurface with diagonal shape operator, constant mean curvature, and three distinct principal curvatures, then it is 1-minimal.

Proof. First, we assume H_2 is non-constant on M and try to get a contradiction. By differentiating (3.3) in direction of e_1 and using the definition of β, we get

$$e_1(H_2)=\frac{4}{3}(2H_1-\lambda_1)e_1(\lambda_2)+\frac{4}{3}(\lambda_1+\lambda_2-2H_1)(\lambda_1-2H_1)\beta. \tag{3.17}$$

By Proposition 3.3 and equalities (3.10), from (3.17) we obtain

$$\begin{aligned}&e_1e_1(H_2)=\tfrac{4}{3}\lambda_1\lambda_2(\lambda_1-\lambda_2)(\lambda_1+2H_1)\\&+\tfrac{4}{3}(4H_1-\lambda_1-2\lambda_2)(\lambda_1-2H_1)(4\lambda_1\lambda_2+{\lambda_1}^2-4H_1\lambda_2-2H_1\lambda_1)\\&+\left[3\beta-4\alpha+2\tfrac{(\lambda_1+\lambda_2-2H_1)\beta-(\lambda_1-\lambda_2)\alpha}{\lambda_1-2H_1}\right]e_1(H_2).\end{aligned} \tag{3.18}$$

Combining (3.11) and (3.18), we get

$$(P_{1,2}\alpha + P_{2,2}\beta)e_1(H_2) = P_{3,6}, \tag{3.19}$$

where the polynomial degree $P_{1,2}$, $P_{2,2}$ and $P_{3,6}$ in terms λ_1 and λ_2 are 2, 2 and 6, respectively.

Differentiating (3.19) along e_1 and using equalities (3.9), (3.10)-(i) and (3.19), we get the following equality

$$P_{4,8}\alpha + P_{5,8}\beta = P_{6,5}e_1(H_2). \tag{3.20}$$

The polynomial degree of $P_{4,8}, P_{5,8}$ and $P_{6,5}$ in terms λ_1 and λ_2 are 8, 8 and 5, respectively.

Combining (3.17) and (3.20), we obtain

$$\begin{aligned}&\Big(P_{4,8} + \tfrac{4}{3}P_{6,5}(\lambda_1 - \lambda_2)(\lambda_1 - 2H_1)\Big)\,\alpha \\ &+ \Big(P_{5,8} - \tfrac{4}{3}P_{6,5}(\lambda_1 + \lambda_2 - 2H_1)(\lambda_1 - 2H_1)\Big)\,\beta = 0.\end{aligned} \tag{3.21}$$

On the other hand, combining (3.17) with (3.19) and using Proposition 3.3, we get

$$P_{2,2}(\lambda_1 + \lambda_2 - 2H_1)(\lambda_1 - 2H_1)\beta^2 - P_{1,2}(\lambda_1 - \lambda_2)(\lambda_1 - 2H_1)\alpha^2 = \zeta,$$

(3.22)

where ζ is given by

$$\zeta = \lambda_2(4H_1 - \lambda_1 - 2\lambda_2)(\lambda_1 - 2H_1)\left(\frac{P}{2,2(\lambda_1 - \lambda_2) - P_{1,2}(\lambda_1 + \lambda_2 - 2H_1)}\right) + \frac{3}{4}P_{3,6}.$$

Using Proposition 3.3 and equality (3.21), we get

$$\begin{aligned}\alpha^2 &= \tfrac{\frac{2}{3}P_{6,5}(\lambda_1-\lambda_4)(\lambda_1-2H_1)+P_{5,8}}{P_{4,8}+\frac{4}{3}P_{6,5}(\lambda_1-\lambda_2)(\lambda_1-2H_1)}\lambda_2\lambda_4, \\ \beta^2 &= \tfrac{\frac{4}{3}P_{6,5}(\lambda_1-\lambda_2)(\lambda_1-2H_1)-P_{4,8}}{P_{5,8}-\frac{2}{3}P_{6,5}(\lambda_1-\lambda_4)(\lambda_1-2H_1)}\lambda_2\lambda_4.\end{aligned} \tag{3.23}$$

