Proyecciones Journal of Mathematics
Vol. 42, N^o 6, pp. 1489-1498, December 2023.
Universidad Católica del Norte
Antofagasta - Chile

10.22199/issn.0717-6279-5440

Characterization and commuting probability of n-centralizer finite rings

Jutirekha Dutta
Tezpur University, India
Dhiren Kumar Basnet
Tezpur University, India
and
Rajat Kanti Nath
Tezpur University, India
Received : April 2022. Accepted : August 2023

Abstract

Let R be a finite ring. The commuting probability of R is the probability that any two randomly chosen elements of R commute. A ring R is called an n-centralizer ring if it has n distinct centralizers. In this paper, we characterize some n-centralizer finite rings and compute their commuting probabilities.

Keywords: *Finite ring, commuting probability, n-centralizer rings.*

Subjclass [2010]: *16U70, 16U80.*

1. Introduction

The commuting probability of a finite ring R is the probability that a randomly chosen pair of elements of R commute. We write $\Pr(R)$ to denote this probability.

It is not difficult to see that

$$\Pr(R) = \frac{|Z(R)|}{|R|} + \frac{1}{|R^2|} \sum_{r \in R \setminus Z(R)} |C_R(r)|, \tag{1.1}$$

where $C_R(r)$ and $Z(R)$ are known as centralizer of $r \in R$ and center of R given by $C_R(r) = \{s \in R : rs = sr\}$ and $Z(R) = \underset{r \in R}{\cap} C_R(r)$.

In 1976, MacHale [29] initiated the study of $\Pr(R)$. However, it gets popularity in the recent years only unlike the commuting probability of a finite group. The study of commuting probability of a finite group was originated from the works of Erdös and Turán [19] published in 1968. Many mathematicians have considered the notion of commuting probability of a finite group in their works (see [11] and the references therein). Fundamental contributions in this area, due to Erdös, Erfanian, Guralnick, Gustafson, Hegarty, Joseph, Lescot, MacHale, Rezaei, Rusin, Russo, Turán and many other authors, can be found in [6, 10, 19, 21, 22, 23, 24, 25, 26, 27, 28, 30, 31, 32] etc. However, only few papers are available on $\Pr(R)$ (including its generalizations) in the literature (see [8, 9, 12, 15, 16, 17, 18, 29] for example).

Central problems in this area are to find all the rational numbers $r \in \mathbf{Q} \cap (0, 1]$ such that $\Pr(R) = r$ for some finite ring R and characterize finite rings such that $\Pr(R) = r$. Recently, Buckley et al. [9] have computed $\Pr(R)$ for several families of finite rings and characterized all finite rings having $\Pr(R) \geq \frac{11}{32}$.

A ring R is called n-centralizer if $|\mathrm{Cent}(R)| = n$, where $\mathrm{Cent}(R) = \{C_R(x) : x \in R\}$. The class of n-centralizer finite rings was introduced and studied by Dutta et al. [13, 14] following the works of Belcastro and Sherman [7], where n-centralizer finite groups were studied. Fundamental results on n-centralizer finite groups can be found in [1, 2, 3, 4, 5, 7]. Existence of n-centralizer finite groups for all positive integers $n \geq 4$ was proved by Ashrafi in [2], answering an open problem of Belcastro and Sherman [7]. In [13, 14], n-centralizer finite rings are characterized for certain positive integers. However, the existence of n-centralizer finite rings for all positive integers $n \geq 4$ is still not clear. In this paper, we characterize n-centralizer finite rings and compute their commuting probabilities for $n \leq 7$.

Throughout this paper R denotes a finite ring. For any subring S of R, we write R/S or $\frac{R}{S}$ to denote the additive quotient group $(S,+)$ in $(R,+)$. The isomorphisms considered in this paper are the additive group isomorphisms. We shall also use the fact that for any non-commutative ring R, the additive group $\frac{R}{Z(R)}$ is not a cyclic group (see [29, Lemma 1]).