From (3.22) and (3.25) we get the following polynomial of degree 22:

$$
\begin{aligned}
&-\lambda_2\lambda_4(\lambda_1+2H_1)(\lambda_2-\lambda_1)P_{1,2}\Big(P_{5,8}-\tfrac{2}{3}P_{6,5}(\lambda_1-\lambda_4)(\lambda_1-2H_1)\Big)^2\\
&-\tfrac{1}{2}\lambda_2\lambda_4(\lambda_1+2H_1)(\lambda_1-\lambda_4)P_{2,2}\Big(P_{4,8}+\tfrac{4}{3}P_{6,5}(\lambda_1-\lambda_2)(\lambda_1-2H_1)\Big)^2\\
&=\zeta\left(P_{5,8}-\tfrac{2}{3}P_{6,5}(\lambda_1-\lambda_4)(\lambda_1-2H_1)\right)\left(P_{4,8}+\tfrac{4}{3}P_{6,5}(\lambda_1-\lambda_2)(\lambda_1-2H_1)\right),
\end{aligned}
$$

(3.24)

Let $\gamma(t), t\in I$ be an integral curve of e_1 passing through $p=\gamma(t_0)$. Then, $e_1(\lambda_1)$ and $e_1(\lambda_2)$ are nonzero and for $i=2,3,4$ we have $e_i(\lambda_1)=e_i(\lambda_2)=0$. We take $\lambda_2=\lambda_2(t)$ and $\lambda_1=\lambda_1(\lambda_2)$ in some neighborhood of $\lambda_0=\lambda_2(t_0)$. Using (3.21), we have

$$
\begin{aligned}
&\tfrac{d\lambda_1}{d\lambda_2}=\tfrac{d\lambda_1}{dt}\tfrac{dt}{d\lambda_2}=\tfrac{e_1(\lambda_1)}{e_1(\lambda_2)}\\
&=2\tfrac{(\lambda_1+\lambda_2-2H_1)\beta-(\lambda_1-\lambda_2)\alpha}{(\lambda_1-\lambda_2)\alpha}\\
&=\tfrac{2\left(P_{4,8}+\frac{4}{3}P_{6,5}(\lambda_1-\lambda_2)(\lambda_1-2H_1)\right)(\lambda_1+\lambda_2-2H_1)}{\left(\frac{4}{3}P_{6,5}(\lambda_1+\lambda_2-2H_1)(\lambda_1-2H_1)-P_{5,8}\right)(\lambda_1-\lambda_2)}-2
\end{aligned}
\tag{3.25}
$$

Now, we differentiate (3.24) with respect to λ_2 and then substitute $\frac{d\lambda_1}{d\lambda_2}$ from (3.25), which gives

$$
f(\lambda_1,\lambda_2)=0,
\tag{3.26}
$$

where $f(\lambda_1,\lambda_2)$ is an algebraic phrase of degree 30 in terms of λ_1 and λ_2.

We rearrange (3.24) and (3.26) as power series in terms of λ_2 as follow.

$$
\begin{aligned}
&(i)\ \sum_{i=0}^{22} f_i(\lambda_1)\lambda_2^i=0,\\
&(ii)\ \sum_{i=0}^{30} g_i(\lambda_1)\lambda_2^i=0.
\end{aligned}
\tag{3.27}
$$

We eliminate λ_2^{30} between (3.27)(i) and (3.27)(ii) we obtain a new polynomial equation in λ_2 of degree 29. Combining obtained equation with (3.27)(i), we obtain a polynomial equation in λ_2 of degree 28. In a similar way, by (3.27)(i) and its consequences we can eliminate λ_2. At last, we obtain a non-trivial algebraic polynomial equation in λ_1 with constant coefficients which implies that λ_1 is constant and then by (3.3), λ_2 and H_2 are constant, which contradicts with the first assumption. Hence, H_2 is constant on M_1^4.

Now, we claim that $H_2=0$. Having assumed $H_2\neq 0$, the condition (2.8)(i) implies that the 3rd mean curvature is constant. So, all mean curvatures are constant (i.e. M_1^4 is isoparametric). By Corollary 2.7 in

[12], an isoparametric Lorentzian hypersurface of type $\mathcal{F}_1$ has at most one nonzero principal curvature, which contradicts with the assumption that, three principal curvatures of M are assumed to be mutually distinct. So $H_2 \equiv 0$. □

4. Nondiagonal cases

Theorem 4.1. Let $x : M_1^4 \to L^5$ be a L_1-biharmonic orientable Lorentzian hypersurface of type $\mathcal{F}_2$. If the 1st mean curvature and one of real principal curvature are constant, then the 2nd mean curvature is constant. Also, in this case M_1^4 is 3-minimal

Proof. Firstly, we prove that 2nd mean curvature is constant. Taking $\mathcal{U} = \{p \in M_1^4 : \nabla H_2^2(p) \neq 0\}$, it is enough to prove $\mathcal{U} = \emptyset$. Assuming that $\mathcal{U}$ is nonempty we try to get a contradiction. M_1^4 is of type $\mathcal{F}_2$ means that with respect to a suitable (local) orthonormal tangent frame $\{e_1, \cdots, e_4\}$ on M, S is of form $\mathcal{F}_2$, such that $Se_1 = \kappa e_1 - \lambda e_2$, $Se_2 = \lambda e_1 + \kappa e_2$, $Se_3 = \eta_1 e_3$, $Se_4 = \eta_2 e_4$ and then, we have $P_2 e_1 = [\kappa(\eta_1 + \eta_2) + \eta_1\eta_2]e_1 + \lambda(\eta_1 + \eta_2)e_2$, $P_2 e_2 = -\lambda(\eta_1 + \eta_2)e_1 + [\kappa(\eta_1 + \eta_2) + \eta_1\eta_2]e_2$, $P_2 e_3 = (\kappa^2 + \lambda^2 + 2\kappa\eta_2)e_3$ and $P_2 e_4 = (\kappa^2 + \lambda^2 + 2\kappa\eta_1)e_4$.

Using the polar decomposition $\nabla H_2 = \sum_{i=1}^{4} \epsilon_i e_i(H_2) e_i$, from condition (2.8)(ii) we get

$$(4.1)\quad \begin{array}{l} (i)(\kappa\eta_1 + \kappa\eta_2 + \eta_1\eta_2 - 9H_2)\epsilon_1 e_1(H_2) - \lambda(\eta_1 + \eta_2)\epsilon_2 e_2(H_2) = 0 \\ (ii)\lambda(\eta_1 + \eta_2)\epsilon_1 e_1(H_2) + (\kappa\eta_1 + \kappa\eta_2 + \eta_1\eta_2 - 9H_2)\epsilon_2 e_2(H_2) = 0 \\ (iii)(\kappa^2 + \lambda^2 + 2\kappa\eta_2 - 9H_2)\epsilon_3 e_3(H_2) = 0, \\ (iv)(\kappa^2 + \lambda^2 + 2\kappa\eta_1 - 9H_2)\epsilon_4 e_4(H_2) = 0, \end{array}$$

Now, assuming H_1 and η_1 to be constant on M, the we prove four simple claims.

Claim: $e_1(H_2) = e_2(H_2) = e_3(H_2) = e_4(H_2) = 0$.
If $e_1(H_2) \neq 0$, dividing equalities (4.1)(i) and (4.1)(ii) by $\epsilon_1 e_1(H_2)$ and putting $u := \frac{\epsilon_2 e_2(H_2)}{\epsilon_1 e_1(H_2)}$ we get

$$(4.2)\quad \begin{array}{l} (i)\kappa(\eta_1 + \eta_2) + \eta_1\eta_2 - 9H_2 = \lambda(\eta_1 + \eta_2)u, \\ (ii)(\kappa(\eta_1 + \eta_2) + \eta_1\eta_2 - 9H_2)u = -\lambda(\eta_1 + \eta_2), \end{array}$$

which gives $\lambda(\eta_1 + \eta_2)(1 + u^2) = 0$, then $\lambda(\eta_1 + \eta_2) = 0$. Since by assumption $\lambda \neq 0$, we get $\eta_1 + \eta_2 = 0$. So, by (4.2)(i), we obtain $\kappa^2 + \lambda^2 = \frac{1}{3}\eta_1^2$. Since one of real principal curvatures η_1 and η_2 is assumed to be constant, we get that $9H_2 = -\eta_1^2 = -\eta_1^2$ is constant. Also, since $H_1 = \frac{1}{2}\kappa$ is assumed to be constant, we get that $H_3 = \frac{-1}{2}\kappa\eta_1^2$ and $H_4 = \frac{-1}{3}\eta_1^4$ are constant. These results are in contradiction with the assumption $e_1(H_2) \neq 0$. Hence, the first claim is proved.

Similarly, if $e_2(H_2) \neq 0$, dividing (4.1)(i) and (4.1)(ii) by $\epsilon_2 e_2(H_2)$ and taking $v := \frac{\epsilon_1 e_1(H_2)}{\epsilon_2 e_2(H_2)}$, we get $\lambda(\eta_1 + \eta_2)(1 + v^2) = 0$, which by a similar way gives the same results in contradiction with the assumption $e_2(H_2) \neq 0$. Hence, the second claim is satisfied.