2. Rings with known central factor

In this section, we deduce some properties of a finite non-commutative ring R having some known central factor. These results will also serve as prerequisites for the results obtained in the next section. We begin with the following result which is obtained by MacHale [29, Theorem 1 and 2] in the year 1974.

Theorem 2.1. *Let R be a finite ring. Then $\frac{R}{Z(R)} \cong \mathbf{Z}_2 \times \mathbf{Z}_2$ if and only if $\Pr(R) = \frac{5}{8}$. Further, if p is the smallest prime divisor of $|R|$ then $\frac{R}{Z(R)} \cong \mathbf{Z}_p \times \mathbf{Z}_p$ if and only if $\Pr(R) = \frac{p^2+p-1}{p^3}$.*

It is worth mentioning that original group theoretic result analogous to Theorem 2.1 is known since long time in the works of some authors already mentioned in the Introduction. If p is any prime divisor of $|R|$, not necessarily the smallest one, then also we have the following result [13, Theorem 2.5].

Theorem 2.2. *Let R be a finite ring and $\frac{R}{Z(R)} \cong \mathbf{Z}_p \times \mathbf{Z}_p$, where p is a prime. Then R is a $(p+2)$-centralizer ring and $\Pr(R) = \frac{p^2+p-1}{p^3}$.*

In [13], we have proved the following two results regarding 4-centralizer and 5-centralizer finite rings.

Theorem 2.3. *Let R be a finite ring. Then*

(a) $\frac{R}{Z(R)} \cong \mathbf{Z}_2 \times \mathbf{Z}_2$ *if and only if R is 4-centralizer* (see [13, Theorem 3.1]).

(b) $\frac{R}{Z(R)} \cong \mathbf{Z}_3 \times \mathbf{Z}_3$ *if and only if R is 5-centralizer* (see [13, Theorem 4.1]).

The original group theoretic result analogous to Theorem 2.3 was proved by Belcastro and Sherman (see [7, Theorems 2 and 4]). The following lemmas are useful in proving the subsequent results.

Lemma 2.4. *Let R be a finite ring. If $\frac{R}{Z(R)} \cong \mathbf{Z}_m \times \mathbf{Z}_n$ then $m = n$.*

Proof. Let $Z := Z(R)$. Suppose $R/Z \cong \mathbf{Z}_m \times \mathbf{Z}_n$ and $m \neq n$. Without loss of generality, we can assume that $m < n$. Then there exist two elements a, b of R such that $ab \neq ba$ and

$$\frac{R}{Z} = \langle Z + a, Z + b \ : \ m(Z + a) = n(Z + b) = Z \rangle.$$

We have $a(mb) = (ma)b = b(ma) = (mb)a$. Therefore, $C_R(mb) = R$ and so $mb \in Z$; which is a contradiction. Hence the lemma follows. □

Lemma 2.5. *Let R be a finite ring. If $\frac{R}{Z(R)} \cong \mathbf{Z}_m \times \mathbf{Z}_m \times \mathbf{Z}_n$ then $m = n$.*

Proof. Let $Z := Z(R)$. Suppose $R/Z \cong \mathbf{Z}_m \times \mathbf{Z}_m \times \mathbf{Z}_n$ and $m \neq n$. Without loss of generality, we can assume that $m < n$. Then there exist three elements a, b, c of R not commuting with each other simultaneously and

$$\frac{R}{Z} = \langle Z + a, Z + b, Z + c \ : \ m(Z + a) = m(Z + b) = m(Z + c) = Z \rangle.$$

We have $a(mc) = (ma)c = c(ma) = (mc)a$ and $b(mc) = (mb)c = c(mb) = (mc)b$. Therefore, $C_R(mc) = R$ and so $mc \in Z$; which is a contradiction. Hence the lemma follows. □

We conclude this section with the following theorem which gives some information regarding 8-centralizer finite rings.