Now, in order to prove the third claim, we assume that $e_3(H_2) \neq 0$. From equality (4.1)(iii) we have $\kappa^2 + \lambda^2 + 2\kappa\eta_2 = 9H_2$, and by a straightforward computation we get

$$-3\kappa^2 + 2(4H_1 - \eta_1)\kappa + 3\eta_1(4H_1 - \eta_1) = -\lambda^2 < 0,$$

then,

$$-2[2\kappa^2 + (\eta_1 - 4H_1)\kappa + 2\eta_1(\eta_1 - 3H_1)] = -(\lambda^2 + \kappa^2 + \eta_1^2) < 0.$$

Remember that the last inequality occurs if and only if we have $\delta < 0$ where

$$\delta = (\eta_1 - 4H_1)^2 - 16\eta_1(\eta_1 - 3H_1) = -15\eta_1^2 + 40\eta_1 H_1 + 16H_1^2.$$

The condition $\delta < 0$ is equivalent to a new inequality $\bar{\delta} < 0$ where

$$\bar{\delta} = (40H_1)^2 + (4 \times 15 \times 16)H_1^2 = 2560H_1^2,$$

which is impossible. So, 3rd claim is true.

To prove the 4th claim, we assume that $e_4(H_2) \neq 0$. From equality (4.1)(iv) we have $\kappa^2 + \lambda^2 + 2\kappa\eta_1 = 9H_2$, and by a straightforward computation we get

$$-11\kappa^2 + (24H_1 - 10\eta_1)\kappa + 12\eta_1 H_1 - 3\eta_1^2 = -\lambda^2 < 0,$$

then,

$$-2[6\kappa^2 + (5\eta_1 - 12H_1)\kappa + 2\eta_1(\eta_1 - 3H_1)] = -(\lambda^2 + \kappa^2 + \eta_1^2) < 0.$$

Remember that the last inequality occurs if and only if we have $\delta < 0$ where

$$\delta = (5\eta_1 - 12H_1)^2 - 48\eta_1(\eta_1 - 3H_1) = -23\eta_1^2 + 24\eta_1 H_1 + 144H_1^2.$$

The condition $\delta < 0$ is equivalent to a new inequality $\bar{\delta} < 0$ where

$$\bar{\delta} = (24H_1)^2 + (4 \times 23 \times 144)H_1^2 = 13824H_1^2,$$

which is impossible. So, 4th claim is affirmed. Therefore, we proved that H_2 is constant on M_1^4.

In the second stage, since the 2nd mean curvature of M_1^4 is constant, we have $L_1H_2 = 0$. Then, by (2.8)(i), we have $9H_1H_2^2 - 3H_2H_3 = 0$. If $H_2 \neq 0$, the last equality implies that $H_3 = 3H_1H_2$ is constant. Also, one can check that we have the identity

$$H_4 = (4H_3 - 6H_2\eta_1)\eta_1 + (4H_1 + \eta_1)\eta_1^3 - 2\eta_1^4,$$

which gives that H_4 is constant. Therefore, M_1^4 is isoparametric, which, by Corollary 2.9 in [12], its shape operator has not more than one non-zero real eigenvalue. Hence, we have $\eta_1\eta_2 = 0$ which gives $H_4 = (\kappa^2 + \lambda^2)\eta_1\eta_2 = 0$. Therefore, M_1^4 is 3-minimal. □

Theorem 4.2. Let $x : M_1^4 \to \mathbf{L^5}$ be an L_1-biharmonic Lorentzian hypersurface of type $\tilde{\mathcal{F}}_3$ with 3 distinct principal curvatures and constant ordinary mean curvature, then it is 1-minimal.

Proof. First, we show that H_2 is constant. It is enough to show that $\mathcal{U} = \{p \in M_1^4 : \nabla H_{k+1}^2(p) \neq 0\}$ is empty. Assuming $\mathcal{U}$ to be nonempty, we try to get a contradiction. M_1^4 is of type $\tilde{\mathcal{F}}_3$ which means that there is an orthonormal basis $\{e_1, \cdots, e_4\}$ such that S is of form $\tilde{\mathcal{F}}_3$. So, we have $Se_1 = (\kappa + \frac{1}{2})e_1 - \frac{1}{2}e_2$, $Se_2 = \frac{1}{2}e_1 + (\kappa - \frac{1}{2})e_2$, $Se_3 = \lambda_1 e_3$ and $Se_4 = \lambda_2 e_4$, and then, for $j = 1, 2, 3$ we have $P_je_1 = [\mu_{1,2;j} + (\kappa - \frac{1}{2})\mu_{1,2;j-1}]e_1 + \frac{1}{2}\mu_{1,2;j-1}e_2$, $P_2e_2 = -\frac{1}{2}\mu_{1,2;j-1}e_1 + [\mu_{1,2;j} + (\kappa - \frac{1}{2})\mu_{1,2;j-1}]e_2$, and $P_2e_3 = \mu_{3;j}e_3$ and $P_2e_4 = \mu_{4;j}e_4$.

Using the polar decomposition $\nabla H_2 = \sum_{i=1}^{4} \epsilon_i e_i(H_2)e_i$, from conditions (2.8)(ii), we get

$$\begin{aligned}
&(i)[\lambda_1\lambda_2 + (\kappa - \tfrac{1}{2})(\lambda_1 + \lambda_2) - 9H_2]\epsilon_1 e_1(H_2) = \tfrac{1}{2}(\lambda_1 + \lambda_2)\epsilon_2 e_2(H_2),\\
&(ii)[\lambda_1\lambda_2 + (\kappa + \tfrac{1}{2})(\lambda_1 + \lambda_2) - 9H_2]\epsilon_2 e_2(H_2) = -\tfrac{1}{2}(\lambda_1 + \lambda_2)\epsilon_1 e_1(H_2),\\
&(iii)(\kappa^2 + 2\kappa\lambda_2 - 9H_2)\epsilon_3 e_3(H_2) = 0,\\
&(iv)(\kappa^2 + 2\kappa\lambda_1 - 9H_2)\epsilon_3 e_4(H_2) = 0.
\end{aligned}$$

(4.3)

Now, we prove the following claim.

Claim: $e_i(H_2) = 0$ for $i = 1, 2, 3, 4$.
If $e_1(H_2) \neq 0$, then dividing equalities (4.3)(i) and (4.3)(ii) by $\epsilon_1 e_1(H_2)$ we get

$$(4.4) \qquad \begin{array}{l} (i) \lambda_1\lambda_2 + (\kappa - \frac{1}{2})(\lambda_1 + \lambda_2) - 9H_2 = \frac{1}{2}(\lambda_1 + \lambda_2)u, \\ (ii) [\lambda_1\lambda_2 + (\kappa + \frac{1}{2})(\lambda_1 + \lambda_2) - 9H_2]u = -\frac{1}{2}(\lambda_1 + \lambda_2), \end{array}$$

where $u := \frac{\epsilon_2 e_2(H_2)}{\epsilon_1 e_1(H_2)}$. By substituting (i) in (ii), we obtain $(\lambda_1+\lambda_2)(1+u)^2 = 0$, then $\lambda_1 + \lambda_2 = 0$ or $u = -1$.
If $\lambda_1 + \lambda_2 = 0$, then, from (4.4)(i) we obtain $9H_2 = -\lambda_1^2$, which gives $3\kappa^2 = -\lambda_1^2$. Since H_1 is assumed to be constant on M, then $\kappa = 2H_1$ is constant on M. Hence, λ_1 and λ_2 are also constant on M. Therefore, M_1^4 is an isoparametric Lorentzian hypersurface of real principal curvatures in E_1^5, which by Corollary 2.7 in [12], cannot has more than one nonzero principal curvature contradicting with the assumptions. So, $\lambda_1 + \lambda_2 \neq 0$ and then $u = -1$.

From $u = -1$, we get $\lambda_1\lambda_2 + \kappa(\lambda_1 + \lambda_2) = 9H_2$, then

$$3\kappa^2 + 4\kappa(\lambda_1 + \lambda_2) + \lambda_1\lambda_2 = 0.$$

Since $4H_1 = 2\kappa + \lambda_1 + \lambda_2$ is assumed to be constant on M, by substituting which in the last equality, we get $\lambda^2 - H_1\lambda - 3H_1^2 = 0$, which means λ, κ and the kth mean curvatures (for $k = 2, 3, 4$) are constant on M. So, we got a contradiction and therefore, the first part of the claim is proved.

By a similar manner, each of assumptions $e_i(H_2) \neq 0$ for $i = 2, 3, 4$, gives the equality $\lambda^2 + 2\kappa\lambda = 9H_2$, which implies the contradiction that H_2 is constant on M. So, the claim is affirmed.