Theorem 2.6. *Let R be a finite ring. If $\frac{R}{Z(R)} \cong \mathbf{Z}_4 \times \mathbf{Z}_4$ then R is 8-centralizer.*

Proof. Firstly suppose that $\frac{R}{Z(R)} \cong \mathbf{Z}_4 \times \mathbf{Z}_4$. Then there exist two elements $a, b \in R$ such that $ab \neq ba$ and

$$\frac{R}{Z(R)} = \langle Z + a, Z + b : 4(Z + a) = 4(Z + b) = Z \rangle,$$

where $Z := Z(R)$. If S/Z is an additive non-trivial subgroup of R/Z then $|S/Z| = 2, 4$ or 8. Therefore, some of the proper additive subgroups of R properly containing Z are

$$
\begin{aligned}
S_m &:= C_R(a+mb) \\
&= Z \cup (Z+(a+mb)) \cup (Z+2(a+mb)) \cup (Z+3(a+mb)), \\
&\qquad \text{where } 1 \le m \le 3, \\
S_4 &:= C_R(2a+b) \\
&= Z \cup (Z+(2a+b)) \cup (Z+2(2a+b))) \cup (Z+3(2a+b)), \\
S_5 &:= C_R(2a+2b) = Z \cup (Z+(2a+2b)), \\
S_6 &:= C_R(a) = Z \cup (Z+a) \cup (Z+2a) \cup (Z+3a) \text{ and} \\
S_7 &:= C_R(b) = Z \cup (Z+b) \cup (Z+2b) \cup (Z+3b).
\end{aligned}
$$

Now for any $x \in R \setminus Z$, we have $Z+x$ is equal to $Z+k$ for some $k \in \{ma, mb, ma+mb : 1 \le m \le 3\}$. Therefore $C_R(x) = C_R(k)$. Again, let $y \in S_j - Z$ for some $j \in \{1, 2, \ldots, 7\}$, then $C_R(y) \neq S_q$, where $1 \le q\,(\neq j) \le 7$. Thus $C_R(y) = S_j$. This shows that R is 8-centralizer. □

3. Characterization and Commuting probability

In this section, we characterize some n-centralizer finite rings and compute their commuting probabilities for $n \le 7$. It is clear that R is 1-centralizer if and only if it is commutative. Therefore, R is 1-centralizer if and only if $\Pr(R) = 1$.

In [13, Theorem 2.1], we have proved that there is no n-centralizer ring for $n = 2, 3$. The following two results give commuting probabilities of 4-centralizer and 5-centralizer finite rings.

Theorem 3.1. *Let R be a finite ring. Then R is 4-centralizer if and only if* $\Pr(R) = \frac{5}{8}$.

Proof. The proof follows from Theorem 2.3(a) and Theorem 2.1. □

Theorem 3.2. *If R is a finite 5-centralizer ring then* $\Pr(R) = \frac{11}{27}$.

Proof. The proof follows from Theorem 2.3(b) and Theorem 2.2. □

The following characterization of finite 6-centralizer rings is useful in computing their commuting probabilities.

Theorem 3.3. *If R is a 6-centralizer finite ring then $\frac{R}{Z(R)}$ is isomorphic to $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$, or $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$.*

Proof. If R is a 6-centralizer finite ring then by Theorem 3.1 in [14], we have $|R : Z(R)| = 8, 12$ or 16.

By fundamental theorem of finite abelian groups, $\frac{R}{Z(R)}$ is isomorphic to $\mathbf{Z}_8, \mathbf{Z}_2 \times \mathbf{Z}_4$, $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2, \mathbf{Z}_{12}$, $\mathbf{Z}_2 \times \mathbf{Z}_6$, $\mathbf{Z}_{16}, \mathbf{Z}_2 \times \mathbf{Z}_8, \mathbf{Z}_4 \times \mathbf{Z}_4, \mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_4$ or $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$.