In second stage we prove that $H_2 = 0$. Since H_1 and H_2 are constant, from (2.8)(i) we obtain that H_3 is constant. Therefore, M_1^4 is isoparametric. On the other hand, by Corollary 2.7 in [12], an isoparametric Lorentzian hypersurface of Case II in the E_1^5 has at most one nonzero principal curvature, so we get $\lambda = 0$ (for example). Then $H_1 = \frac{1}{2}\kappa$, $H_2 = \frac{1}{6}\kappa^2$ and $H_3 = 0$, hence, by (2.8)(i), we get $\kappa = 0$. Therefore $H_2 = 0$. □

Theorem 4.3. Let $x : M_1^4 \to \mathbf{L^5}$ be a L_1-biharmonic orientable Lorentzian hypersurface of type $\tilde{\mathcal{F}}_3$ with one constant principal curvature. Then its second mean curvature is constant. Additionally, if its ordinary mean curvature is constant, then it is 1-minimal.

Proof. First we prove that H_2 is constant. In fact, we show that $\mathcal{U} = \{p \in M_1^4 : \nabla H_2^2(p) \neq 0\}$ has no member. Assuming $\mathcal{U}$ to be nonempty we try to get a contradiction. Since M_1^4 is of type $\tilde{\mathcal{F}}_3$, there exists an orthonormal tangent frame $\{e_1, \cdots, e_4\}$ on M_1^4 such that the shape operator is of form $\tilde{\mathcal{F}}_3$ and we have $Se_1 = (\kappa + \frac{1}{2})e_1 - \frac{1}{2}e_2$, $Se_2 = \frac{1}{2}e_1 + (\kappa - \frac{1}{2})e_2$, $Se_3 = \lambda_1 e_3$ and $Se_4 = \lambda_2 e_4$, and then, we have $P_2 e_1 = [\lambda_1\lambda_2 + (\kappa - \frac{1}{2})(\lambda_1 + \lambda_2)]e_1 + \frac{1}{2}(\lambda_1 + \lambda_2)e_2$, $P_2 e_2 = -\frac{1}{2}(\lambda_1 + \lambda_2)e_1 + [\lambda_1\lambda_2 + (\kappa + \frac{1}{2})(\lambda_1 + \lambda_2)]e_2$, and $P_2 e_3 = (\kappa^2 + 2\kappa\lambda_2)e_3$ and $P_2 e_4 = (\kappa^2 + 2\kappa\lambda_1)e_4$.

Using the polar decomposition $\nabla H_2 = \sum_{i=1}^{4} \epsilon_i e_i(H_2)e_i$, from condition (2.8)(ii) we get

$$(4.5)\quad \begin{aligned} &(i)[\lambda_1\lambda_2 + (\kappa - \tfrac{1}{2})(\lambda_1 + \lambda_2) - 9H_2]\epsilon_1 e_1(H_2) = \tfrac{1}{2}(\lambda_1 + \lambda_2)\epsilon_2 e_2(H_2)\\ &(ii)[\lambda_1\lambda_2 + (\kappa + \tfrac{1}{2})(\lambda_1 + \lambda_2) - 9H_2]\epsilon_2 e_2(H_2) = \tfrac{1}{2}(\lambda_1 + \lambda_2)\epsilon_1 e_1(H_2),\\ &(iii)(\kappa^2 + 2\kappa\lambda_2 - 9H_2)\epsilon_3 e_3(H_2) = 0,\\ &(iv)(\kappa^2 + 2\kappa\lambda_1 - 9H_2)\epsilon_3 e_4(H_2) = 0. \end{aligned}$$

Now, we prove some simple claims.

Claim: $e_1(H_2) = e_2(H_2) = e_3(H_2) = e_4(H_2) = 0$.

If $e_1(H_2) \neq 0$, then by dividing equalities (4.5)(i, ii) by $\epsilon_1 e_1(H_2)$ we get

$$(4.6)\quad \begin{aligned} &(i)\lambda_1\lambda_2 + (\kappa - \tfrac{1}{2})(\lambda_1 + \lambda_2) - 9H_2 = \tfrac{1}{2}(\lambda_1 + \lambda_2)u,\\ &(ii)[\lambda_1\lambda_2 + (\kappa + \tfrac{1}{2})(\lambda_1 + \lambda_2) - 9H_2]u = -\tfrac{1}{2}(\lambda_1 + \lambda_2), \end{aligned}$$

where $u := \frac{\epsilon_2 e_2(H_2)}{\epsilon_1 e_1(H_2)}$. By substituting (i) in (ii), we obtain $\frac{1}{2}(\lambda_1 + \lambda_2)(1 + u)^2 = 0$, Then $\lambda_1 + \lambda_2 = 0$ or $u = -1$. If $\lambda_1 + \lambda_2 = 0$, then, by assumption we get that $\kappa = 2H_1$ is constant, and also, from (4.4(i)) we obtain $H_2 = \frac{-1}{9}\lambda_1^2$ which gives $\frac{1}{6}(\kappa^2 - \lambda_1^2) = \frac{-1}{9}\lambda_1^2$ and then $\lambda_1^2 = 3\kappa^2$. Hence, we get $H_2 = \frac{-1}{3}\kappa^2$, which means H_2 is constant.

Also, by assumption $\lambda_1 + \lambda_2 \neq 0$ we get $u = -1$, which, using (4.6)(i) and $4H_1 = 2\kappa + \lambda_1 + \lambda_2$, gives $5\kappa^2 - 16\kappa H_1 - \lambda_1(4H_1 - 2\kappa - \lambda_1) = 0$. Without loss of generality, we assume that λ_1 is constant on M. So, from the last equation we get that κ, λ_2 and H_2 are constant on $\mathcal{U}$, which is a contradiction. Therefore, the first claim is proved. The second claim (i.e. $e_2(H_2) = 0$) can be proven by a similar manner.

Now, if $e_3(H_2) \neq 0$, then using (4.5)(iii) and $4H_1 = 2\kappa + \lambda_1 + \lambda_2$ and by assuming λ_1 to be constant on M, we get

$$\kappa^2 - (\frac{16}{3}H_1 - \frac{2}{3}\lambda_1)\kappa - 4\lambda_1 H_1 + \lambda_1^2 = 0,$$

which gives that κ, λ_2 and H_2 are constant on $\mathcal{U}$, which is a contradiction. Therefore, the third claim is proved.

The forth claim (i.e. $e_4(H_2) = 0$) can be proven by a manner exactly similar to third one. Therefore, we proved that H_2 is constant.

Now, we show that $H_2 = 0$. Since H_1 and H_2 are constant, from (2.8)(i) we obtain that H_3 is constant. Therefore, M_1^4 is isoparametric. Since, by Corollary 2.7 in [12], an isoparametric Lorentzian hypersurface of real principal curvatures in $\mathbf{L^5}$ has at most one nonzero principal curvature, we get $H_2 = 0$. $\square$

Theorem 4.4. Let $x : M_1^4 \to \mathbf{L^5}$ be a L_1-biharmonic orientable Lorentzian hypersurface of type $\tilde{\mathcal{F}}_4$. Then, its second mean curvature is constant. In addition, if its ordinary mean curvature is constant, then it is 1-minimal.

Proof. First we prove that H_2 is constant. In fact, we show that $\mathcal{U} = \{p \in M_1^4 : \nabla H_2^2(p) \neq 0\}$ has no member. Assuming $\mathcal{U}$ to be nonempty we try to get a contradiction. Since M_1^4 is of type $\tilde{\mathcal{F}}_4$, there exists an orthonormal tangent frame $\{e_1, \cdots, e_4\}$ on M_1^4 such that the shape operator is of form $\tilde{\mathcal{F}}_4$ and we have $Se_1 = \kappa e_1 - \frac{\sqrt{2}}{2} e_3$, $Se_2 = \kappa e_2 - \frac{\sqrt{2}}{2} e_3$, $Se_3 = \frac{\sqrt{2}}{2} e_1 - \frac{\sqrt{2}}{2} e_2 + \kappa e_3$ and $Se_4 = \lambda e_4$ and then, we have $P_2 e_1 = (\kappa^2 + 2\kappa\lambda - \frac{1}{2}) e_1 + \frac{1}{2} e_2 + \frac{\sqrt{2}}{2}(\kappa + \lambda) e_3$, $P_2 e_2 = \frac{-1}{2} e_1 + (\kappa^2 + 2\kappa\lambda + \frac{1}{2}) e_2 + \frac{\sqrt{2}}{2}(\kappa + \lambda) e_3$, $P_2 e_3 = \frac{-\sqrt{2}}{2}(\kappa + \lambda) e_1 + \frac{\sqrt{2}}{2}(\kappa + \lambda) e_2 + (\kappa^2 + 2\kappa\lambda) e_3$ and $P_2 e_4 = 3\kappa^2 e_4$.