Since R is non-commutative, $\frac{R}{Z(R)}$ is not isomorphic to $\mathbf{Z}_8, \mathbf{Z}_{12}$ and $\mathbf{Z}_{16}$. Using Lemma 2.4 and Lemma 2.5, we have $\frac{R}{Z(R)}$ is not isomorphic to $\mathbf{Z}_2 \times \mathbf{Z}_4, \mathbf{Z}_2 \times \mathbf{Z}_6$, $\mathbf{Z}_2 \times \mathbf{Z}_8$ and $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_4$. Also by Theorem 2.6, we have $\frac{R}{Z(R)}$ is not isomorphic to $\mathbf{Z}_4 \times \mathbf{Z}_4$. Hence the result follows.

□

Theorem 3.4. *If R is a 6-centralizer finite ring such that $\frac{R}{Z(R)}$ is not isomorphic to $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$ then* $\Pr(R) = \frac{7}{16}$.

Proof. Since R is 6-centralizer finite ring, by Theorem 3.3, we have $\frac{R}{Z} \cong \mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$ where $Z := Z(R)$.

Therefore, there exist three non-central elements a, b, c of R not commuting with each other simultaneously and

$$\frac{R}{Z} = \langle Z + a, Z + b, Z + c \ : \ 2(Z + a) = 2(Z + b) = 2(Z + c) = Z \rangle.$$

If S/Z is an additive non-trivial subgroup of R/Z then $|S/Z| = 2$ or 4. Since $|\mathrm{Cent}(R)| = 6$, we have

$$C_R(k) = C_R(l) = C_R(k + l) = Z \cup (Z + k) \cup (Z + l) \cup (Z + (k + l))$$

for some $k, l \in \{a, b, c, a+b, a+c, b+c, a+b+c\}$ and $k \neq l$. Without loss of generality, we can assume that $C_R(b) = C_R(c)$. Hence, some of the proper additive subgroups of R properly containing Z are

$$\begin{aligned}
S_1 &:= C_R(a) = Z \cup (Z + a),\\
S_2 &:= C_R(b) = C_R(c) = C_R(b + c)\\
&= Z \cup (Z + b) \cup (Z + c) \cup (Z + (b + c)),\\
S_3 &:= C_R(a + b) = Z \cup (Z + (a + b)),\\
S_4 &:= C_R(a + c) = Z \cup (Z + (a + c)) \text{ and}\\
S_5 &:= C_R(a + b + c) = Z \cup (Z + (a + b + c)).
\end{aligned}$$

Now for any $x \in R \setminus Z$, we have $Z + x$ is equal to $Z + k$ for some $k \in \{a, b, c, a + b, a + c, b + c, a + b + c\}$. Therefore $C_R(x) = C_R(k)$. Again, let $y \in S_j - Z$ for some $j \in \{1, 2, \ldots, 5\}$, then $C_R(y) \neq S_q$, where $1 \leq q\,(\neq j) \leq 5$. Thus $C_R(y) = S_j$. Hence, by (1.1) we have

$$\Pr(R) = \frac{|Z|}{8|Z|} + \frac{(4 \times 2)|Z|^2 + (3 \times 4)|Z|^2}{8^2|Z|^2} = \frac{7}{16}.$$

This completes the proof. □

Now we compute commuting probability of some finite 7-centralizer rings. The following result is useful in this regard.

Theorem 3.5. *Let R be a finite ring. Then R is 7-centralizer if and only if*

$$\frac{R}{Z(R)} \cong \mathbf{Z}_5 \times \mathbf{Z}_5.$$

Proof. If $\frac{R}{Z(R)} \cong \mathbf{Z}_5 \times \mathbf{Z}_5$ then, by Theorem 2.2, it follows that R is 7-centralizer.

If R is a 7-centralizer finite ring then by Theorem 4.3 in [14], we have $|R : Z(R)| = 12, 18, 20,$ 24 or 25.

Therefore by fundamental theorem of finite abelian groups, $\frac{R}{Z(R)}$ is isomorphic to $\mathbf{Z}_{12}, \mathbf{Z}_2 \times \mathbf{Z}_6, \mathbf{Z}_{18}$, $\mathbf{Z}_3 \times \mathbf{Z}_6$, $\mathbf{Z}_{20}, \mathbf{Z}_2 \times \mathbf{Z}_{10}$, $\mathbf{Z}_{24}, \mathbf{Z}_2 \times \mathbf{Z}_{12}, \mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_6$, $\mathbf{Z}_{25}$ or $\mathbf{Z}_5 \times \mathbf{Z}_5$.

Since R is non-commutative, $\frac{R}{Z(R)}$ is not isomorphic to $\mathbf{Z}_{12}, \mathbf{Z}_{18}, \mathbf{Z}_{20}, \mathbf{Z}_{24}$ and $\mathbf{Z}_{25}$.

Also by Lemma 2.4 and Lemma 2.5, we have $\frac{R}{Z(R)}$ is not isomorphic to $\mathbf{Z}_2 \times \mathbf{Z}_6, \mathbf{Z}_2 \times \mathbf{Z}_{10}, \mathbf{Z}_3 \times \mathbf{Z}_6$, $\mathbf{Z}_2 \times \mathbf{Z}_{12}$, $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_6$.

Hence, the theorem follows. □

We conclude this paper by the following result.

Theorem 3.6. *If R is a finite 7-centralizer ring then* $\Pr(R) = \frac{29}{125}$.

Proof. The result follows from Theorem 3.5 and Theorem 2.2. □

References

[1] A. Abdollahi, S. M. J. Amiri and A. M. Hassanabadi, “Groups with specific number of centralizers”, *Houston J. Math.*, vol. 33, no. 1, pp. 43-57, 2007.

[2] A. R. Ashrafi, “On finite groups with a given number of centralizers”, *Algebra Colloq.,* vol. 7, no. 2, pp. 139-146, 2000.

[3] A. R. Ashrafi, “Counting the centralizers of some finite groups”, *Korean J. Comput. Appl. Math.,* vol. 7, no. 1, pp. 115-124, 2000.

[4] A. R. Ashrafi and B. Taeri, “On finite groups with exactly seven element centralizers”, *J. Appl. Math. Comput.,* vol. 22, no. 1-2, pp. 403-410, 2006.

[5] S. J. Baishya, “On finite groups with specific number of centralizers”, *Int. Electron. J. Algebra,* vol. 13, pp. 53-62, 2013.

[6] F. Barry, D. MacHale and A. N. Shé, “Some supersolvability conditions for finite groups”, *Math. Proc. R. Ir. Acad.,* vol. 106, no. A2, pp. 163-177, 2006.

[7] S. M. Belcastro and G. J. Sherman, “Counting centralizers in finite groups”, *Math. Magazine,* vol. 67, no. 5, pp. 366-374, 1994.

[8] S. M. Buckley and D. Machale, “Contrasting the commuting probabilities of groups and rings”. [On line]. Available: https://bit.ly/3sxa8b1

[9] S. M. Buckley, D. Machale and A. N. Shé, “Finite rings with many commuting pairs of elements”. [On line]. Available: https://bit.ly/3QMAWfe

[10] A. K. Das and R. K. Nath, “A characterization of certain finite groups of odd order”, *Math. Proc. Royal Irish Acad.,* vol. 111, no. A2, pp. 69-78, 2011.

[11] A. K. Das, R. K. Nath and M. R. Pournaki, “A survey on the estimation of commutativity in finite groups”, *Southeast Asian Bull. Math.,* vol. 37, pp. 161-180, 2013.

[12] J. Dutta and D. K. Basnet, “Some bounds for commuting probability of finite rings”, *Proc. Math. Sci.,* vol. 129, no. 1, pp. 1-6, 2019.

[13] J. Dutta, D. K. Basnet and R. K. Nath, “Characterizing some rings of finite order”, *Tamkang J. Math.,* vol. 53, no. 2, pp. 97-108, 2022.