Using the polar decomposition $\nabla H_2 = \sum_{i=1}^{4} \epsilon_i e_i(H_2) e_i$, from condition (2.8)(ii) we get

$$
\begin{aligned}
&(i)(\kappa^2 + 2\kappa\lambda - \tfrac{1}{2} - 9H_2)\epsilon_1 e_1(H_2) - \tfrac{1}{2}\epsilon_2 e_2(H_2) - \tfrac{\sqrt{2}}{2}(\kappa + \lambda)\epsilon_3 e_3(H_2) = 0\\
&(ii)\tfrac{1}{2}\epsilon_1 e_1(H_2) + (\kappa^2 + 2\kappa\lambda + \tfrac{1}{2} - 9H_2)\epsilon_2 e_2(H_2) + \tfrac{\sqrt{2}}{2}(\kappa + \lambda)\epsilon_3 e_3(H_2) = 0\\
&(iii)\tfrac{\sqrt{2}}{2}(\kappa + \lambda)(\epsilon_1 e_1(H_2) + \epsilon_2 e_2(H_2)) + (\kappa^2 + 2\kappa\lambda - 9H_2)\epsilon_3 e_3(H_2) = 0,\\
&(iv)(3\kappa^2 - 9H_2)\epsilon_4 e_4(H_2) = 0.
\end{aligned} \tag{4.7}
$$

Now, we prove some simple claims.

Claim: $e_1(H_2) = e_2(H_2) = e_3(H_2) = e_4(H_2) = 0$.
If $e_1(H_2) \neq 0$, then by dividing both sides of equalities (4.7)(i, ii, iii) by $\epsilon_1 e_1(H_2)$, and using the identity $2H_2 = \kappa^2 + \kappa\lambda$ in Case $\tilde{\mathcal{F}}_4$, putting $u_1 := \frac{\epsilon_2 e_2(H_2)}{\epsilon_1 e_1(H_2)}$ and $u_2 := \frac{\epsilon_3 e_3(H_2)}{\epsilon_1 e_1(H_2)}$, we get

$$
\begin{aligned}
&(i) - \tfrac{1}{2} - \tfrac{7}{2}\kappa^2 - \tfrac{5}{2}\kappa\lambda - \tfrac{1}{2}u_1 - \tfrac{\sqrt{2}}{2}(\kappa + \lambda)u_2 = 0\\
&(ii)\tfrac{1}{2} + (\tfrac{1}{2} - \tfrac{7}{2}\kappa^2 - \tfrac{5}{2}\kappa\lambda)u_1 + \tfrac{\sqrt{2}}{2}(\kappa + \lambda)u_2 = 0\\
&(iii)\tfrac{-\sqrt{2}}{2}(\kappa + \lambda)(1 + u_1) - (\tfrac{7}{2}\kappa^2 + \tfrac{5}{2}\kappa\lambda)u_2 = 0,
\end{aligned} \tag{4.8}
$$

which, by comparing (i) and (ii), gives $\frac{-1}{2}\kappa(7\kappa+5\lambda)(1+u_1)=0$. If $\kappa=0$, then $H_2=0$. Assuming $\kappa\neq 0$, we get $u_1=-1$ or $\lambda=-\frac{7}{5}\kappa$. If $u_1\neq -1$ then $\lambda=-\frac{7}{5}\kappa$, then by (4.8)$(iii)$ we obtain $u_1=-1$, which is a contradiction. Hence we have $u_1=-1$, which by (4.8)(i,iii) implies $u_2=0$.

Now we discuss on two cases $\lambda=-\frac{7}{5}\kappa$ or $\lambda\neq-\frac{7}{5}\kappa$. If $\lambda=-\frac{7}{5}\kappa$, then, $\kappa=\frac{5}{2}H_1$, $H_2=\frac{-1}{5}\kappa^2$, $H_3=\frac{-4}{5}\kappa^3$ and $H_4=\frac{-7}{5}\kappa^4$ are all constants on $\mathcal{U}$. Also, the case $\lambda\neq-\frac{7}{5}\kappa$ is in contradiction with (4.8)(ii).

Hence, the first claim $e_1(H_2)\equiv 0$ is affirmed. Similarly, the second claim (i.e. $e_2(H_2)=0$) can be proved.

Now, applying the results $e_1(H_2)=e_2(H_2)=0$, from (4.8)(ii) and (4.8)(iii) we get $e_3(H_2)=0$.

The final claim (i.e. $e_4(H_2)=0$), can be proved using (4.8)(iv), in a straightforward manner.

In second stage, we prove that $H_2=0$. By (2.8)(i), we have $L_1H_2=9H_1H_2^2-3H_2H_3=0$. If $H_2=0$, it remains nothing to prove. By assumption $H_2\neq 0$, we get $3H_1H_2=H_3$, which gives $\kappa(\kappa^2-3H_1\kappa+3H_1^2)=0$, where $\kappa^2-3H_1\kappa+3H_1^2>0$, Hence, $\kappa=0$. Therefore, $H_2=H_3=H_4=0$. $\square$

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