[14] J. Dutta, D. K. Basnet and R. K. Nath, “A note on n-centralizer finite rings”, *An. Stiint. Univ. Al. I. Cuza Iasi Mat.,* vol. 64, no. 1, pp. 161-171, 2018.

[15] J. Dutta, D. K. Basnet and R. K. Nath, “On commuting probability of finite rings,” *Indag. Math.*, vol. 28, no. 2, pp. 372-382, 2017.

[16] P. Dutta and R. K. Nath, “A generalization of commuting probability of finite rings”, *Asian-European J. Math.*, vol. 11, Art. ID. 1850023, 2018.

[17] P. Dutta and R. K. Nath, “On relative commuting probability of finite rings”, *Miskolc Math. Notes*, vol. 20, no. 1, pp. 225-232, 2019.

[18] P. Dutta and R. K. Nath, “On r-commuting probability of finite rings”, *Indian Journal of Mathematics*, vol. 62, no. 3, pp. 287-297, 2020.

[19] P. Erdös and P. Turán, “On some problems of a statistical group-theory IV”, *Acta. Math. Acad. Sci. Hungar.*, vol. 19, pp. 413-435, 1968.

[20] A. Erfanian, R. Rezaei and P. Lescot, “On the relative commutativity of a subgroup of a finite group”, *Comm. Algebra*, vol. 35, pp. 4183-4197, 2007.

[21] I. V. Erovenko and B. Sury, “Commutativity degrees of wreath products of finite abelian groups”, *Bull. Aust. Math. Soc.*, vol. 77, no. 1, pp. 31-36, 2008.

[22] R. M. Guralnick and G. R. Robinson, “On the commuting probability in finite groups”, *J. Algebra*, vol. 300, no. 2, pp. 509-528, 2006.

[23] W. H. Gustafson, “What is the probability that two group elements commute?”, *Amer. Math. Monthly*, vol. 80, pp. 1031-1034, 1973.

[24] P. Hegarty, “Limit points in the range of the commuting probability function on finite groups”, *J. Group Theory*, vol. 16, no. 2, pp. 235-247, 2013.

[25] K. H. Hofmann and F. G. Russo, “The probability that x and y commute in a compact group”, *Math. Proc. Cambridge Phil. Soc.*, vol. 153, pp. 557-571, 2012.

[26] K. S. Joseph, *Commutativity in Non-Abelian Groups*, Ph.D. thesis, University of California, Los Angeles, 1969.

[27] P. Lescot, “Isoclinism classes and commutativity degrees of finite groups”, *J. Algebra*, vol. 177, pp. 847-869, 1995.

[28] P. Lescot, "Central extensions and commutativity degree", *Comm. Algebra,* vol. 29, no. 10, pp. 4451-4460, 2001.

[29] D. MacHale, "Commutativity in finite rings", *Amer. Math. Monthly*, vol. 83, pp. 30-32, 1976.

[30] R. K. Nath, "Commutativity degree of a class of finite groups and consequences", *Bull. Aus. Math. Soc.*, vol. 88, no. 3, pp. 448-452, 2013.

[31] R. K. Nath and A. K. Das, "On a lower bound of commutativity degree", *Rend. Circ. Mat. Palermo*, vol. 59, pp. 137-142, 2010.

[32] D. J. Rusin, "What is the probability that two elements of a finite group commute?", *Pacific J. Math.*, vol. 82, pp. 237-247, 1979.

Jutirekha Dutta
Department of Applied Sciences,
Tezpur University,
Napaam-784028,
Sonitpur, Assam,
India
e-mail: jutirekhadutta@yahoo.com

Dhiren Kumar Basnet
Department of Mathematical Sciences,
Tezpur University,
Napaam-784028, Sonitpur, Assam,
India
e-mail: dbasnet@tezu.ernet.in

and

Rajat Kanti Nath
Department of Mathematical Sciences,
Tezpur University,
Napaam-784028, Sonitpur, Assam,
India
e-mail: rajatkantinath@yahoo.com
Corresponding author